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An Inequality for Multiplicative Functions

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Let k be a positive integer and f a multiplicative function with $0 < f(p) \leqslant 1/k$ for all primes p. Then, for squarefree n, we have

$$\sum_{\substack{l \mid n \\ l \leqslant n^{1/(k+1)}}} f(l) \geqslant \frac{1}{k+1} \sum_{l \mid n} f(l)$$

This improves some recent results of Alladi, Erdős, and Vaaler.

1. INTRODUCTION

In [1], Alladi, Erdős, and Vaaler proved the following theorem, which generalises the well known

$$\sum_{\substack{d \mid n \\ d \leqslant \sqrt{n}}} 1 = \frac{d(n)}{2} + O(1).$$

Namely, they proved

THEOREM 1. *Let k be a positive integer and f a multiplicative function with $0 < f(p) \leqslant 1/k$ for all primes p. Then, for squarefree n, we have*

$$(2k+2+o(1)) \sum_{\substack{l \mid n \\ l \leqslant n^{1/(k+1)}}} f(l) \geqslant \sum_{l \mid n} f(l)$$

where $o(1) \to 0$ as $\omega(n) = \sum_{p \mid n} 1 \to \infty$.

The object of this paper is to improve and extend Theorem 1. The proof that we present is also simpler than that of [1].

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2. Notation

Here, we explain the notation used.

Let $t \geqslant 0$ and F_t be the set of multiplicative functions $F: \{1, 2, 3, ...\} \to [0, \infty)$, which satisfy $F(p) \geqslant t$ for all primes p. Also, we denote by G_t the set of all multiplicative functions $G: \{1, 2, 3, ...\} \to [0, \infty)$ which satisfy $t \geqslant G(p) \geqslant 0$ for all primes p.

For squarefree integers n, we write

$$a(t, n) = \inf \left\{ \left(\sum_{\substack{d \mid n \\ d \geqslant n^{t/(t+1)}}} F(d) \right) \left(\sum_{d \mid n} F(d) \right)^{-1} : F \in F_t \right\}$$

$$b(t, n) = \sup \left\{ \left(\sum_{\substack{d \mid n \\ d > n^{t/(t+1)}}} G(d) \right) \left(\sum_{d \mid n} G(d) \right)^{-1} : G \in G_t \right\}.$$

Also we write,

$$A(t) = \inf\,(a(t, m): m \text{ squarefree})$$

$$B(t) = \sup\,(b(t, m): m \text{ squarefree}).$$

3. Statement of Results

Clearly, for all $F \in F_t$, we have

$$\sum_{\substack{d \mid n \\ d \geqslant n^{t/(t+1)}}} F(d) \geqslant A(t) \sum_{d \mid n} F(d),$$

where n is squarefree. It is obvious that Theorem 1 is equivalent to saying

$$A(k) \geqslant \frac{1}{2k + 2 + o(1)}$$

for integers $k \geqslant 1$.

Improving this, we prove

Theorem 2. *For all $t \geqslant 0$, we have*

$$A(t+1) \geqslant \frac{A(t)}{A(t) + 1}.$$

In particular

$$A(k) \geqslant \frac{1}{k+1}$$

for non-negative integers k.

Regarding $B(t)$, we prove

THEOREM 3. *For all $t \geqslant 0$, we have*

$$B(t+1) \leqslant \frac{1}{2-B(t)}.$$

In particular, we have

$$B(k) = \frac{k}{k+1}$$

for positive integers k.

We use Theorem 3 to extend Theorem 1 for rational numbers $k \geqslant 0$. We prove

THEOREM 4. *For all $t > 0$, we have*

$$A(1/t) + B(t) = 1.$$

Further, if $k > 0$ is rational and $[a_0, a_1, ..., a_r]$ is the continued fraction expansion of k, then we have

$$A(k) \geqslant \frac{1}{1 + a_0 + a_1 + \cdots + a_r}$$

and

$$B(k) \leqslant \frac{a_0 + a_1 + \cdots + a_r}{1 + a_0 + a_1 + \cdots + a_r}.$$

4. PROOF OF THEOREM 2

Let F be any element of F_{t+1} ($t \geqslant 0$). We write $F(n) = \sum_{d \mid n} g(d)$ where g is a multiplicative function. Clearly $g \in F_t$. Now

$$\sum_{\substack{d \mid n \\ d \geqslant n^{(t+1)/(t+2)}}} F(d) = \sum_{\substack{d \mid n \\ d \geqslant n^{(t+1)/(t+2)}}} \sum_{ab=d} g(b)$$

$$= \sum_{a \mid n} \sum_{\substack{b \mid n/a \\ b \geqslant (n^{(t+1)/(t+2)}/a)}} g(b)$$

$$\geqslant \sum_{\substack{a \mid n \\ a \geqslant n^{1/(t+2)}}} \sum_{\substack{b \mid n/a \\ b \geqslant (n^{(t+1)/(t+2)}/a)}} g(b)$$

$$\geqslant \sum_{\substack{a \mid n \\ a \geqslant n^{1/(t+2)}}} \sum_{\substack{b \mid n/a \\ b \geqslant (n/a)^{(t/(t+1))}}} g(b)$$

$$\geqslant A(t) \sum_{\substack{a \mid n \\ a \geqslant n^{1/(t+2)}}} F(n/a) = A(t) \Bigg(\sum_{\substack{a \mid n \\ a \leqslant n^{(t+1)/(t+2)}}} F(a) \Bigg)$$

$$\geqslant A(t) \Bigg(\sum_{a \mid n} F(a) - \sum_{\substack{a \mid n \\ a \geqslant n^{(t+1)/(t+2)}}} F(a) \Bigg)$$

and so $A(t+1) \geqslant A(t)/(A(t)+1)$. Now clearly, $A(0)=1$ and hence it follows that $A(k) \geqslant 1/(k+1)$ for all non-negative integers k. This proves Theorem 2.

5. Proof of Theorem 3

Let G be any element of G_{t+1} $(t \geqslant 0)$. As before, we write $G(n) = \sum_{d \mid n} h(d)$ where h is multiplicative and clearly belongs to G_t. Now

$$\sum_{\substack{d \mid n \\ d > n^{(t+1)/(t+2)}}} G(d) = \sum_{\substack{ab \mid n \\ ab > n^{(t+1)/(t+2)}}} h(b) = \sum_{a \mid n} \sum_{\substack{b \mid n/a \\ b > (n^{(t+1)/(t+2)}/a)}} h(b)$$

$$= \sum_{\substack{a \mid n \\ a < n^{1/(t+2)}}} \sum_{\substack{b \mid n \\ b > (n^{(t+1)/(t+2)}/a)}} h(b) + \sum_{\substack{a \mid n \\ a \geqslant n^{1/(t+2)}}} \sum_{\substack{b \mid n/a \\ b > (n^{(t+1)/(t+2)}/a)}} h(b)$$

$$\leqslant B(t) \sum_{\substack{a \mid n \\ a < n^{1/(t+2)}}} G(n/a) + \sum_{\substack{a \mid n \\ a \geqslant n^{1/(t+2)}}} G(n/a)$$

(by the definition of $B(t)$)

$$\leqslant B(t) \sum_{\substack{a \mid n \\ a > n^{(t+1)/(t+2)}}} G(a) + \sum_{a \mid n} G(a) - \sum_{\substack{a \mid n \\ a > n^{(t+1)/(t+2)}}} G(a)$$

and hence $B(t+1) \leqslant 1/(2 - B(t))$. Clearly $B(1) = 1/2$, and so $B(k) \leqslant k/(k+1)$ for all positive integers k.

For a fixed positive integer k, consider a squarefree number n having a prime factor, p, greater than $n^{k/(k+1)}$. Consider the function $G'(d) = k^{\omega(d)}$ which belongs to G_k. Clearly

$$\sum_{\substack{d \mid n \\ d \geqslant n^{k/(k+1)}}} k^{\omega(d)} = \sum_{pd \mid n} k^{\omega(d)+1} = k \sum_{d \mid n/p} k^{\omega(d)}$$

$$= k(k+1)^{\omega(n)-1}$$

and

$$\sum_{d \mid n} k^{\omega(d)} = (k+1)^{\omega(d)}, \qquad \text{whence}$$

$$B(k) \geqslant b(k, n) \geqslant k/(k+1).$$

From this and our earlier statement $B(k) \leqslant k/(k+1)$ it follows that $B(k) = k/(k+1)$, which completes the proof of Theorem 3.

6. Proof of Theorem 4

Let G be an element of G_t and let $H(l) = 1/(G(l))$ for all squarefree l. Then $H \in F_{(1/t)}$ $(t > 0)$. Consider,

$$\sum_{\substack{l \mid n \\ l > n^{t/(t+1)}}} G(l) = \sum_{\substack{l \mid n \\ l < n^{1/(t+1)}}} G(n)/G(l) = \sum_{\substack{l \mid n \\ l < n^{1/(t+1)}}} G(n)\, H(l)$$

$$= G(n) \Bigg(\sum_{l \mid n} H(l) - \sum_{\substack{l \mid n \\ l \geqslant n^{1/(t+1)}}} H(l) \Bigg)$$

and so

$$\Bigg(\sum_{\substack{l \mid n \\ l > n^{t/(t+1)}}} G(l) \Bigg) \Bigg(\sum_{l \mid n} G(l) \Bigg)^{-1} + \Bigg(\sum_{\substack{l \mid n \\ l \geqslant n^{1/(t+1)}}} H(l) \Bigg) \Bigg(\sum_{l \mid n} H(l) \Bigg)^{-1} = 1.$$

From this, it easily follows that

$$A(1/t) + B(t) = 1 \qquad \text{for} \quad t > 0.$$

To prove the second part of the theorem, it suffices to consider k in $(0, 1]$, by Theorems 2 and 3. We use induction on the denominator of k. The theorem is easily seen to be true for all rational numbers with denominator 1. Suppose the theorem is true for all rational numbers with denominator less than the denominator of k. Let $[0, a_1, ..., a_r]$ be the continued fraction expansion of $k \in (0, 1]$ and $[a_1, ..., a_r]$ that of $1/k$. Now $1/k$ has a lower denominator than k, and the result follows from the induction hypothesis and the fact

$$A(k) + B(1/k) = B(k) + A(1/k) = 1.$$

This proves the theorem.

7. Concluding Remarks

If t is an irrational number, then by considering a rational approximation a/q to t from above, with $q \leqslant \sqrt{\omega(n)}$ and $0 < a/q - t \leqslant 2/(\omega(n))$, we can see (using Theorem 4) that

$$a(t, n) \geqslant \frac{1}{10\sqrt{\omega(n)}}$$

and

$$b(t, n) \leqslant 1 - \frac{1}{10\sqrt{\omega(n)}}.$$

This improves Theorem 2 of [1] which gives $a(t, n) \geqslant t/((t+1)\,\omega(n))$. However, in view of Theorem 4, we would expect bounds for $a(t, n)$ and $b(t, n)$ which are independent of $\omega(n)$.

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Reference

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