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PHYSICS LETTERS B
Physics Letters B 570 (2003) 251–259
www.elsevier.com/locate/npe

Classical $O(N)$ nonlinear sigma model on the half line: a study on consistent Hamiltonian description

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Received 20 June 2003; accepted 20 July 2003

Editor: T. Yanagida

Abstract

The problem of consistent Hamiltonian structure for $O(N)$ nonlinear sigma model in the presence of five different types of boundary conditions is considered in detail. For the case of Neumann, Dirichlet and the mixture of these two types of boundaries, the consistent Poisson brackets are constructed explicitly, which may be used, e.g., for the construction of current algebras in the presence of boundary. While for the mixed boundary conditions and the mixture of mixed and Dirichlet boundary conditions, we prove that there is no consistent Poisson brackets, showing that the mixed boundary conditions are incompatible with all nontrivial subgroups of $O(N)$.

PACS: 11.10.Ef; 11.10.Lm; 11.10.Kk

Keywords: Poisson brackets; $O(N)$ sigma model; Boundary condition

1. Introduction

Field theories with boundaries have been attracting the attention of theoretical physicists for a number of reasons, especially from the quantum point of view. For example, the existence of boundaries is responsible for the Casimir effect and surface phenomena, fundamental excitations in the bulk may have interesting behavior when scattered off the boundaries [1], and sometimes boundary bound state might appear, etc. Another important aspect of boundaries appear in the study of string theory, where they are used to distinguish different types of string theories and are also regarded as the reason for the occurrence of noncommutativity on the D-branes.

The introduction of boundary interactions into the Lagrangian also causes some problem at the classical level, since the boundary conditions would in general spoil the naive Poisson structure. In order to describe classical field theories with boundaries as consistent Hamiltonian systems, many authors prefer to use the Dirac method for treating constraints [2–4]. However, as pointed out in [5], the direct application of Dirac method in boundary systems has some problems, mostly due to the fact that boundary conditions regarded as constraints have functional mea-

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doi:10.1016/j.physletb.2003.07.058

sure 0 in the space of fields. To overcome these problems, some authors prefer to use modified versions of Dirac method [6], or first turn the field theories with boundaries into mechanical systems with infinite many degrees of freedom by use of either Fourier mode expansion or lattice approximation and then use the Dirac approach [7–10]. Other methods, including symplectic quantization [11,12] and lightcone quantization [13], are also used to treat boundary problems. However, none of the above mentioned methods is applicable systematically to all field theories with boundaries. The approach which works for one particular model with ease may become very cumbersome, or even completely inapplicable to use for another. In [5], we proposed a novel method for treating the boundary constraints. Our method is based on a very simple idea, i.e., the principle of locality: since the boundary conditions are constraints only at the boundaries, they should modify the naive Poisson structure only at the boundaries. By directly modifying the naive Poisson brackets at the boundaries with some test operators and checking the compatibility with boundary constraints, we can obtain conditions to determine the test operators. This method is used in the subsequent works [14] and [15] to study the problem of open string quantization in background NS–NS B-field, and is proven to be very powerful and easy to use.

In this Letter, we are aimed at using the method of [5] to study the problem of consistent Hamiltonian description for $O(N)$ nonlinear sigma model in the presence of integrable boundary conditions [16,17]. Besides getting more concrete examples for the application of our method, there are more direct motivations to study the Hamiltonian description for this model. In the literatures, $O(N)$ nonlinear sigma model is often taken as a typical model of field theories with certain nice geometrical properties [18], it is also a frequently used toy model for stimulating nonabelian gauge theories [19–22], and a theoretical laboratory for exploring Poisson–Lie geometry and current algebras [23]. In a number of problems in statistical physics, condensed matter systems [24,25] and/or high energy physics, e.g., quantum antiferromagnetism, large N behavior and asymptotic freedom in strong interactions, $O(N)$ nonlinear sigma model is often found to be a simplified version of the underlying field theoretic description. Another area in which nonlinear sigma model found important applications is string theory. There the model is often used to describe D-brane dynamics in curved backgrounds [26]. The exact integrability of $O(N)$ nonlinear sigma model on the half line [16,17,27,28] provides more direct motivations for the mathematical physicists to study this theory. In this respect, the study of consistent Hamiltonian description of $O(N)$ nonlinear sigma model is quite essential, because the quantum analysis on the factorized scattering in the bulk as well as off the boundary based on quantum inverse scattering method needs semiclassical support, for which the classical Hamiltonian description of the model is a starting point. Even from a pure classical integrable system point of view, a consistent Hamiltonian description is still a key structure because it is needed to prove that the integrals of motion are pairwise in involution under the correct Poisson brackets. However, to our knowledge, a systematical analysis on the Hamiltonian structure of the $O(N)$ nonlinear sigma model in the presence of integrable boundary conditions is still not undertaken, at least in the form we shall present. That's why we start our analysis from now on.

2. The model on the half-line

The action for $O(N)$ nonlinear sigma model in $(1+1)$-spacetime dimensions reads

$$S = \frac{1}{2}\int d^2x \left[\partial_\mu \mathbf{n}^{\mathrm{T}} \cdot \partial^\mu \mathbf{n} + \omega\big(\mathbf{n}^{\mathrm{T}} \cdot \mathbf{n} - 1\big)\right], \tag{1}$$

where the field $\mathbf{n} = (n_1, n_2, n_3, \ldots, n_N)^{\mathrm{T}}$ obey the $O(N)$ condition $\mathbf{n}^{\mathrm{T}} \cdot \mathbf{n} = 1$, thanks to the Lagrangian multiplier ω. We use the superscript T to represent matrix transpose. The spacetime metric we adopt is $(\eta_{\mu\nu}) = \mathrm{diag}(1, -1)$, and summation over repeated indices is assumed throughout.

The variation of (1) with respect to $\mathbf{n}$ leads to the equation of motion

$$\partial_\mu \partial^\mu \mathbf{n}^{\mathrm{T}} - \omega \mathbf{n}^{\mathrm{T}} = 0. \tag{2}$$

By use of the $O(N)$ condition $\mathbf{n}^{\mathrm{T}} \cdot \mathbf{n} = 1$, (2) can be rewritten as

$$\partial_\mu \partial^\mu \mathbf{n} + \big(\partial_\mu \mathbf{n}^{\mathrm{T}} \cdot \partial^\mu \mathbf{n}\big)\mathbf{n} = 0. \tag{3}$$

In the Hamiltonian description, the fundamental dependent variables are the fields ("canonical coordinates") and their conjugate momenta. The conjugate momenta in the bulk are defined as

$$\pi_i \equiv \frac{\delta \mathcal{L}_{\rm B}}{\delta(\partial_t n^i)} = \partial_t n_i. \tag{4}$$

Since the $O(N)$ condition $\mathbf{n}^{\rm T} \cdot \mathbf{n} = 1$ is a constraint, the correct Poisson brackets for the fields n_i and the conjugate momenta π_i must be obtained by use of the standard Dirac method. The results read

$$\{n_i(x), n_j(y)\} = 0, \tag{5}$$

$$\{n_i(x), \pi_j(y)\} = (\delta_{ij} - n_i n_j)\delta(x-y), \tag{6}$$

$$\{\pi_i(x), \pi_j(y)\} = (\pi_i n_j - n_i \pi_j)\delta(x-y). \tag{7}$$

This finishes the description of the model in the bulk.

In the presence of a boundary, the form of the Lagrangian is kept unchanged, but the spatial integration in (1) is restricted on the half line $x \in [0, \infty)$. Several types of boundary conditions are *claimed* to be integrable in the literatures [16,17]. They are:

(i) Neumann boundary conditions along all target space directions, i.e., $\partial_x n_i|_{x=0} = 0$, $i = 1, \ldots, N$. We denote this set of boundary conditions as (AN) (i.e., all Neumann);

(ii) Dirichlet boundary conditions along all target space directions, i.e., $\partial_t n_i|_{x=0} = 0$, $i = 1, \ldots, N$. This set of boundary conditions is denoted as (AD) (i.e., all Dirichlet);

(iii) A mixture of Neumann and Dirichlet boundary conditions, i.e., $\partial_x n_i|_{x=0} = 0$ for $i = 1, \ldots, p$ and $\partial_t n_i|_{x=0} = 0$ for $i = p+1, \ldots, N$. This set of boundary conditions is denoted as (ND) (i.e., mixed Neumann and Dirichlet);

(iv) Mixed boundary conditions along all target space directions, i.e., $(\partial_x n_i + M_{ij}\partial_t n_j)|_{x=0} = 0$ for $i = 1, \ldots, N$, where M is a real invertible antisymmetric matrix of the form

$$M = g_1(i\sigma^2) \oplus g_2(i\sigma^2) \oplus \cdots \oplus g_K(i\sigma^2), \tag{8}$$

in which σ^2 is the second Pauli matrix, g_1 through g_K are free parameters (boundary coupling constants). Notice that this type of boundary conditions is only possible for even $N = 2K$, because otherwise M cannot not be invertible. This set of boundary conditions is actually not found in [16,17], but is a simple generalization of the non-diagonal boundary conditions proposed there (the non-diagonal boundary condition in [16,17] contains only one $i\sigma^2$ block). We shall refer to this set of boundary conditions as (AM) (all mixed);

(v) A mixture of mixed and Dirichlet boundary conditions, i.e., $(\partial_x n_i + \mathcal{M}_{ij}\partial_t n_j)|_{x=0} = 0$ for $i, j = 1, \ldots, p$ $(p = 2K)$ and $\partial_t n_i = 0$ for $i = p+1, \ldots, N$, where $\mathcal{M}$ is given as the M in (8). This last set of boundary conditions is denoted as (MD). It has been mentioned in [16,17] that the mixture of mixed and Neumann boundary conditions (MN) is not integrable, at least on the quantum level. We thus exclude this case from our consideration.

To put things together, it is useful to introduce another matrix

$$W = \begin{pmatrix} \mathcal{W} & \\ & 0_{N-p} \end{pmatrix}, \tag{9}$$

in which $\mathcal{W} = \mathcal{M}^{-1}$, the inverse of $\mathcal{M}$. Then the MD boundary conditions can be written in the following unified form:

$$(\partial_t n_i + W_{ij}\partial_x n_j)\big|_{x=0} = 0, \quad i = 1, \ldots, N. \tag{10}$$

Moreover, the form of (10) also contains the other 4 types of boundary conditions mentioned above as special degenerated cases, if we allow the matrix $\mathcal{W}$ to take different forms. Concretely, (10) will be reduced into AD boundaries for $p = 0$, into AM boundaries for $p = N = 2K$ and $W = M^{-1}$; for generic p with $\mathcal{W}$ diagonal and all $\mathcal{W}_{ii} \to \infty$, (10) will be reduced into ND boundaries; and for $p = N$ with $\mathcal{W}$ diagonal and all $\mathcal{W}_{ii} \to \infty$, it will be reduced into AN boundaries. We therefore will take (10) as the starting point for our analysis.

It should be remarked that, in the presence of the boundary conditions (10), there is some ambiguity in the definition of canonical conjugate momenta, because the mixed boundary conditions can be realized via variational principle by adding a boundary term to the action which contains $\partial_t n_i$. The additional boundary term makes the canonical momenta defined as variations of the *complete Lagrangian* $\mathcal{L}$ with respect to the time derivatives of the fields n_j differ from those defined as variations of the *bulk Lagrangian* $\mathcal{L}_{\rm B}$. For our purpose, it is more convenient to stick to the bulk momenta π_i, because there is already a set of known

Poisson brackets (5)–(7) which can be taken as the basis of our analysis. Using the phase space variables n_i and π_i, we can rewrite the boundary conditions (10) as

$$(\pi_i + W_{ij}\partial_x n_j)\big|_{x=0} = 0. \tag{11}$$

It can be seen that, since the boundary conditions (11) identify $\partial_x n_i$ with some specific linear combination of π_i, the Poisson brackets (5)–(7) would no longer hold. In the next section, we shall try to construct consistent Poisson brackets which are compatible with (11). However, it will turn out that only for AD, AN and ND boundaries we can make a success. For AM and MD boundaries we can find no consistent Poisson brackets, which indicates that the mixed boundary conditions are not allowed for $O(N)$ nonlinear sigma model.

3. Boundary constraints and general compatibility conditions

Following the method of [5], the very first step in getting consistent modifications of the Poisson brackets (5)–(7) would be introducing the *boundary constraints*

$$G_i \equiv \int_0^\infty dx\,\delta(x)(\pi_i + W_{ij}\partial_x n_j) \simeq 0. \tag{12}$$

This is just another way of writing the boundary conditions (11), in which the δ-function is a slightly regularized one [5], satisfying $\int_0^\infty dx\,\delta(x) = 1$.

Since the constraints G_i are strong zeros beyond the boundary at $x = 0$, it is tempting to think that there is no need to modify (5)–(7) except at $x = 0$, and it was indeed so in the cases of [5,14,15]. However, at this point, we would prefer to keep things as general as possible. Therefore, assuming that the consistent bulk Poisson brackets take the form

$$\begin{aligned}
\{n_i(x), n_j(y)\} &= \mathfrak{A}_{ij}(n,\pi)\delta(x-y),\\
\{n_i(x), \pi_j(y)\} &= \mathfrak{B}_{ij}(n,\pi)\delta(x-y),\\
\{\pi_i(x), \pi_j(y)\} &= \mathfrak{C}_{ij}(n,\pi)\delta(x-y),
\end{aligned} \tag{13}$$

and adding boundary modifications, the most general form for the potential consistent Poisson brackets will be

$$\begin{aligned}
&\{n_i(x), n_j(y)\}_{\mathrm{M}}\\
&\quad= \mathfrak{A}_{ij}(n,\pi)\delta(x-y) + \mathcal{A}_{ij}\delta(x+y),
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\{n_i(x), \pi_j(y)\}_{\mathrm{M}}\\
&\quad= \mathfrak{B}_{ij}(n,\pi)\delta(x-y) + \mathcal{B}_{ij}\delta(x+y),
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\{\pi_i(x), \pi_j(y)\}_{\mathrm{M}}\\
&\quad= \mathfrak{C}_{ij}(n,\pi)\delta(x-y) + \mathcal{C}_{ij}\delta(x+y),
\end{aligned} \tag{16}$$

where the suffix M denotes modified Poisson brackets, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are some *known* functions in the phase space with $\mathfrak{A}_{ij}$ and $\mathfrak{C}_{ij}$ antisymmetric in $i \leftrightarrow j$, and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are some operators acting on the variable y which are yet to be determined by consistency requirements. Since the Poisson brackets are antisymmetric, the operators $\mathcal{A}_{ij}$ and $\mathcal{C}_{ij}$ *must* also *be antisymmetric* in $i \leftrightarrow j$.

At first sight, it may look strange that we assume the odd form (13) for the bulk Poisson brackets rather than use (5)–(7) directly. The reason for this will be clear in the next section when we try to find solutions for the compatibility conditions which we now derive.

In order to determine the values of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we first apply the compatibility conditions

$$\{G_i, n_j(y)\}_{\mathrm{M}} = 0, \tag{17}$$

$$\{G_i, \pi_j(y)\}_{\mathrm{M}} = 0. \tag{18}$$

Straightforward calculations yield

$$\begin{aligned}
&\{G_i, n_j(y)\}_{\mathrm{M}}\\
&\quad= \int_0^\infty dx\,\delta(x)\{\pi_i + W_{ik}\partial_x n_k, n_j(y)\}_{\mathrm{M}}\\
&\quad= \int_0^\infty dx\,\delta(x)\big(\{\pi_i, n_j(y)\}_{\mathrm{M}} + \{W_{ik}\partial_x n_k, n_j(y)\}_{\mathrm{M}}\big)\\
&\quad= \int_0^\infty dx\,\delta(x)\big[-\mathfrak{B}_{ji}(n,\pi)\delta(x-y) - \mathcal{B}_{ji}\delta(x+y)\\
&\qquad + W_{ik}\partial_x\{\mathfrak{A}_{kj}(n,\pi)\delta(x-y) + \mathcal{A}_{kj}\delta(x+y)\}\big]\\
&\quad= -\big[(\mathfrak{A} - \mathcal{A})W\partial_y + (\mathfrak{B} + \mathcal{B})\big]_{ji}\delta(y),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\{G_i, \pi_j(y)\}_{\mathrm{M}}\\
&\quad= \int_0^\infty dx\,\delta(x)\{\pi_i + W_{ik}\partial_x n_k, \pi_j(y)\}_{\mathrm{M}}
\end{aligned}$$

$$= \int_0^\infty dx\, \delta(x)\big(\{\pi_i, \pi_j(y)\}_{\mathrm{M}} + \{W_{ik}\partial_x n_k, \pi_j(y)\}_{\mathrm{M}}\big)$$
$$= \int_0^\infty dx\, \delta(x)\big[\mathfrak{C}_{ij}(n,\pi)\delta(x-y) + \mathcal{C}_{ij}\delta(x+y)$$
$$+ W_{ik}\partial_x\big(\mathfrak{B}_{ij}(n,\pi)\delta(x-y) + \mathcal{B}_{ij}\delta(x+y)\big)\big]$$
$$= \big[\mathfrak{C} + \mathcal{C} - W(\mathfrak{B} - \mathcal{B})\partial_y\big]_{ij}\delta(y), \qquad (20)$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \ldots, \pi_N)^{\mathrm{T}}$. Comparing (19), (20) to the compatibility conditions (17) and (18), we get the following equation for the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,

$$(\mathfrak{A} - \mathcal{A})W\partial_y + (\mathfrak{B} + \mathcal{B}) = 0, \qquad (21)$$
$$\mathfrak{C} + \mathcal{C} - W(\mathfrak{B} - \mathcal{B})\partial_y = 0. \qquad (22)$$

The compatibility between the test Poisson brackets and the boundary constraints do not provide the complete set of compatibility conditions for the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. In order that the test Poisson brackets (14)–(16) be fully consistent, they are also required to satisfy Jacobi identities. For the canonical variables n_i, π_j, there are totally 4 different types of Jacobi identities to check, i.e., the ones for $\{n_i, n_j, n_k\}$, $\{n_i, n_j, \pi_k\}$, $\{n_i, \pi_j, \pi_k\}$ and $\{\pi_i, \pi_j, \pi_k\}$ respectively. These identities hold identically beyond the boundary, because the bulk Poisson brackets (13) are already consistent before implementing the boundary constraints. Therefore, what we need to check are only the Jacobi identities at the boundary. Using (14)–(16), we get from the above mentioned Jacobi identities the following equations:

$$\frac{\delta(\mathfrak{A}+\mathcal{A})_{ij}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mk} - \frac{\delta(\mathfrak{A}+\mathcal{A})_{ij}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{km}$$
$$+ \frac{\delta(\mathfrak{A}+\mathcal{A})_{jk}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mi}$$
$$- \frac{\delta(\mathfrak{A}+\mathcal{A})_{jk}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{im}$$
$$+ \frac{\delta(\mathfrak{A}+\mathcal{A})_{ki}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mj}$$
$$- \frac{\delta(\mathfrak{A}+\mathcal{A})_{ki}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{jm} = 0, \qquad (23)$$

$$\frac{\delta(\mathfrak{A}+\mathcal{A})_{ij}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mk} + \frac{\delta(\mathfrak{A}+\mathcal{A})_{ij}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mk}$$
$$+ \frac{\delta(\mathfrak{B}+\mathcal{B})_{jk}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mi}$$
$$- \frac{\delta(\mathfrak{B}+\mathcal{B})_{jk}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{im}$$
$$- \frac{\delta(\mathfrak{B}+\mathcal{B})_{ik}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mj}$$
$$+ \frac{\delta(\mathfrak{B}+\mathcal{B})_{ik}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{jm} = 0, \qquad (24)$$

$$\frac{\delta(\mathfrak{B}+\mathcal{B})_{ij}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mk} + \frac{\delta(\mathfrak{B}+\mathcal{B})_{ij}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mk}$$
$$+ \frac{\delta(\mathfrak{C}+\mathcal{C})_{jk}}{\delta n_m}(\mathfrak{A}+\mathcal{A})_{mi} - \frac{\delta(\mathfrak{C}+\mathcal{C})_{jk}}{\delta \pi_m}(\mathfrak{B}+\mathcal{B})_{im}$$
$$- \frac{\delta(\mathfrak{B}+\mathcal{B})_{ik}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mj}$$
$$- \frac{\delta(\mathfrak{B}+\mathcal{B})_{ik}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mj} = 0, \qquad (25)$$

$$\frac{\delta(\mathfrak{C}+\mathcal{C})_{ij}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mk} + \frac{\delta(\mathfrak{C}+\mathcal{C})_{ij}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mk}$$
$$+ \frac{\delta(\mathfrak{C}+\mathcal{C})_{jk}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mi} + \frac{\delta(\mathfrak{C}+\mathcal{C})_{jk}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mi}$$
$$+ \frac{\delta(\mathfrak{C}+\mathcal{C})_{ki}}{\delta n_m}(\mathfrak{B}+\mathcal{B})_{mj}$$
$$+ \frac{\delta(\mathfrak{C}+\mathcal{C})_{ki}}{\delta \pi_m}(\mathfrak{C}+\mathcal{C})_{mj} = 0. \qquad (26)$$

Once the equations (23)–(26) are satisfied, the Jacobi identities for any functions on the phase space will hold consistently, because n_i, π_j form a basis for the phase space of the model. Therefore we conclude that the system of equations (21)–(26) is the complete set of conditions which the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ must obey. As long as a solution $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$ to the above system of operator equations is found, we will get a consistent Hamiltonian description for $O(N)$ nonlinear sigma model with the boundary conditions (10). However, since the system of equations (21)–(26) is over determined, the existence of a solution is not guaranteed in general. When no solution to (21)–(26) can be found, the nonexistence of a solution should be considered as a signature that the corresponding boundary conditions are incompatible with the bulk dynamics. In the next section, we shall show that the AM and MD boundaries belong to this forbidden class of boundaries. The other three types of boundaries, i.e., AD, AN and ND boundaries, will all give rise to consistent solutions to the compatibility equations (21)–(26).

4. Consistent Poisson brackets

In this section, we shall try to find explicit solutions for the system of equations (21)–(26) under each of the five different types of boundary conditions mentioned earlier. The basic strategy in getting these special solutions is like this: we shall first try to get solutions to the relatively simpler equations (21), (22) and then check that they are consistent with the rest equations, (23)–(26). All solutions to the system of equations (21)–(26) can in principle be obtained in this manner.

4.1. $O(N)$ symmetric boundaries AD and AN

The first types of boundaries we shall consider are the AD and AN boundaries, which can be easily seen to preserve the complete $O(N)$ symmetry of the model. We shall treat both of these two types of boundary conditions in a unified way by use of the boundary constraints (12) and requiring p to be either 0 or equal to N. Doing so we are seemingly to be considering the AD, AN and AM boundaries in a unified manner. However, it will be clear shortly that the AM case is distinguished from the AD and AN cases, because AM is actually symmetry breaking.

Now let us look at the equations (21), (22) in more detail. Since we are now considering symmetry preserving boundaries, there is no problem to identify the bulk Poisson brackets (13) with (5)–(7), i.e., to choose $\mathfrak{A}_{ij} = 0$, $\mathfrak{B}_{ij} = \delta_{ij} - n_i n_j$ and $\mathfrak{C}_{ij} = \pi_i n_j - \pi_j n_i$. Then (21), (22) will become

$$\mathcal{A}_{im} W_{mj} \partial_y - \big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}} + \mathcal{B}\big)_{ij} = 0, \tag{27}$$

$$\big(\boldsymbol{\pi} \cdot \mathbf{n}^{\mathrm{T}} - \mathbf{n} \cdot \boldsymbol{\pi}^{\mathrm{T}} + \mathcal{C}\big)_{ij} - W_{im}\big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}} - \mathcal{B}\big)_{mj} \partial_y = 0. \tag{28}$$

To solve the last two equations, we need to consider three different cases, i.e., (a) $p = 0$ or effectively $W = 0$; (b) $p = N$ with W diagonal and $W_{ii} \to \infty$ for all i; (c) $p = N = 2K$ and $W = M^{-1}$ with M given in (8). In case (a) we get from (27) and (28) the result

$$\mathcal{B}_{ij} = -\big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\big)_{ij},$$
$$\mathcal{C}_{ij} = -\big(\boldsymbol{\pi} \cdot \mathbf{n}^{\mathrm{T}} - \mathbf{n} \cdot \boldsymbol{\pi}^{\mathrm{T}}\big)_{ij};$$

in case (b) we have

$$\mathcal{A}_{ij} = 0, \qquad \mathcal{B}_{ij} = \big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\big)_{ij};$$

and, in case (c), since the first term in (28) is antisymmetric in $i \leftrightarrow j$ while the second term is not, we must require both terms to vanish separately, yielding

$$\mathcal{C}_{ij} = -\big(\boldsymbol{\pi} \cdot \mathbf{n}^{\mathrm{T}} - \mathbf{n} \cdot \boldsymbol{\pi}^{\mathrm{T}}\big)_{ij}, \qquad \mathcal{B}_{ij} = \big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\big)_{ij}.$$

It then follows from (27) that $\mathcal{A}_{ij} = 2(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}})_{im}(W^{-1})_{mj}(\partial_y)^{-1}$, which is not acceptable because it is not antisymmetric in $i \leftrightarrow j$. Therefore, we conclude that there is no solution to the equations (27), (28) with $W = M^{-1}$. This implies that the AM boundaries are not compatible with the bulk $O(N)$ symmetry, which has been used to obtain the Poisson brackets (5)–(7) upon which the equations (27), (28) are based. Therefore, we shall temporarily restrict ourselves to the cases (a) and (b).

By use of the equations (23)–(26), we find that, for the case (a), i.e., AD boundaries, the following operators constitute a consistent set of solution to (21)–(26),

$$\mathcal{A}_{ij} = 0, \qquad \mathcal{B}_{ij} = -\big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\big)_{ij},$$
$$\mathcal{C}_{ij} = -\big(\boldsymbol{\pi} \cdot \mathbf{n}^{\mathrm{T}} - \mathbf{n} \cdot \boldsymbol{\pi}^{\mathrm{T}}\big)_{ij}. \tag{29}$$

For the case (b), i.e., AN boundaries, the solution to (21)–(26) is found to be

$$\mathcal{A}_{ij} = 0, \qquad \mathcal{B}_{ij} = \big(I - \mathbf{n} \cdot \mathbf{n}^{\mathrm{T}}\big)_{ij},$$
$$\mathcal{C}_{ij} = \big(\boldsymbol{\pi} \cdot \mathbf{n}^{\mathrm{T}} - \mathbf{n} \cdot \boldsymbol{\pi}^{\mathrm{T}}\big)_{ij}. \tag{30}$$

Substituting the solutions (29) and (30) back into the test Poisson brackets (14)–(16), we get the following Poisson brackets, which are consistent with AD and AN boundary conditions respectively and satisfy all Jacobi identities simultaneously,

$$\big\{n_i(x), n_j(y)\big\}_{\mathrm{M}} = 0,$$
$$\big\{n_i(x), \pi_j(y)\big\}_{\mathrm{M}} = (\delta_{ij} - n_i n_j)\big[\delta(x-y) - \delta(x+y)\big],$$
$$\big\{\pi_i(x), \pi_j(y)\big\}_{\mathrm{M}} = (\pi_i n_j - n_i \pi_j)\big[\delta(x-y) - \delta(x+y)\big], \tag{31}$$

$$\{n_i(x), n_j(y)\}_{\mathrm{M}} = 0,$$
$$\{n_i(x), \pi_j(y)\}_{\mathrm{M}} = (\delta_{ij} - n_i n_j)\big[\delta(x-y) + \delta(x+y)\big],$$
$$\{\pi_i(x), \pi_j(y)\}_{\mathrm{M}} = (\pi_i n_j - n_i \pi_j)\big[\delta(x-y) + \delta(x+y)\big]. \tag{32}$$

The action (1), together with the consistent Poisson brackets (31) (resp. (32)), form a complete Hamiltonian description for classical $O(N)$ nonlinear sigma model in the presence of AD (resp. AN) boundary conditions.

4.2. The symmetry breaking boundary ND

ND boundaries correspond to $1 < p < N$ in (9) and $\mathcal{W}$ diagonal with $\mathcal{W}_{ii} \to \infty$ for all i. Since $O(N)$ transformations cannot transform Neumann boundary conditions into Dirichlet ones, ND boundaries explicitly break the $O(N)$ symmetry into the subgroup $O(p) \times O(N-p)$. Consequently, while considering the consistent Hamiltonian description of the model in the presence of ND boundaries, we need to modify not only the Poisson brackets at the boundary, but also in the bulk. In fact, that the ND boundary conditions break not only the $O(N)$ symmetry at the boundary but also in the bulk is an important conclusion of our study, since it can be seen that the direct substitution of the $O(N)$ conditions $\mathfrak{A}_{ij} = 0$, $\mathfrak{B}_{ij} = \delta_{ij} - n_i n_j$ and $\mathfrak{C}_{ij} = \pi_i n_j - \pi_j n_i$ together with the matrix W in (9)—with $\mathcal{W}$ diagonal and $\mathcal{W}_{ii} \to \infty$ for all i—into the equations (21) and (22) would lead to contradictory results.

For convenience we divide the suffices i, j, etc., of the fields into two disjoint sets, labeled respectively by Latin and Greek letters. Latin indices a, b run from 1 to p and Greek indices α, β run from $p+1$ to N. We also introduce the notations $\mathbf{n}^{(1)} = (n_1, \ldots, n_p)^{\mathrm{T}}$, $\mathbf{n}^{(2)} = (n_{p+1}, \ldots, n_N)^{\mathrm{T}}$ and similarly $\boldsymbol{\pi}^{(1)} = (\pi_1, \ldots, \pi_p)^{\mathrm{T}}$, $\boldsymbol{\pi}^{(2)} = (\pi_{p+1}, \ldots, \pi_N)^{\mathrm{T}}$. Then the $O(p) \times O(N-p)$ symmetric bulk in the presence of ND boundaries can be described by the fields $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ obeying, respectively, $\mathbf{n}^{(1)T} \cdot \mathbf{n}^{(1)} = u$, $\mathbf{n}^{(2)T} \cdot \mathbf{n}^{(2)} = v$, where the constants u and v satisfy $u + v = 1$. The bulk Poisson brackets in this case are characterized by (13) with the following functions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$,

$$\mathfrak{A}_{ab} = \mathfrak{A}_{a\beta} = \mathfrak{A}_{\alpha b} = \mathfrak{A}_{\alpha\beta} = 0,$$
$$\mathfrak{B}_{ab} = \delta_{ab} - n_a n_b, \qquad \mathfrak{B}_{a\beta} = 0, \qquad \mathfrak{B}_{\alpha b} = 0,$$
$$\mathfrak{B}_{\alpha\beta} = \delta_{\alpha\beta} - n_\alpha n_\beta,$$
$$\mathfrak{C}_{ab} = \pi_a n_b - \pi_b n_a, \qquad \mathfrak{C}_{a\beta} = 0, \qquad \mathfrak{C}_{\alpha b} = 0,$$
$$\mathfrak{C}_{\alpha\beta} = \pi_\alpha n_\beta - \pi_\beta n_\alpha. \tag{33}$$

Substituting (33) into (21), (22) and setting $\mathcal{W}_{ij} = 0$ for $i \neq j$ and $\mathcal{W}_{ii} \to \infty$ for all i, we get, from (21)–(26), the following consistent solution,

$$\mathcal{A}_{ab} = 0, \qquad \mathcal{B}_{ab} = \delta_{ab} - n_a n_b,$$
$$\mathcal{C}_{ab} = \pi_a n_b - \pi_b n_a,$$
$$\mathcal{A}_{\alpha\beta} = 0, \qquad \mathcal{B}_{\alpha\beta} = n_\alpha n_\beta - \delta_{\alpha\beta},$$
$$\mathcal{C}_{\alpha\beta} = -\pi_\alpha n_\beta + \pi_\beta n_\alpha,$$
$$\mathcal{A}_{a\beta} = \mathcal{A}_{\alpha b} = \mathcal{B}_{a\beta} = \mathcal{B}_{\alpha b} = \mathcal{C}_{a\beta} = \mathcal{C}_{\alpha b} = 0. \tag{34}$$

The Poisson brackets (13) with $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ given in (33) and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ in (34) are nothing but the union of consistent Poisson brackets for an $O(p)$ nonlinear sigma model with AN boundaries and those of an $O(N-p)$ nonlinear sigma model with AD boundaries, as they should be.

4.3. The forbidden boundaries AM and MD

That the AM boundaries are not compatible with the $O(N)$ symmetry in the bulk has already been mentioned earlier in this section. This fact can also be seen from another point of view. Following [16] and with a straightforward generalization, we can see that the AM boundary conditions (10) with $W = M^{-1}$ can be realized on the Lagrangian level by adding to the bulk action (1) with the boundary term

$$S_b = \int dt\, M_{ij} n_i \partial_t n_j \bigg|_{x=0}. \tag{35}$$

It can be easily seen that, under the global $O(N)$ transformation $n_i \to O_{ij} n_j$, M will transform as $M_{ij} \to O_{ik} M_{kl} O^{\mathrm{T}}_{lj}$. That M does not commute with the generic element O of the group $O(N)$ is an explicit signature that the boundary term (35) is not invariant under $O(N)$. In fact, the maximal subgroup of $O(N)$ which may leave the boundary term (35) invariant is $O(2)^{\otimes K}$, an Abelian subgroup, in which case M must be given in the form of (8). This explains our choice of M in (8).

Since the bulk $O(N)$ symmetry is broken by the AM boundary conditions into $O(2)^{\otimes K}$, we may introduce the fields $\mathbf{n}^{(\ell)} = (n_{2\ell-1}, n_{2\ell})^{\mathrm{T}}$ and their conjugate momenta to describe the bulk system as a union of K $O(2)$ nonlinear sigma models, each obeys $\mathbf{n}^{(\ell)\mathrm{T}} \cdot \mathbf{n}^{(\ell)} = u_\ell$, with the constants u_ℓ satisfying $\sum_{\ell=1}^{K} u_\ell = 1$. Accordingly, the Poisson brackets which are consistent in the bulk are just (13) with the matrix functions $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ given, respectively, by

$$\mathfrak{A} = 0, \qquad \mathfrak{B} = \bigoplus_{\ell=1}^{K} \mathfrak{B}^{(\ell)}, \qquad \mathfrak{C} = \bigoplus_{\ell=1}^{K} \mathfrak{C}^{(\ell)}, \tag{36}$$

where $\mathfrak{B}^{(\ell)}$ and $\mathfrak{C}^{(\ell)}$ are all 2×2 matrices given as

$$\mathfrak{B}^{(\ell)} = I_{2\times 2} - \mathbf{n}^{(\ell)} \cdot \mathbf{n}^{(\ell)\mathrm{T}},$$
$$\mathfrak{C}^{(\ell)} = \boldsymbol{\pi}^{(\ell)} \cdot \mathbf{n}^{(\ell)\mathrm{T}} - \mathbf{n}^{(\ell)} \cdot \boldsymbol{\pi}^{(\ell)\mathrm{T}}. \tag{37}$$

Now substituting (36) and (37) into (21) and (22), we get, at the ℓth diagonal block, the following equations:

$$\mathcal{A}_{im} W_{mj}^{(\ell)} \partial_y - \left(I - \mathbf{n}^{(\ell)} \cdot \mathbf{n}^{(\ell)\mathrm{T}} + \mathcal{B}\right)_{ij} = 0, \tag{38}$$

$$\left(\boldsymbol{\pi}^{(\ell)} \cdot \mathbf{n}^{(\ell)\mathrm{T}} - \mathbf{n}^{(\ell)} \cdot \boldsymbol{\pi}^{(\ell)\mathrm{T}} + \mathcal{C}\right)_{ij} - W_{im}^{(\ell)} \left(I - \mathbf{n}^{(\ell)} \cdot \mathbf{n}^{(\ell)\mathrm{T}} - \mathcal{B}\right)_{mj} \partial_y = 0, \tag{39}$$

where $i, j = 2\ell - 1$ or 2ℓ, $W^{(\ell)}$ is the ℓth diagonal block of W, which is given in (8) through $W = M^{-1}$. It follows that there is no solution to (38) and (39), since the first term in (38) is diagonal, while the second term cannot be diagonal. Similarly, the first term in (39) is anti-diagonal, but the second term cannot be anti-diagonal.

Now we are forced to answer the following questions: What happens to the mixed boundary conditions? Why couldn't we find any consistent Poisson brackets for the $O(N)$ nonlinear sigma model in the presence of AM boundaries? Two contradictory answers might be in order, which are (1) the AM boundaries are completely incompatible with any orthogonal symmetry, i.e., even the $O(2)$'s cannot survive after AM boundary conditions are applied; (2) the method we are using to construct the consistent boundary Poisson brackets fails for the mixed boundaries for $O(N)$ nonlinear sigma model. Our choice is the answer (1). To support our choice, we now consider the simplest case of $K = 1$, i.e., a single $O(2)$ nonlinear sigma model with mixed boundary conditions $(\partial_x n_i + M_{ij} \partial_t n_j)|_{x=0} = 0$, $M = g\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is exactly the original boundary conditions studied in [16,17]. Expanding the above boundary conditions in component form, we get

$$(\partial_x n_1 - g\partial_t n_2)\big|_{x=0} = 0,$$
$$(\partial_x n_2 + g\partial_t n_1)\big|_{x=0} = 0. \tag{40}$$

On the other hand, from the $O(2)$ condition at the boundary, $(n_1^2 + n_2^2)_{x=0} = 1$, we can get

$$(n_1 \partial_t n_1 + n_2 \partial_t n_2)\big|_{x=0} = 0, \tag{41}$$
$$(n_1 \partial_x n_1 + n_2 \partial_x n_2)\big|_{x=0} = 0. \tag{42}$$

Substituting (40) into (42), it follows that

$$(n_1 \partial_t n_2 - n_2 \partial_t n_1)\big|_{x=0} = 0. \tag{43}$$

Combining (41) and (43) with the $O(2)$ condition $(n_1^2 + n_2^2)_{x=0} = 1$, we get both $\partial_x n_i|_{x=0} = 0$ and $\partial_t n_i|_{x=0} = 0$. In other words, if the mixed boundaries are applied, the fields n_i will obey both Neumann and Dirichlet boundary conditions simultaneously. This is certainly impossible, so we end up with the surprising conclusion that the mixed boundaries are actually not allowed in $O(N)$ nonlinear sigma model, not to say their integrability. This conclusion removes the AM as well as MD boundary conditions from the allowed list of integrable boundaries.

5. Discussions

Using the method proposed in [5] and developed in [14] and [15], we analyzed the problem of consistent Poisson brackets for classical $O(N)$ nonlinear sigma model in the presence of five different sets of boundary conditions, i.e., the AD, AN, ND, AM and MD boundaries. Only in the presence of AD, AN and ND boundaries we have found consistent Poisson brackets, while for AM and MD boundaries, no consistent Poisson brackets can be found, showing that the mixed boundary conditions are completely incompatible with any orthogonal symmetry.

Through the analysis of ND boundaries, we find that the idea underlying our method needs a significant modification. The original statement that in the presence of boundary constraints the Poisson brackets need to be modified only at the boundary is only valid if the boundary conditions preserve all the bulk symmetries. On the other hand, if the boundary conditions are symmetry breaking, they will also affect the bulk part of the Poisson brackets, so that the final consistent Poisson brackets have the same symmetry in the bulk and at the boundary.

The result of this Letter not only widens the scope of applicability of the method of [5], but also has important applications in the study of $O(N)$ nonlinear sigma model itself. A straightforward application might be in the study of current algebra in the presence of boundary conditions, which is an important ingredient in the classical integrable structure of the model. For instance, the Poisson algebra calculations made in [29] should be reexamined using our result (32), because the bulk Poisson brackets (5)–(7) are no longer consistent in the presence of Neumann boundaries as used in [29].

Acknowledgement

This work is supported in part by the National Natural Science Foundation of China.

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