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Lipschitz quasistability of impulsive differential-difference equations with variable impulsive perturbations

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Abstract

In the present paper, by means of a suitable comparison lemma sufficient conditions for uniform Lipschitz stability of an arbitrary solution of an impulsive system of differential-difference equations with variable impulsive perturbations are obtained.

Keywords: Lipschitz quasistability; Impulsive differential-difference equations; Variable impulsive perturbations

1. Introduction

The impulsive differential and differential-difference equations are an adequate apparatus for mathematical simulation of numerous real processes and phenomena studied in the theory of optimal control, physics, chemistry, biology, bioengineering sciences, technology, medicine, etc.

On the other hand, however, the mathematical theory of the impulsive differential-difference equations is much more complicated in comparison with the corresponding theory of the impulsive ordinary differential equations (without delay) and the theory of the differential-difference equations (without impulses). This is the reason why their theory is developing rather slowly [4].

In the present paper the notion of uniform Lipschitz stability of an arbitrary solution of an impulsive system of differential-difference equations with variable impulsive perturbations is defined. By means of a suitable comparison method sufficient conditions for uniform Lipschitz stability of a fixed solution of such a system are found. Since the impulses take place at the moments when the integral curves meet some previously fixed hypersurfaces of the extended phase

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space, then for this type of equations phenomena may appear such as "beating" of the solutions, bifurcation, merging of the solutions, loss of the property of autonomy, etc. This is the reason why for such equations one cannot speak of Lipschitz stability of an arbitrary solution in the classical sense [1]. In relation to this, in the present paper the sense in which the notion of uniform Lipschitz stability of a given solution of an impulsive differential-difference equation with variable impulsive perturbations should be understood is made precise introducing the notion of uniform Lipschitz quasistability.

We shall note that similar investigations for impulsive ordinary differential equations (without delay) were carried out [2, 3].

2. Preliminary notions and definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|\cdot|$; Ω be a domain in $\mathbb{R}^n$, $\Omega \neq \emptyset$; $h > 0$; $t_0 \in \mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$. Consider the initial value problem

$$\dot{x}(t) = f(t, x(t), x(t-h)), \quad t \neq \tau_k(x(t)), \ t > t_0, \tag{1}$$

$$x(t) = \mathcal{S}_0(t), \quad t \in [t_0 - h, t_0], \tag{2}$$

$$\Delta x(t)|_{t=\tau_k(x(t))} = I_k(x(t)), \quad t > t_0, \ k = 1, 2, \ldots, \tag{3}$$

where $f:(t_0, \infty) \times \Omega \times \Omega \to \mathbb{R}^n$; $I_k: \Omega \to \mathbb{R}^n$; $\tau_k: \Omega \to (t_0, \infty)$, $k = 1, 2, \ldots$; $\Delta x(t) = x(t+0) - x(t-0)$; $\mathcal{S}_0: [t_0 - h, t_0] \to \mathbb{R}^n$.

Introduce the following notation: $\sigma_k = \{(t, x) \in (t_0, \infty) \times \Omega: t = \tau_k(x)\}$, i.e., σ_k, $k = 1, 2, \ldots$, are hypersurfaces with equations $t = \tau_k(x(t))$; $C_0 = C[[t_0 - h, t_0], \mathbb{R}^n]$ is the class of all continuous functions $\mathcal{S}: [t_0 - h, t_0] \to \mathbb{R}^n$; $\| \mathcal{S} \| = \max_{t \in [t_0 - h, t_0]} |\mathcal{S}(t)|$ is the norm of the function $\mathcal{S} \in C_0$; $x(t) = x(t; t_0, \mathcal{S}_0)$ is the solution of problem (1)–(3); $\mathcal{T}^+(t_0, \mathcal{S}_0)$ is the maximal interval of type $[t_0, \beta)$ in which the solution $x(t; t_0, \mathcal{S}_0)$ is defined; $\tilde{x}(t) = x(t-h)$, $t > t_0$.

Let $\mathcal{S}_0 \in C_0$, $\tau_0(x) \equiv t_0$ for $x \in \Omega$.

We shall make a description of the solution $x(t)$ of problem (1)–(3):

(1) For $t_0 - h \leqslant t \leqslant t_0$ the solution $x(t)$ coincides with the function $\mathcal{S}_0 \in C_0$.

(2) Let $t_1, t_2, \ldots$ $(t_0 < t_1 < t_2 < \ldots)$ be the moments at which the integral curve $(t, x(t))$ of problem (1)–(3) meets the hypersurfaces $\{\sigma_k\}_{k=1}^{\infty}$, i.e., each of the points $t_1, t_2, \ldots$ is a solution of one of the equations $t = \tau_k(x(t))$, $k = 1, 2, \ldots$. Let $t_l^h = t_l + h$, $l = 0, 1, 2, \ldots$.

Construct the sequence $\{\tau_i\}_{i=0}^{\infty}$ observing the following rules:

(a) $\{\tau_i\}_{i=0}^{\infty} = \{t_k\}_{k=0}^{\infty} \cup \{t_l^h\}_{l=0}^{\infty}$.

(b) $\tau_0 \equiv t_0$.

(c) The sequence $\{\tau_i\}_{i=0}^{\infty}$ is monotone increasing.

We shall note that in general it is possible that $\{t_k\}_{k=1}^{\infty} \cap \{t_l^h\}_{l=0}^{\infty} \neq \emptyset$.

(2.1) For $\tau_0 < t \leqslant \tau_1$ the solution of problem (1)–(3) coincides with the solution of problem (1)–(2).

(2.2) For $\tau_i < t \leqslant \tau_{i+1}$, $i = 1, 2, \ldots$, one of the following three cases may occur:

(a) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \setminus \{t_l^h\}_{l=0}^{\infty}$, $\tau_i = t_k$ and j_k is the number of the hypersurface met by the integral curve $(t, x(t))$ at the moment t_k, then the solution $x(t)$ of problem (1)–(3) coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h)), \tag{4}$$

$$y(t_k) = x(t_k) + I_{j_k}(x(t_k)). \tag{5}$$

(b) If $\tau_i \in \{t_l^h\}_{l=0}^{\infty} \setminus \{t_k\}_{k=1}^{\infty}$, then the solution $x(t)$ coincides with the solution of the problem

$$\dot{y}(t) = f(t, y(t), x(t-h+0)), \tag{6}$$

$$y(\tau_i) = x(\tau_i). \tag{7}$$

(c) If $\tau_i \in \{t_k\}_{k=1}^{\infty} \cap \{t_l^h\}_{l=0}^{\infty}$ and $\tau_i = t_k$, then the solution $x(t)$ of problem (1)–(3) coincides with the solution of problem (6), (5).

(3) If the point $x(t_k) + I_{jk}(x(t_k)) \bar{\in} \Omega$, then the solution $x(t)$ of problem (1)–(3) is not defined for $t > t_k$.

(4) The function $x(t)$ is piecewise continuous in $\mathscr{T}^+(t_0, \mathscr{S}_0)$, continuous from the left at the points $t_1, t_2, \ldots$ in $\mathscr{T}^+(t_0, \mathscr{S}_0)$ and $x(t_k + 0) = x(t_k) + I_{j_k}(x(t_k))$.

Together with problem (1)–(3) we shall consider the problem

$$\dot{x}^*(t) = f(t, x^*(t), x^*(t-h)), \quad t \neq \tau_k(x^*(t)), \ t > t_0^*, \tag{8}$$

$$x^*(t) = \mathscr{S}^*(t), \quad t \in [t_0^* - h, t_0^*], \tag{9}$$

$$\Delta x^*(t)|_{t=\tau_k(x^*(t))} = I_k(x^*(t)), \quad t > t_0^*, \ k = 1, 2, \ldots, \tag{10}$$

where

$$t_0^* \in [t_0, \infty), \quad \mathscr{S}^* \in C[[t_0^* - h, t_0^*], \mathbb{R}^n].$$

Introduce the following notation: $x^*(t; t_0^*, \mathscr{S}^*)$ is the solution of problem (8)–(10); $x^*(t) = x^*(t; t_0, \mathscr{S}^*)$; $t_1^*, t_2^*, \ldots$ ($t_0^* < t_1^* < t_2^* < \cdots$) are the moments at which the integral curve $(t, x^*(t))$ meets the hypersurfaces σ_k, $k = 1, 2, \ldots$; $\tilde{x}^* = x^*(t-h)$, $t > t_0$.

Remark 1. If $t_0 \equiv t_0^*$, $\mathscr{S}_0(t) \equiv \mathscr{S}^*(t)$ for $t \in [t_0 - h, t_0]$, then problem (1)–(3) is equivalent to problem (8)–(10).

Introduce the following definition of uniform Lipschitz quasistability:

Definition 2. The solution $x(t) = x(t; t_0, \mathscr{S}_0)$ of problem (1)–(3) is said to be *uniformly Lipschitz quasistable* if

$$(\exists M > 0)(\forall \eta > 0)(\exists \delta = \delta(\eta) > 0)$$

$$(\forall \mathscr{S}^* \in C_0 \colon \| \mathscr{S}^* - \mathscr{S}_0 \| < \delta)(\forall t_0 \in \mathbb{R})$$

$$(\forall t > t_0 \colon |t - t_k| > \eta, k = 1, 2, \ldots)\colon$$

$$|x^*(t) - x(t)| \leqslant M \| \mathscr{S}^* - \mathscr{S}_0 \|.$$

Introduce the following conditions:
(H1) $f \in C[(t_0, \infty) \times \Omega \times \Omega, \mathbb{R}^n]$.
(H2) $|f(t, x, \tilde{x})| \leqslant L, L > 0, (t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$.
(H3) $I_k \in C[\Omega, \mathbb{R}^n], k = 1, 2, \ldots$.
(H4) $\tau_k \in C^1[\Omega, (t_0, \infty)], k = 1, 2, \ldots$.
(H5) $t_0 < \tau_1(x) < \tau_2(x) < \ldots, x \in \Omega$.
(H6) $\tau_k(x) \to \infty$ as $k \to \infty$ uniformly in $x \in \Omega$.
(H7) $I + I_k : \Omega \to \Omega, k = 1, 2, \ldots$, where I is the identity in Ω.
(H8) For any $(t_0^*, \mathscr{S}^*) \in [t_0, \infty) \times C[[t_0^* - h, t_0^*], \mathbb{R}^n]$ the solution of the problem without impulses (8), (9) does not leave the domain Ω for $t \in \Delta$, where

$$\Delta = \begin{cases} (t_0^*, \infty) & \text{if } t_k^* \text{ are a finite number,} \\ \bigcup_{k=1}^{\infty} (t_{k-1}^*, t_k^*] & \text{if } t_k^* \text{ are infinitely many.} \end{cases}$$

(H9) $\mathscr{T}^+(t_0, \mathscr{S}_0) = (t_0, \infty)$.
We shall note that for the impulsive differential equations it is possible that the so-called "beating" of the solutions occurs, i.e., a phenomenon where the integral curve $(t, x(t))$ meets several or infinitely many times one and the same hypersurface. In the present paper we shall consider problems of the type (1)–(3) for which "beating" of the solutions is absent.
Introduce the following condition:
(H10) The integral curve of each solution of problem (1)–(3) meets for $t > t_0$ successively each one of the hypersurfaces $\sigma_1, \sigma_2, \ldots$ exactly once.
For impulsive functional differential equations this phenomenon has been studied in detail. Effective sufficient conditions were found for the absence of "beating" of the solutions of such systems of equations [5].

3. Comparison lemma

Since the moments of impulse effect for the solutions $x(t)$ and $x^*(t)$ of problems (1)–(3) and (8)–(10) are different, then in the estimation of the difference of these solutions a number of obstacles appear. In order to overcome these obstacles we shall use a suitable comparison lemma.
Consider the scalar impulsive differential equation

$$\dot{u}(t) = g(t, u(t)), \quad t \bar{\in} (\underline{t}_k, \bar{t}_k), \; k = 1\ 2, \ldots, \tag{11}$$

$$u(\bar{t}_k + 0) = \psi_k(u(\underline{t}_k)), \quad k = 1, 2, \ldots, \tag{12}$$

$$u(t_0 + 0) = u_0, \tag{13}$$

where $g : (t_0, \infty) \times \mathbb{R} \to \mathbb{R}$; $\psi_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots$; $t_0 < \underline{t}_1 \leqslant \bar{t}_1 < \underline{t}_2 \leqslant \bar{t}_2 < \cdots < \underline{t}_k \leqslant \bar{t}_k < \cdots$ and $\lim_{k \to \infty} \underline{t}_k = \infty$; $u_0 \in \mathbb{R}$.
Introduce the following notation: $u(t) = u(t; t_0, u_0)$ is the solution of (11)–(13); $\mathscr{T}^+(t_0, u_0)$ is the maximal interval of type (t_0, ω) in which the solution $u(t; t_0, u_0)$ is defined.

The solution $u(t; t_0, u_0)$ of (11)–(13) is defined in the following way:

$$u(t; t_0, u_0) = \begin{cases} u_0(t; t_0, u_0), & t_0 < t \leqslant \underline{t}_1, \\ u_1(t; \bar{t}_1, u_1^+), & \bar{t}_1 < t \leqslant \underline{t}_2, \\ \cdots & \\ u_k(t; \bar{t}_k, u_k^+), & \bar{t}_k < t \leqslant \underline{t}_{k+1}, \\ \cdots & \end{cases}$$

where $u_k(t; \bar{t}_k, u_k^+)$, $k = 1, 2, \dots$, is the solution of Eq. (11) for which $u_k(\bar{t}_k; \bar{t}_k, u_k^+) = u_k^+$ and $u_k^+ = \psi_k(u_{k-1}(\underline{t}_k; \bar{t}_{k-1}, u_{k-1}^+))$, $k = 2, 3, \dots$, and $u_0(t; t_0, u_0)$ is the solution of (11) for which $u_0(t; t_0, u_0) = u_0$ and $u_1^+ = \psi_1(u_0(\underline{t}_1; t_0, u_0))$.

Definition 3. The solution $r: \mathscr{T}^+(t_0, u_0) \to \mathbb{R}$ of (11)–(13) ($r(t) = r(t; t_0, u_0)$) is said to be a *maximal solution* of (11)–(13) if any other solution $u: (t_0, \tilde{\omega}) \to \mathbb{R}$ of (11)–(13) satisfies the inequality $r(t) \geqslant u(t)$ for $t \in \mathscr{T}^+(t_0, u_0) \cap (t_0, \tilde{\omega})$.

Lemma 4 (Lakshmikantham et al. [3]). *Let the following conditions hold:*

(1) *The function $m: (t_0, \infty) \to \mathbb{R}$ is piecewise continuous in (t_0, ∞) with points of discontinuity of the first kind $t = \underline{t}_k$ and $t = \bar{t}_k$ at which it is continuous from the left.*

(2) $t_0 < \underline{t}_1 \leqslant \bar{t}_1 < \underline{t}_2 \leqslant \bar{t}_2 < \cdots < \underline{t}_k \leqslant \bar{t}_k < \cdots$.

(3) $\lim_{k \to \infty} \underline{t}_k = \infty$.

(4) *For $k = 1, 2, \dots$ the following inequalities are valid:*

$$D^+ m(t) \leqslant g(t, m(t)), \quad t \bar{\in} (\underline{t}_k, \bar{t}_k],$$

$$m(\bar{t}_k + 0) \leqslant \psi_k(m(\underline{t}_k)),$$

$$m(t_0 + 0) \leqslant u_0,$$

where $g \in C[(t_0, \infty) \times \mathbb{R}, \mathbb{R}]$, $\psi_k \in C[\mathbb{R}, \mathbb{R}]$, $\psi_k(u)$ is nondecreasing with respect to u and

$$D^+ m(t) = \lim_{\sigma \to 0+} \sup \frac{1}{\sigma} [m(t + \sigma) - m(t)].$$

(5) *The maximal solution $r(t; t_0, u_0)$ of (11)–(13) is defined in the set $\mathscr{T} = (t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k]$. Then $m(t) \leqslant r(t; t_0, u_0)$ for $t \in \mathscr{T}$.*

4. Main results

Introduce the following conditions:

(H11) $\mathscr{T}^+(t_0, u_0) = (t_0, \infty)$.

(H12) $\mathscr{T}^+(t_0, \mathscr{S}^*) = (t_0, \infty)$.

Theorem 5. *Let the following conditions hold*:

(1) *Conditions* (H1)–(H12) *are met.*

(2) *For* $(t, x^*, \tilde{x}^*)$, $(t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$, $t \neq t_k^*$, $t \neq t_k$, $k = 1, 2, \ldots$, *the inequality*

$$[x^* - x, f(t, x^*, \tilde{x}^*) - f(t, x, \tilde{x})]_+ \leqslant g(t, |x^* - x|)$$

is valid, where $g \in C[[t_0, \infty) \times \mathbb{R}_+, \mathbb{R}]$ *and*

$$[x, y]_+ = \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma}[|x + \sigma y| - |x|], \quad x, y \in \mathbb{R}^n.$$

(3) *For* $t \in \sigma_k$, $k = 1, 2, \ldots$, *the inequalities*

$$|x^*(t) - x(t) + I_k(x^*(t)) - I_k(x(t))| \leqslant \gamma_k(|x^*(t) - x(t)|)$$

are valid, where $\gamma_k \in C[\mathbb{R}_+, \mathbb{R}_+]$ *and* $\gamma_k(u)$ *is nondecreasing with respect to* u, $k = 1, 2, \ldots$.

(4) *For* $(t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$ *and* $k = 1, 2, \ldots$ *the following inequalities are valid*:

$$\frac{\partial \tau_k(x)}{\partial x} f(t, x, \tilde{x}) \leqslant 0.$$

(5) *For* $x^*, x \in \Omega$ *and* $k = 1, 2, \ldots$ *the inequalities*

$$|\tau_k(x^*) - \tau_k(x)| \leqslant \beta_k |x^* - x|$$

are valid, where $0 < \beta_k = \mathrm{const}$.

(6) *The functions* $\psi_k : \mathbb{R}_+ \to \mathbb{R}_+$ *and*

$$\psi_k(u) = \gamma_k((1 + L\beta)u) + L\beta u, \quad k = 1, 2, \ldots.$$

(7) *There exist constants* $M > 0$ *and* $\delta_1 > 0$ *such that for the solution* $r(t; t_0, u_0)$ *of* (11)–(13) *with* ψ_k *defined in condition* (6) *of Theorem* 5 *the inequality* $r(t; t_0, u_0) \leqslant M u_0$ *is valid for* $0 \leqslant u_0 < \delta_1$, $t \in (t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k]$.

Then the solution $x(t) = x(t; t_0, \mathcal{S}_0)$ *of problem* (1)–(3) *is uniformly Lipschitz quasistable.*

Proof. Let $\eta > 0$. Choose $\delta = \delta(\eta) = \min(\delta_1, \eta/(2M\beta + 1))$. Let $\mathcal{S}^* \in C_0$, $\|\mathcal{S}^* - \mathcal{S}_0\| < \delta$ and $x^*(t) = x^*(t; t_0, \mathcal{S}^*)$ be the solution of problem (8)–(10) for which $x^*(t) = \mathcal{S}^*(t)$ for $t \in [t_0 - h, t_0]$.

From condition (H10) it follows that $(t, x(t))$ meets successively the hypersurfaces $\sigma_1, \sigma_2, \ldots$ respectively at the moments $t_1, t_2, \ldots$. Since in the interval $(t_k, t_{k+1}]$ $x(t)$ coincides with the solution of problem (4), (5) ($j_k = k$), we conclude that for $t_k < t \leqslant t_{k+1}$ the function $x(t)$ satisfies the integral equation

$$x(t) = x(t_k) + I_k(x(t_k)) + \int_{t_k}^{t} f(s, x(s), x(s - h))\, \mathrm{d}s. \tag{14}$$

Let $t_1^*, t_2^*, \ldots$ be the moments at which the integral curve $(t, x^*(t; t_0, \mathcal{S}^*))$ meets the hypersurfaces $\sigma_1, \sigma_2, \ldots$. Analogously to (14) for the solution $x^*(t)$ we obtain

$$x^*(t) = x^*(t_k^*) + I_k(x^*(t_k^*)) + \int_{t_k^*}^{t} f(s, x^*(s), x^*(s - h))\, \mathrm{d}s, \quad t_k^* < t \leqslant t_{k+1}^*. \tag{15}$$

Introduce the notation $m(t) = |x^*(t) - x(t)|$, $u_0 = \| \mathscr{S}^* - \mathscr{S}_0 \|$, $\underline{t}_k = \min(t_k, t_k^*)$ and $\bar{t}_k = \max(t_k, t_k^*)$.

From condition (2) of Theorem 5 it follows that for $t \bar{\in} (\underline{t}_k, \bar{t}_k]$, $k = 1, 2, \dots$, the following inequalities are valid:

$$\begin{aligned}
D^+ m(t) &= \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma} [m(t+\sigma) - m(t)] \\
&= \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma} [|x^*(t+\sigma) - x(t+\sigma)| - |x^*(t) - x(t)|] \\
&\leqslant \lim_{\sigma \to 0^+} \sup \Bigg| \left(\frac{1}{\sigma}\right) [(x^*(t+\sigma) - x(t+\sigma)) - (x^*(t) - x(t))] \\
&\qquad - [f(t, x^*(t), x^*(t-h)) - f(t, x(t), x(t-h))] \Bigg| \\
&\quad + \lim_{\sigma \to 0^+} \sup \frac{1}{\sigma} \{ |[x^*(t) - x(t)] + \sigma [f(t, x^*(t), x^*(t-h)) - f(t, x(t), x(t-h))]| \\
&\qquad - |x^*(t) - x(t)| \} \\
&= [x^*(t) - x(t), f(t, x^*(t), x^*(t-h)) - f(t, x(t), x(t-h))]_+ \\
&\leqslant g(t, |x^*(t) - x(t)|) = g(t, m(t)).
\end{aligned} \tag{16}$$

We shall estimate the expression $m(\bar{t}_k + 0) = |x^*(\bar{t}_k + 0) - x(\bar{t}_k + 0)|$ for an arbitrary positive integer k.

In the case $\bar{t}_k = t_k^*$ and $\underline{t}_k = t_k$ from conditions (H1), (H2), condition (3) of Theorem 5 and (14) we obtain

$$\begin{aligned}
m(\bar{t}_k + 0) &= |x^*(\bar{t}_k) + I_k(x^*(\bar{t}_k)) - x(\bar{t}_k)| \\
&\leqslant |x^*(\bar{t}_k) - x(\underline{t}_k) + I_k(x^*(\bar{t}_k)) - I_k(x(\underline{t}_k))| + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x(s), x(s-h))| \, ds \\
&\leqslant \gamma_k(|x^*(\bar{t}_k) - x(\underline{t}_k)|) + L(\bar{t}_k - \underline{t}_k).
\end{aligned}$$

On the other hand, for the expression $|x^*(\bar{t}_k) - x(\underline{t}_k)|$ we obtain the estimate

$$\begin{aligned}
&|x^*(\bar{t}_k) - x(\underline{t}_k)| \\
&\quad \leqslant |x^*(\underline{t}_k) - x(\underline{t}_k)| + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x^*(s), x^*(s-h))| \, ds \leqslant m(\underline{t}_k) + L(\bar{t}_k - \underline{t}_k).
\end{aligned}$$

From condition (4) of Theorem 5 it follows that

$$\tau_k(x^*(\bar{t}_k)) \leqslant \tau_k(x^*(\underline{t}_k)).$$

Then from condition (5) of Theorem 5 we obtain

$$\begin{aligned}
0 &\leqslant \bar{t}_k - \underline{t}_k = \tau_k(x^*(\bar{t}_k)) - \tau_k(x(\underline{t}_k)) \\
&\leqslant \tau_k(x^*(\underline{t}_k)) - \tau_k(x(\underline{t}_k)) \leqslant \beta |x^*(\underline{t}_k) - x(\underline{t}_k)| = \beta m(\underline{t}_k).
\end{aligned} \tag{17}$$

Hence

$$m(\bar{t}_k + 0) \leqslant \gamma_k(1 + L\beta)m(\underline{t}_k) + L\beta m(\underline{t}_k) = \psi_k(m(\underline{t}_k)). \tag{18}$$

In the case when $\bar{t}_k = t_k$ and $\underline{t}_k = t_k^*$ we again use (H1), (H2), condition (3) of Theorem 5 and (15) and obtain

$$\begin{aligned} m(\bar{t}_k + 0) &\leqslant |x^*(\underline{t}_k) - x(\bar{t}_k) + I_k(x^*(\underline{t}_k)) - I_k(x(\bar{t}_k))| + \int_{\underline{t}_k}^{\bar{t}_k} |f(s, x^*(s), x^*(s-h))|\,ds \\ &\leqslant \gamma_k(|x^*(\underline{t}_k) - x(\bar{t}_k)|) + L(\bar{t}_k - \underline{t}_k). \end{aligned}$$

On the other hand,

$$|x^*(\underline{t}_k) - x(\bar{t}_k)| \leqslant m(\underline{t}_k) + L(\bar{t}_k - \underline{t}_k),$$

and from conditions (4) and (5) of Theorem 5 we obtain the estimate

$$\begin{aligned} 0 \leqslant \bar{t}_k - \underline{t}_k &= \tau_k(x(\bar{t}_k)) - \tau_k(x^*(\underline{t}_k)) \\ &\leqslant \tau_k(x(\underline{t}_k)) - \tau_k(x^*(\underline{t}_k)) \leqslant \beta|x(\underline{t}_k) - x^*(\underline{t}_k)| = \beta m(\underline{t}_k). \end{aligned} \tag{19}$$

Hence

$$m(\bar{t}_k + 0) \leqslant \gamma_k(1 + L\beta)m(\underline{t}_k) + L\beta m(\underline{t}_k) = \psi_k(m(\underline{t}_k)). \tag{20}$$

From inequalities (18) and (20) there follows the estimate

$$m(\bar{t}_k + 0) \leqslant \psi_k(m(\underline{t}_k)), \quad k = 1, 2, \ldots . \tag{21}$$

We estimate the expression $m(t_0 + 0)$:

$$m(t_0 + 0) = |x^*(t_0 + 0) - x(t_0 + 0)| = |x^*(t_0) - x(t)| \leqslant \| \mathscr{S}^* - \mathscr{S}_0 \| = u_0. \tag{22}$$

Inequalities (16), (21) and (22) show that the conditions of Lemma 4 are satisfied. Then

$$|x^*(t) - x(t)| = m(t) \leqslant r(t; t_0, \| \mathscr{S}^* - \mathscr{S}_0 \|), \tag{23}$$

for $t \in (t_0, \infty) \backslash \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k]$, where $r(t; t_0, \| \mathscr{S}^* - \mathscr{S}_0 \|)$ is the solution of (11)–(13) for $u_0 = \| \mathscr{S}^* - \mathscr{S}_0 \|$.

From (23) and condition (7) of Theorem 5 it follows that

$$|x^*(t) - x(t)| \leqslant M \| \mathscr{S}^* - \mathscr{S}_0 \| \quad \text{for } t \in (t_0, \infty) \backslash \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k].$$

Moreover, from (17), (19) and the choice of δ we obtain

$$\begin{aligned} 0 \leqslant \bar{t}_k - \underline{t}_k &\leqslant \beta |x^*(\underline{t}_k) - x(\underline{t}_k)| \\ &\leqslant \beta M \| \mathscr{S}^* - \mathscr{S}_0 \| \leqslant M\beta\delta < \tfrac{1}{2}\eta. \end{aligned}$$

From the above estimate it follows that

$$\{t \in (t_0, \infty) \colon |t - t_k| > \eta\} \subset (t_0, \infty) \backslash \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k].$$

Hence

$$|x^*(t) - x(t)| \leqslant M \|\mathscr{S}^* - \mathscr{S}_0\| \quad \text{for } \|\mathscr{S}^* - \mathscr{S}_0\| < \delta,\ t > t_0,\ |t - t_k| > \eta,\ k = 1, 2, \ldots .$$

Theorem 5 is proved. □

Corollary 6. *Let the following conditions hold:*

(1) *Conditions* (H1)–(H12) *are met.*

(2) *For* $(t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$ *and* $(t, x^*, \tilde{x}^*) \in S(x, \tilde{x}, \rho) = \{(t, x^*, \tilde{x}^*) \in (t_0, \infty) \times \Omega \times \Omega$: $|x^* - x| < \rho,\ |\tilde{x}^* - \tilde{x}| < \rho\}$, $\rho > 0$, $t \neq t_k$, $t \neq t_k^*$, $k = 1, 2, \ldots$, *the following inequality is valid:*

$$[x^* - x, f(t, x^*, \tilde{x}^*) - f(t, x, \tilde{x})]_+ \leqslant 0.$$

(3) *For* $x^* \in S(\rho) = \bigcup_{t \in (t_0, \infty)} \{x^* \in \Omega\colon |x^*(t) - x(t)| < \rho\}$ *and* $k = 1, 2, \ldots$, *the following inequalities are valid:*

$$|x^*(t_k^*) - x(t_k) + I_k(x^*(t_k^*)) - I_k(x(t_k))| \leqslant \gamma_k |x^*(t_k^*) - x(t_k)|; \quad |I_k(x^*)| \leqslant \tfrac{1}{3}\rho,$$

where $\gamma_k \geqslant 0$ *are constants.*

(4) *For* $(t, x^*, \tilde{x}^*) \in S(x, \tilde{x}, \rho)$ *and* $k = 1, 2, \ldots$, *the following inequalities are valid:*

$$\frac{\partial \tau_k(x^*)}{\partial x^*} f(t, x^*, \tilde{x}^*) \leqslant 0.$$

(5) *For* $x^*, y^* \in S(\rho)$ *and* $k = 1, 2, \ldots$, *the inequalities*

$$|\tau_k(x^*) - \tau_k(y^*)| \leqslant \beta |x^* - y^*|$$

are valid, where $0 < \beta = \text{const}$.

(6) $\prod_{k=1}^{\infty} [\gamma_k + (1 + \gamma_k) L\beta] < \infty$.

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ *of problem* (1)–(3) *is uniformly Lipschitz quasistable.*

Theorem 7. *Let the following conditions hold:*

(1) *Conditions* (1)–(3) *of Theorem* 5 *are satisfied.*

(2) *For* $x, x^* \in \Omega$ *and* $k = 1, 2, \ldots$ *the following inequalities are valid:*

$$|\tau_k(x^*) - \tau_k(x)| \leqslant \beta_k |x^* - x|,$$

where $\beta_k \geqslant 0$ *are constants.*

(3) *For* $k = 1, 2, \ldots$ *the inequalities*

$$L\beta_k < 1,$$

$$\beta_k (1 - L\beta_k)^{-1} \leqslant \beta$$

are valid, where $0 < \beta = \text{const}$.

(4) *There exist constants* $M > 0$ *and* $\delta_1 > 0$ *such that for any solution* $u(t; t_0, u_0)$ *of* (11)–(13) *for which*

$$\psi_k(u) = \gamma_k (1 - L\beta_k)^{-1} u + L\beta_k (1 - L\beta_k)^{-1} u,$$

the inequality

$$u(t; t_0, u_0) \leqslant M u_0$$

is valid for $0 \leqslant u_0 < \delta_1$, $t \in (t_0, \infty) \setminus \bigcup_{k=1}^{\infty} (\underline{t}_k, \bar{t}_k]$.

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ *of problem* (1)–(3) *is uniformly Lipschitz stable.*

The proof of Theorem 7 is analogous to the proof of Theorem 5.

Corollary 8. *Let the following conditions hold:*

(1) *Conditions* (1)–(3) *of Corollary* 6 *are met.*

(2) *Conditions* (2) *and* (3) *of Theorem* 7 *are satisfied.*

(3) $\prod_{k=1}^{\infty} (\gamma_k + L\beta_k)(1 - L\beta_k)^{-1} < \infty$.

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ *of problem* (1)–(3) *is uniformly Lipschitz quasistable.*

Theorem 9. *Let the conditions of Theorem* 5 *hold, condition* (2) *being replaced by the following condition:*

(2a) *For* $(t, x^*, \tilde{x}^*), (t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$, $t \neq t_k^*$, $t \neq t_k$, $k = 1, 2, \ldots$, *the following inequality is valid:*

$$|x^* - x + \sigma(f(t, x^*, \tilde{x}^*) - f(t, x, \tilde{x}))| \leqslant |x^* - x| + \sigma g(t, |x^* - x|) + \varepsilon(\sigma),$$

where $\sigma > 0$ *is small enough and* $\varepsilon(\sigma)/\sigma \to 0$ *as* $\sigma \to 0$.

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ *of problem* (1)–(3) *is uniformly Lipschitz quasistable.*

The proof of Theorem 9 is analogous to the proof of Theorem 5. The fact is used that from condition (2a) there follow the inequalities

$$\begin{aligned} D^+ m(t) &= \limsup_{\sigma \to 0^+} \frac{1}{\sigma} [|x^*(t+\sigma) - x(t+\sigma)| - |x^*(t) - x(t)|] \\ &\leqslant \limsup_{\sigma \to 0^+} \frac{1}{\sigma} [|x^*(t+\sigma) - x(t+\sigma)| + \varepsilon(\sigma) \\ &\qquad - |x^*(t) - x(t) - \sigma(f(t, x^*(t), x^*(t-h)) - f(t, x(t), x(t-h)))|] \\ &\leqslant \limsup_{\sigma \to 0^+} \frac{\varepsilon(\sigma)}{\sigma} + \limsup_{\sigma \to 0^+} \frac{1}{\sigma} |x^*(t+\sigma) - x^*(t) - x(t+\sigma) + x(t) \\ &\qquad - f(t, x^*(t), x^*(t-h)) + f(t, x(t), x(t-h))| \\ &= 0. \end{aligned} \tag{24}$$

Corollary 10. *Let the conditions of Corollary* 6 *hold, condition* (2) *being replaced by the condition:*

(2b) *For* $(t, x, \tilde{x}) \in (t_0, \infty) \times \Omega \times \Omega$ *and* $(t, x^*, \tilde{x}^*) \in S(x, \tilde{x}, \rho)$, $t \neq t_k$, $t \neq t_k^*$, $k = 1, 2, \ldots$, *the following inequality is valid:*

$$|x^* - x + \sigma(f(t, x^*, \tilde{x}^*) - f(t, x, \tilde{x}))| \leqslant |x^* - x| + \varepsilon(\sigma),$$

where $\sigma > 0$ is small enough and $\varepsilon(\sigma)/\sigma \to 0$ as $\sigma \to 0$.

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ of problem (1)–(3) *is uniformly Lipschitz quasistable.*

Theorem 11. *Let the conditions of Theorem* 7 *hold, condition* (2) *of Theorem* 5 *being replaced by condition* (2a).

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ of problem (1)–(3) *is uniformly Lipschitz quasistable.*

The proof of Theorem 11 is analogous to the proof of Theorem 5. Inequalities (24) are used.

Corollary 12. *Let the conditions of Corollary* 8 *hold, condition* (2) *of Corollary* 6 *being replaced by condition* (2b).

Then the solution $x(t) = x(t; t_0, \mathscr{S}_0)$ of problem (1)–(3) *is uniformly Lipschitz quasistable.*

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