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The Chromatic Polynomial of an Unlabeled Graph

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We investigate the chromatic polynomial $\chi(G, \lambda)$ of an unlabeled graph G. It is shown that $\chi(G, \lambda) = (1/|A(g)|) \sum_{\pi \in A(g)} \chi(g, \pi, \lambda)$, where g is any labeled version of G, $A(g)$ is the automorphism group of g and $\chi(g, \pi, \lambda)$ is the chromatic polynomial for colorings of g fixed by π. The above expression shows that $\chi(G, \lambda)$ is a rational polynomial of degree $n = |V(G)|$ with leading coefficient $1/|A(g)|$. Though $\chi(G, \lambda)$ does not satisfy chromatic reduction, each polynomial $\chi(g, \pi, \lambda)$ does, thus yielding a simple method for computing $\chi(G, \lambda)$. We also show that the number $N(G)$ of acyclic orientations of G is related to the argument $\lambda = -1$ by the formula $N(G) = (1/|A(g)|) \sum_{\pi \in A(g)} (-1)^{s(\pi)} \chi(g, \pi, -1)$, where $s(\pi)$ is the number of cycles of π. This information is used to derive Robinson's ("Combinatorial Mathematics V" (Proc. 5th Austral. Conf. 1976), Lecture Notes in Math. Vol. 622, pp. 28–43, Springer-Verlag, New York/Berlin, 1977) cycle index sum equations for counting unlabeled acyclic digraphs. © 1985 Academic Press, Inc.

1. PRELIMINARY DEFINITIONS

A *labeled graph g of order n* is a pair $(V(g), E(g))$, where $V(g)$ is an n-set and $E(g)$ is a set of 2-element subsets of $V(g)$. The elements of $V(g)$ and $E(g)$ are called *points* and *edges*, respectively. For simplicity we assume that $V(g) = \{1, 2, \ldots, n\}$. Let γ_n denote the set of labeled graphs with n points.

The group S_n acts on γ_n in a natural way. If $\pi \in S_n$ and $g \in \gamma_n$ then $\pi(g)$ is the graph with edge set

$$\pi(E(g)) = \{\{\pi(i), \pi(j)\} : \{i, j\} \in E(g)\}.$$

When $\pi(g) = g$ (i.e., when $\pi(E(g)) = E(g)$) we say that π is an *automorphism of g*. The group of all automorphisms of g is denoted $A(g)$.

Let g be a labeled graph with n points and let S be a subset of $E(g)$. Let

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$g\backslash S$ be the graph obtained from g by deleting the edges in S and let g/S be the graph obtained by contracting the edges of S (and then deleting any loops or multiple edges). Note that g/S has fewer vertices and edges than g, whereas $g\backslash S$ has fewer edges and the same number of vertices.

Now assume that π is an automorphism of g and that S is a π-invariant subset of $E(g)$. Then clearly $\pi(E(g)-S)=E(g)-S$ so π is an automorphism of $g\backslash S$. Also there is an induced action of π on $V(g/S)$. It is simple to check that with this induced action π is an automorphism of g/S. In what follows we will at times refer to π as an automorphism of $g\backslash S$ and g/S.

Lastly assume π is an automorphism of a labeled graph g with disjoint cycles $C_1,\dots, C_s$ (as a permutation of $V(g)$). The *quotient of g with respect to π* is the graph $g:\pi$ with $V(g:\pi)=\{C_1,\dots, C_s\}$ and with C_i adjacent to C_j if there exist $v\in C_i$ and $w\in C_j$ with v adjacent to w in g.

2. Chromatic Polynomials

Let g be a labeled graph. A *coloring of g* with k colors is a function σ mapping $V(g)$ into $\{1, 2,\dots, k\}$ with the property that $\sigma(i)$ is different from $\sigma(j)$ whenever i is adjacent to j. It is well known that there is a polynomial $\chi(g, \lambda)$ with the property that $\chi(g, k)$ is the number of colorings of g with k colors for each positive integer k. This polynomial, called the *chromatic polynomial* of g, is a monic polynomial of degree $n=|V(g)|$ with integer coefficients. To compute $\chi(g, \lambda)$ one can use the following three rules;

(a) If g is the one point graph then $\chi(g, \lambda)=\lambda$.

(b) Let g and h be labeled graphs and let $g+h$ be their disjoint union. Then

$$\chi(g+h, \lambda)=\chi(g, \lambda)\,\chi(h, \lambda).$$

(c) Let g be a labeled graph and let e be any edge of g. Then

$$\chi(g, \lambda)=\chi(g\backslash e, \lambda)-\chi(g/e, \lambda).$$

Rules (a), (b), and (c) give a simple procedure to find $\chi(g, \lambda)$. This procedure, known as *chromatic reduction* uses rule (c) to express $\chi(g, \lambda)$ in terms of the chromatic polynomials of graphs with fewer points or fewer edges. By repeatedly applying rule (c) one can eventually express $\chi(g, \lambda)$ in terms of chromatic polynomials which can be evaluated using rules (a) and (b).

Let π be an automorphism of g. Define a *(π, k)-coloring of g* to be a coloring σ of g with k colors having the property that $\sigma(\pi(v))=\sigma(v)$ for all

$v \in V(g)$. For each $k \geqslant 0$ define $\chi(g, \pi, k)$ to be the number of (π, k)-colorings of g. For any π in S_n let $s(\pi)$ denote the number of cycles of π as a permutation of $\{1, 2, \ldots, n\}$. The following theorem states some basic properties of $\chi(g, \pi, k)$.

THEOREM 2.1. *Let g be a labeled graph, let π be an automorphism of g and let k be a non-negative integer.*

(a) *If any vertex cycle of π contains adjacent points, then $\chi(g, \pi, k) = 0$ for all $k \geqslant 1$.*

(b) *Otherwise $\chi(g, \pi, k) = \chi(g : \pi, k)$, where $\chi(g : \pi, \lambda)$ is the chromatic polynomial of the labeled graph $g : \pi$.*

Proof. To prove (a), suppose σ is a (π, k)-coloring of g and suppose that v and w are adjacent points which lie in the same cycle of π. Then $w = \pi^i v$ for some $i \geqslant 1$ so $\sigma(v) = \sigma(w)$, which is a contradiction. So no such σ exist and $\chi(g, \pi, k) = 0$.

The proof of (b) is straightforward. All vertices in the same cycle of π receive the same color in any (π, k)-coloring of g. So there is a natural correspondence between the colorings counted by $\chi(g, \pi, k)$ and those counted by $\chi(g : \pi, k)$. We leave it to the reader to verify that this correspondence is a bijection.

COROLLARY 2.1. *Let g be a labeled graph and π an automorphism of g. Then*

(a) *There exists a polynomial $\chi(g, \pi, \lambda)$ such that the number of (π, k)-colorings of g is given by $\chi(g, \pi, k)$. This polynomial is either* 0 *or a monic, integral polynomial of degree $s(\pi)$ depending on whether or not a cycle of π contains adjacent points.*

(b) (*Chromatic Reduction for $\chi(g, \pi, \lambda)$*). *The polynomial $\chi(g, \pi, \lambda)$ defined in part* (a) *satisfies the following three rules*:

(i) *If g has no lines and if π permutes the vertices of g transitively then $\chi(g, \pi, \lambda) = \lambda$.*

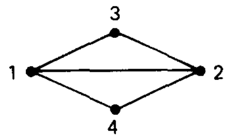
(ii) *If $g = g_1 + g_2$, where g_1 and g_2 are π-invariant subgraphs of g then $\chi(g, \pi, \lambda) = \chi(g_1, \pi, \lambda)\, \chi(g_2, \pi, \lambda)$.*

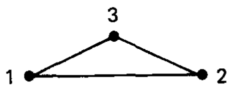
(iii) *Let C_1 and C_2 be vertex cycles of π and let E be the set of edges of g joining points in C_1 to points in C_2. If E is nonempty then*

$$\chi(g, \pi, \lambda) = \chi(g \backslash E, \pi, \lambda) - \chi(g/E, \pi, \lambda).$$

We henceforth call $\chi(g, \pi, \lambda)$ *the* π-chromatic polynomial of g. *A very important property of these polynomials is that for all* $\rho \in S_n$ *we have*

$$\chi(g, \pi, \lambda) = \chi(\rho g, \rho\pi\rho^{-1}, \lambda).$$

EXAMPLE 2.2. Let g be the graph  . Let

$\pi_1 = (1, 2)\ (3)\ (4)$ and let $\pi_2 = (1)\ (2)\ (3, 4)$. By Theorem 2.1(a) we have $\chi(g, \pi_1, \lambda) = 0$ and by Theorem 2.1(b) we have $\chi(g, \pi_2, \lambda) = \chi(h, \lambda)$, where $h = g : \pi_2$ is the graph  . Applying chromatic reduction to $\chi(h, \lambda)$ one finds

$$\chi(g, \pi_2, \lambda) = \lambda(\lambda - 1)(\lambda - 2) = \lambda^3 - 3\lambda^2 + 2\lambda.$$

We end this section with a generalization of Corollary 1.3 of Stanley [3]. Let g be a labeled graph. An *orientation* of g is an assignment of direction $v \to w$ to each edge $\{v, w\}$ of g. The orientation is *acyclic* if g contains no directed cycles. We will use the following fact about acyclic orientations.

THEOREM 2.2 (Stanley [3]). *Let g be a labeled graph with n points. Then the number of acyclic orientations of g is* $(-1)^n \chi(g, -1)$.

Let π be an automorphism of g, and let δ be an orientation of g. We say δ is a *π-orientation* if for each edge $\{v, w\} \in E(g)$ we have $v \to w$ iff $\pi(v) \to \pi(w)$. The following lemma about acyclic π-orientations will be important later.

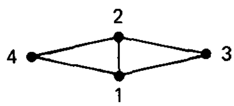
LEMMA. *Let δ be an acyclic π-orientation of g and let C and D be (vertex) cycles of π. Suppose u_0 and v_0 are adjacent vertices with $u_0 \in C$, $v_0 \in D$, and $u_0 \to v_0$. If u and v are any pair of adjacent points in C and D, respectively then $u \to v$.*

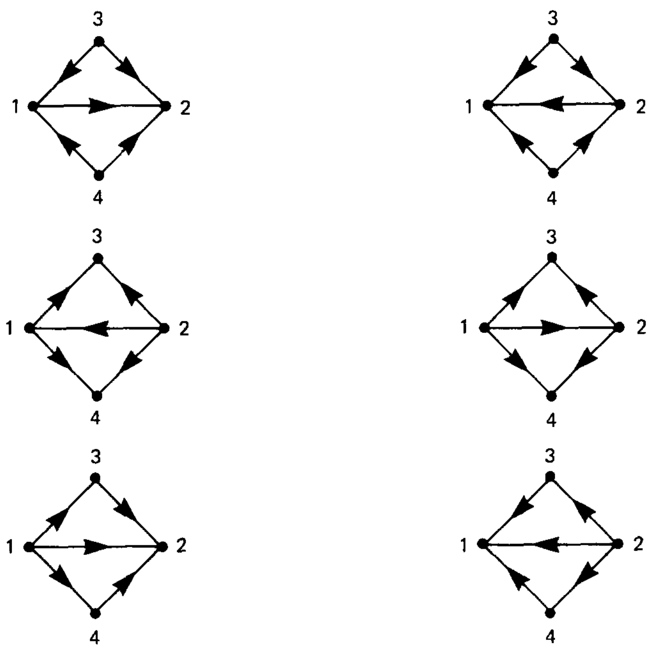
Proof. Assume to the contrary that u and v are adjacent points in C and D with $v \to u$. Choose $i_0 \geqslant 0$ such that $\pi^{i_0} v = v_0$ and let $u_1 = \pi^{i_0} u$. Since δ is a π-orientation we have $v_0 \to u_1$. Next choose $j_0 \geqslant 0$ such that $\pi^{j_0} u_0 = u_1$ and let $v_1 = \pi^{j_0} v_0$. As before we have $u_1 \to v_1$. Continuing in this way we obtain $u_0, u_1, \ldots \in C$ and $v_0, v_1, \ldots \in D$ with $u_i \to v_i$ and $v_i \to u_{i+1}$. Since C and D are finite these lists eventually must repeat previous elements giving us a directed cycle.

This lemma tells us that if C and D are vertex cycles and E is the set of lines joining points in C to points in D then every edge in E must be directed from C to D or from D to C. If we let $C = D$ in the proof of the preceding lemma we have that if a cycle of π contains adjacent points then there are no acyclic π-orientations of g. The next result is a generalization of Theorem 2.2.

THEOREM 2.3. *Let g be a labeled graph and π an automorphism of g. Then the number of acyclic π-orientations of g is $(-1)^{s(\pi)}\chi(g, \pi, -1)$.*

Proof. This is clear if any cycle of π contains adjacent points (since in this case both numbers are 0). Assume no cycle of π contains adjacent points. Define a map ψ from the set of acyclic π-orientations of g to the set of acyclic orientations of $g:\pi$ as follows: if C and D are vertex cycles of π which are adjacent in $g:\pi$ then direct C to D if there exists u in C and v in D with $u \to v$ in δ. The preceding lemma shows that ψ is well defined and it is easy to check that the orientation of $g:\pi$ obtained from δ is acyclic. It is also easy to check that ψ is bijective and so the number of acyclic π-orientations of g equals the number of acyclic orientations of $g:\pi$. By Theorem 2.2 this number is $(-1)^{s(\pi)}\chi(g:\pi, -1)$ which by Theorem 2.1(b) is $(-1)^{s(\pi)}\chi(g, \pi, -1)$. This completes the proof.

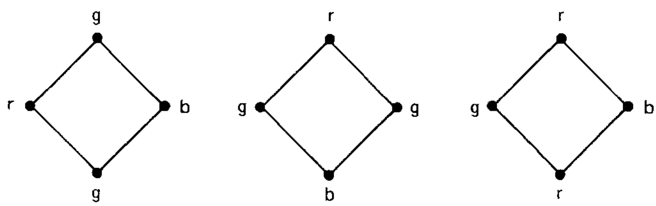
EXAMPLE 2.3. Let g be the graph  and let π be the automorphism (1) (2) (3, 4). Then $(-1)^{s(\pi)}\chi(g, \pi, -1) = (-1)^3((-1)^3 - 3(-1)^2 + 2(-1)) = 6$ (see Example 2.2 for the computation of $\chi(g, \pi, \lambda)$). The six acyclic π-orientations of g appear below.



3. Coloring Unlabeled Graphs

An unlabeled graph is a labeled graph whose labels have been removed. Two unlabeled graphs are the same if and only if they have the same structure. There is an intuitive notion of what we mean to k-color an unlabeled graph G. This is simply an assignment of the colors $\{1, 2, \ldots, k\}$ to the unlabeled points of G in such a way that adjacent points are assigned different colors. Two of these colorings are the same if they are the same up to the structure of the graph (the colors are *not* interchangeable). For example, the first coloring of the graph below is the same as the second but different from the third.

Example 3.1.



Given an unlabeled graph G one can define the function $\chi(G, \lambda)$ whose value at a positive integer k is the number of k-colorings of G. A few examples show that $\chi(G, \lambda)$ does not satisfy any sort of obvious chromatic reduction. Nonetheless $\chi(G, \lambda)$ is a rational polynomial in λ of degree $n = |V(G)|$ and with leading coefficient $1/|A(G)|$.

In this section we discuss properties of $\chi(G, \lambda)$ and show how it can be computed. First we must make precise definitions of "unlabeled graph" and "coloring of an unlabeled graph." The reader should convince himself that these formal definitions capture the intuitive notions discussed above.

Recall from Section 1 that the group S_n acts on the set γ_n of labeled graphs with n points. An *unlabeled graph* of order n is an orbit of this action. Thus an unlabeled graph G is a set of labeled graphs and so we can speak of a labeled graph g as lying in an unlabeled graph G. In this case we sometimes call g a *labeled version* of G.

Next let g be a labeled graph of order n, let σ be a k-coloring of g and let π be an element of S_n. We claim that $\sigma\pi^{-1}$ is a k-coloring of the graph πg. To see this suppose u and v are adjacent points of πg. Then $\pi^{-1}u$ and $\pi^{-1}v$ are adjacent points of g so $\sigma(\pi^{-1}u)$ is different from $\sigma(\pi^{-1}v)$, i.e., $(\sigma\pi^{-1})\, u \neq (\sigma\pi^{-1})\, v$. Thus π maps the pair (g, σ) to a pair (h, τ), where τ is a k-coloring of h and where g and h lie in the same unlabeled graph. Let G be an unlabeled graph and let $u(G, k)$ denote the set of pairs (g, σ) such that σ is a k-coloring of g and such that g lies in the unlabeled graph G. As noted above there is a natural action $\pi((g, \sigma)) = (\pi g, \sigma\pi^{-1})$ of S_n on the set $u(G, k)$. A *k-coloring of G* is an orbit of S_n acting on $u(G, k)$. For each positive integer k let $\chi(G, k)$ denote the number of k-colorings of G. The following theorem, which gives a means of computing $\chi(G, k)$ was obtained independently by Allen Schwenk. His results will appear elsewhere.

THEOREM 3.1. *Let g be any labeled version of G. Then* $\chi(G, k) = (1/|A(g)|) \sum_{\pi \in A(g)} \chi(g, \pi, k)$.

Proof. Let Ω be a finite group acting on a finite set Λ. A well-known result (often referred to as Burnsides lemma) asserts that N_Λ, the number of orbits of this action, is

$$N_\Lambda = \frac{1}{|\Omega|} \sum_{w \in \Omega} |\Lambda_w|,$$

where $\Lambda_w = \{p \in \Lambda: pw = p\}$. For a derivation of Burnside's lemma see [1, p. 39]. Applying that result to the present situation we obtain,

$$\chi(G, k) = \frac{1}{n!} \sum_{\pi \in S_n} N(\pi), \tag{3.1}$$

where $N(\pi)$ equals the number of pairs $(g, \sigma) \in u(G, k)$ with $\pi((g, \sigma)) = (g, \sigma)$. Note that $\pi((g, \sigma)) = (g, \sigma)$ if and only if π is an automorphism of g and σ is a π-coloring of g. Hence we can rearrange the sum on the right of (3.1) to get

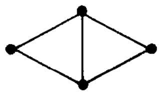
$$\chi(G, k) = \frac{1}{n!} \sum_{g \in G} \sum_{\pi \in A(g)} \chi(g, \pi, k). \tag{3.2}$$

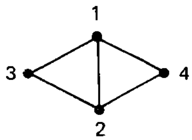
If g and h are graphs in G they are isomorphic hence $\sum_{\pi \in A(g)} \chi(g, \pi, k) = \sum_{\pi \in A(h)} \chi(h, \pi, k)$. Thus selecting a particular $g \in G$ we can rewrite (3.2) as

$$\chi(G, k) = \frac{|G|}{n!} \sum_{\pi \in A(g)} \chi(g, \pi, k). \tag{3.3}$$

Since $|G| = n!/|A(g)|$, the result follows.

For an unlabeled graph G define the *chromatic polynomial* $\chi(G, \lambda)$ of G as follows: choose a labeled version g of G and let $\chi(G, \lambda)$ be $(1/|A(g)|) \sum_{\pi \in A(g)} \chi(g, \pi, \lambda)$. Theorem 3.1 shows that $\chi(G, \lambda)$ evaluated at $\lambda = k$ is the number of k-colorings of G. Note that $\chi(G, \lambda)$ is a polynomial in λ of degree $n = |V(G)|$ with leading coefficient $1/|A(g)|$. This follows because $\chi(g, \pi, \lambda)$ has degree n if and only if π is the identity.

EXAMPLE 3.1. Let $G=$  . Let g be the labeled version

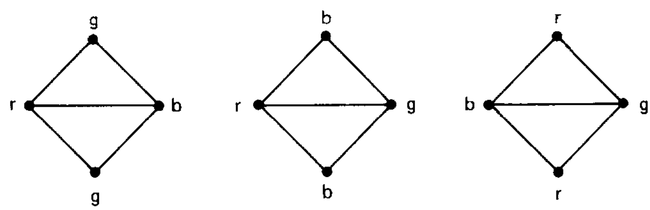


. Thus $A(g)=\{\pi_1, \pi_2, \pi_3, \pi_4\}$, where $\pi_1=(1)\ (2)\ (3)\ (4)$, $\pi_2=(1, 2)\ (3)\ (4)$, $\pi_3=(1)\ (2)\ (3, 4)$, and $\pi_4=(1, 2)\ (3, 4)$. By Theorem 2.1 $\chi(g, \pi_2, \lambda)=\chi(g, \pi_4, \lambda)=0$.

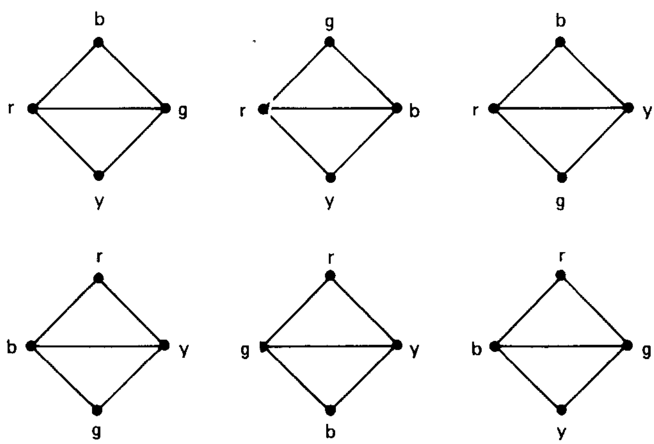
Referring to Example 2.2 one sees that $\chi(g, \pi_3, \lambda)=\lambda^3-3\lambda^2+2\lambda$. Last, one can check that $\chi(g, \pi_1, \lambda)=\lambda(\lambda-1)(\lambda-2)^2$. Thus

$$\chi(G, \lambda)=\tfrac{1}{4}(\lambda(\lambda-1)(\lambda-2)^2+\lambda(\lambda-1)(\lambda-2))=\tfrac{1}{4}\lambda(\lambda-1)^2(\lambda-2).$$

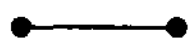
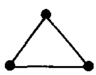
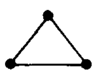
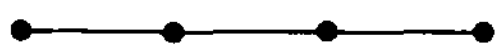
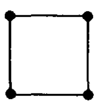
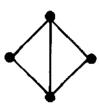
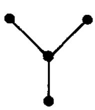
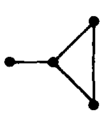
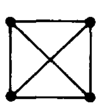
If $\lambda=3$ we find that there are three 3-colorings of G.



If $\lambda=4$ we have that there are 18 4-colorings of G. Of these, 12 are colorings which use exactly 3 of the 4 colors and the remaining 6 are



The following is a list of $\chi(G, \lambda)$ for the connected graphs G with 4 or fewer points:

G	$\chi(G, \lambda)$
	λ
	$\frac{1}{2}(\lambda^2 - \lambda)$
	$\frac{1}{2}(\lambda^3 - \lambda^2)$
	$\frac{1}{6}(\lambda^3 - 3\lambda^2 + 2\lambda)$
	$\frac{1}{2}(\lambda^4 - 3\lambda^3 + 3\lambda^2 - \lambda)$
	$\frac{1}{8}(\lambda^4 - 2\lambda^3 + 3\lambda^2 - 2\lambda)$
	$\frac{1}{4}(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda)$
	$\frac{1}{6}(\lambda^4 - \lambda^2)$
	$\frac{1}{2}(\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda)$
	$\frac{1}{24}(\lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda)$.

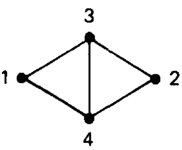
4. Counting Unlabeled Acyclic Digraphs

We begin this section with a discussion of cycle indices. For a permutation π in S_n, let $j_i(\pi)$ denote the number of i-cycles of π as a permutation of $\{1, 2, \ldots, n\}$. Note that $s(\pi) = \sum_i j_i(\pi)$. Let g be a labeled graph. Define the *cycle index of* g, denoted $Z(g)$, and the *chromatic cycle index of* g, denoted $Z_\chi(g)$, by

$$Z(g) = \frac{1}{|A(g)|} \sum_{\pi \in A(g)} x_1^{j_1(\pi)} \cdots x_n^{j_n(\pi)}$$

and

$$Z_\chi(g) = \frac{1}{|A(g)|} \sum_{\pi \in A(g)} \chi(g, \pi, \lambda)\, x_1^{j_1(\pi)} \cdots x_n^{j_n(\pi)}.$$

Example 4.1. Let $g =$  . Then

$$Z(g) = \tfrac{1}{4}(x_1^4 + 2x_1^2 x_2 + x_2^2)$$

and

$$Z_\chi(g) = \tfrac{1}{4}(x_1^4 \lambda(\lambda-1)(\lambda-2)^2 + x_1^2 x_2 \lambda(\lambda-1)(\lambda-2)).$$

It is easy to check that if g and h are isomorphic then $Z(g) = Z(h)$ and $Z_\chi(g) = Z_\chi(h)$. So if G is an unlabeled graph we define $Z(G)$ and $Z_\chi(G)$ to be $Z(g)$ and $Z_\chi(g)$ for g a labeled version of G.

Let G be an unlabeled graph and let $a(G)$ be the set of pairs (g, σ) such that g is a labeled version of G and such that σ is an acyclic orientation of g. It is easy to check that $\pi((g, \sigma)) = (\pi g, \sigma\pi^{-1})$ defines an action of S_n on $a(G)$. An *acyclic orientation* of G is an orbit of this action. Equivalently, an acyclic orientation of G is an unlabeled acyclic digraph whose underlying undirected graph is G. Let $\Delta(G)$ denote the set of acyclic orientations of G.

For $d \in a(G)$, an *automorphism of* d is a permutation $\pi \in S_n$ such that $\pi d = d$. Note that if $d = (g, \sigma)$ then $\pi d = d$ if and only if π is an automorphism of g which preserves the orientation σ. This is true if and only if π is an automorphism of d considered as an acyclic digraph (in general, $A(d)$ is some proper subgroup of $A(g)$). Let $Z(d)$ denote the cycle index of the group $A(d)$. As usual we have that $Z(d_1) = Z(d_2)$ if d_1 and d_2 are in the same orbit of S_n acting on $a(G)$. So for D an acyclic orientation of G we can define $Z(D)$ to be $Z(d)$ for d in the orbit D. For G an unlabeled graph define $Z(\Delta(G))$ to be the cycle index sum of the set $\Delta(G)$, i.e., $Z(\Delta(G))$ is the sum of the cycle indices of all unlabeled acyclic digraphs whose underlying undirected graph is G.

THEOREM 4.1. *For any unlabeled graph G we have*

$$Z(\Delta(G)) = Z_\chi(G)[\lambda \to -1, x_l \to -x_l].$$

Proof. By definition we have

$$Z(\Delta(G)) = \sum_{D \in \Delta(G)} Z(D) = \sum_{D \in \Delta(G)} \frac{1}{|D|} \sum_{d \in D} Z(d)$$
$$= \sum_{D \in \Delta(G)} \sum_{d \in D} \frac{1}{|D|} \frac{1}{|A(d)|} \sum_{\pi \in A(d)} Z(\pi).$$

As $|D|\ |A(d)| = n!$ we can rewrite the above as

$$Z(\Delta(G)) = \frac{1}{n!} \sum_{d \in a(G)} \sum_{\pi \in A(d)} Z(\pi).$$

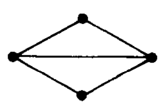
Considering d as a pair (g, σ) we have by Theorem 2.3,

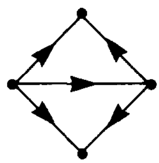
$$Z(\Delta(G)) = \frac{1}{n!} \sum_{g \in G} \sum_{\pi \in A(g)} Z(\pi)(-1)^{s(\pi)} \chi(g, \pi, -1).$$

As $n! = |G| \cdot |A(g)|$ for any $g \in G$,

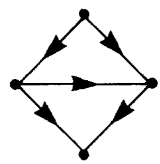
$$Z(\Delta(G)) = \frac{1}{|G|} \sum_{g \in G} Z_\chi(g)[\lambda \to -1, x_l \to -x_l]$$
$$= Z_\chi(G)[\lambda \to -1, x_l \to -x_l].$$

COROLLARY 4.1. *Let G be an unlabeled graph. Then the number of unlabeled acyclic digraphs whose underlying undirected graph is G equals $Z_\chi(G)[\lambda \to -1, x_l \to -1]$.*

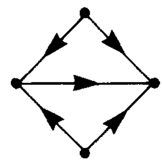
EXAMPLE 4.2. Let $G =$ . Then $Z_\chi(G)[\lambda \to -1, x_l \to -1] = \frac{1}{4}(18 + 6) = 6$ (see Example 4.1). The 6 unlabeled acyclic digraphs with G as underlying graph appear below.



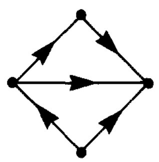
$\frac{1}{2}(x_1^4 + x_1^2 x_2)$



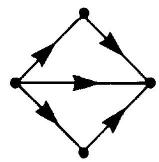
x_1^4



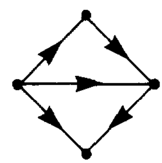
$\frac{1}{2}(x_1^4 + x_1^2 x_2)$.



x_1^4



$\frac{1}{2}(x_1^4 + x_1^2 x_2)$



x_1^4

Below each of these digraphs appears the cycle index of its automorphism group. The reader can verify that the sum of these cycle indices is $Z_\chi(G)$ $[\lambda \to -1, x_l \to -x_l]$.

Let R be the vector space $\mathbb{Q}[[x_1, x_2, \ldots]]$. Define a multiplication $*$ on R according to the following rules:

(a) For monomials $x_1^{a_1} \cdots x_m^{a_m}$ and $x_1^{b_1} \cdots x_m^{b_m}$,

$$(x_1^{a_1} \cdots x_m^{a_m}) * (x_1^{b_1} \cdots x_m^{b_m}) = 2^L x_1^{a_1 + b_1} \cdots x_m^{a_m + b_m},$$

where $L = \sum_{i,j} a_i b_j \gcd(i, j)$.

(b) Extend $*$ linearly to the entire vector space R.

LEMMA. *The operation $*$ is commutative and associative and is distributive over the operation $+$.*

Proof. Commutativity and distributivity are clear. To prove associativity, it is enough to prove associativity for monomials. This follows easily from the identity

$$\gcd(\gcd(i, j), l) = \gcd(i, \gcd(j, l)).$$

THEOREM 4.2. *For all positive integers k,*

$$\sum_{n=0}^{\infty} \sum_{G \in \Gamma_n} Z_\chi(G, k) = \left(\exp\left(\sum_{l=1} \frac{x_l}{l}\right)\right)^k$$

where Γ_n is the set of unlabeled graphs with n vertices and where the kth power on the right is taken in the ring $(R, +, *)$.

Proof. Fix k. By a weighted form of Burnside's lemma (see [1, p. 163]) we have

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{G \in \Gamma_n} Z_\chi(G, k) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{g \in \gamma_n} \sum_{\pi \in A(g)} \chi(g, \pi, k)\, x_1^{j_1(\pi)} \cdots x_n^{j_n(\pi)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\pi \in S_n} x_1^{j_1(\pi)} \cdots x_n^{j_n(\pi)} \sum_{\substack{g \in \gamma_n \\ \pi \in A(g)}} \chi(g, \pi, k); \qquad (4.1)\end{aligned}$$

here γ_n denotes the set of all labeled graphs on n vertices. Recall that $\chi(g, \pi, k)$ is the number of ways to π-color g with k colors. With this in mind, we compute (4.1) in a different way.

Begin by dividing the points $\{1, 2, ..., n\}$ into k color classes. If c_i is the number of points in the ith class then this division can be done in $\binom{n}{c_1, ..., c_k}$ ways. Next choose permutations $\pi_1, ..., \pi_k$ of the classes and let π be the permutation of $\{1, 2, ..., n\}$ obtained from $\pi_1, ..., \pi_k$. Last, construct all π-colored graphs g on k-colors subject to the constraints:

(a) g admits π as an automorphism

(b) the color classes are the ones chosen above.

In constructing such graphs we can only join points in distinct color classes by condition (b). Condition (a) restricts the number of choices we have when inserting edges between color classes. In particular the number of ways that we can insert edges between the points of an i-cycle in one color class and an l-cycle in another is 2^{L_1}, where $L_1 = \gcd(i, l)$. Thus the number of ways to insert edges between the points in color class u and the points in color class v is 2^L, where

$$L = \sum_{i,l} j_i(\pi_u)\, j_l(\pi_v) \gcd(i, l).$$

So the right side of (4.1) can be rewritten as

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{c_1, ..., c_k} \binom{n}{c_1, ..., c_k} \sum_{\pi_1, ..., \pi_k} Z(\pi_1) * \cdots * Z(\pi_k) \\ &\quad = \sum_{n=0}^{\infty} \sum_{c_1, ..., c_k} \left(\frac{1}{c_1!} \sum_{\pi_1 \in S_{c_1}} Z(\pi_1) \right) * \cdots * \left(\frac{1}{c_k!} \sum_{\pi_k \in S_{c_k}} Z(\pi_k) \right) \\ &\quad = \left(\sum_{c_1=0}^{\infty} Z(S_{c_1}) \right) * \cdots * \left(\sum_{c_k=0}^{\infty} Z(S_{c_k}) \right). \qquad (4.2)\end{aligned}$$

Using the identity $\sum_{c=0}^{\infty} Z(S_c) = \exp(\sum_{l=1}^{\infty}(x_l/l))$ (see Harary and Palmer [1, p. 52] we have that (4.2) is equal to $(\exp(\sum_{l=1}^{\infty}(x_l/l)))^k$.

COROLLARY 4.2. *For any integer* k,

$$\sum_{n=0}^{\infty} \sum_{G \in \Gamma_n} Z_\chi(G, k) = \left(\exp\left(\sum_{l=1}^{\infty} \frac{x_l}{l}\right)\right)^k. \tag{4.3}$$

Proof. Fix a monomial $x_1^{a_1} \cdots x_n^{a_n}$. The coefficient of this monomial on the left-hand side of (4.3) is clearly a polynomial in k, call it $f(k)$. Write $\exp(\sum_{l=1}^{\infty}(x_l/l))$ as $1 + \bigcup(x_1, x_2, \ldots)$ and expand the right-hand side of (4.3) using the binomial theorem (here we need that $*$ is commutative and associative). The right-hand side of (4.3) becomes $\sum_{a=0}^{k}\binom{k}{a} \bigcup(x_1, x_2, \ldots)^a$. Hence the coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ on the right-hand side of (4.3) is $\sum_{a=0}^{k}\binom{k}{a} C(a)$, where $C(a)$ is the coefficient of $x_1^{a_1} \cdots x_n^{a_n}$ in $\bigcup(x_1, \ldots)^a$. Let $D = a_1 + a_2 + \cdots + a_n$. Then clearly $C(a) = 0$ for $a > D$ (since $\bigcup(x_1, x_2, \ldots)$ has no constant term). Thus we can rewrite $\sum_{a=0}^{k}\binom{k}{a} C(a)$ as $\sum_{a=0}^{D}\binom{k}{a} C(a)$. This latter expression is a polynomial in k, call it $g(k)$. By Theorem 4.2 we know that $f(k) = g(k)$ for all positive integers k so $f(k)$ and $g(k)$ are identical. This proves Corollary 4.2.

COROLLARY 4.3 (Robinson [2]). *Let A be the set of unlabeled acyclic digraphs and let $Z(A)$ be the cycle index sum for the set A, i.e., $Z(A) = \sum_{D \in A} Z(D)$. Then $Z(A) * \exp(-\sum_{l=1}^{\infty}(x_l/l)) = 1$.*

Proof. Let $k = -1$ and replace x_l by $-x_l$ $(l = 1, 2, \ldots)$ in Corollary 4.2. We obtain

$$\sum_{n=0}^{\infty} \sum_{G \in \Gamma_n} Z_x(G)[\lambda \to -1, x_l \to -x_l] = \left(\exp\left(\sum_{l=1}^{\infty} \frac{-x_l}{l}\right)\right)^{-1}.$$

By Theorem 4.1 the left-hand side is equal to $Z(A)$ and the result follows upon noting that 1 is the multiplicative identity in $(R, +, *)$.

Using the equation from Corollary 4.3, Dr. Paul Butler of the University of Newcastle computed the number of unlabeled acyclic digraphs with n points for $n \leqslant 18$. These results appear in [2].

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