GENUS AND CANCELLATION

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In Hilton and Roitberg [4], an H-space \( E_{\gamma_w} \) was constructed with the properties that \( E_{\gamma_w} \times S^3 \simeq \text{Sp}(2) \times S^3 \) and \( E_{\gamma_w} \times E_{\gamma_w} \simeq \text{Sp}(2) \times \text{Sp}(2) \), but yet \( E_{\gamma_w} \neq \text{Sp}(2) \). That is, cancellation does not hold for the semi-group operation of cartesian product on finite H-spaces, even up to homotopy type. It was clear from the construction however, that \( E_{\gamma_w} \) and \( \text{Sp}(2) \) had homotopy equivalent \( p \)-localizations for all primes \( p \). This prompted the definition by Mislin [7] that two nilpotent CW complexes \( X \) and \( Y \) are in the same genus (\( G(X) = G(Y) \)) if and only if their \( p \)-localizations \( X_p \) and \( Y_p \) are homotopy equivalent for all primes \( p \).

It was conjectured by Mislin [7] and Hilton that the relation between genus and non-cancellation is always as illustrated by the \( E_{\gamma_w} \) example. Zabrodsky [10] has recently shown that if \( X \) is a finite H-space such that its genus \( G(X) \) contains more than one homotopy type, non-cancellation examples involving \( X \) occur. The result proved here is a converse. (In the following, \( p \) is a fixed prime number.)

**THEOREM 1.**

1. If \( X \) is a one-connected H-space with finitely generated homology, then \( X_p \simeq \prod_{i=1}^r X_i \), where the \( X_i \) are irreducible \( p \)-local H-spaces and are unique up to order.

2. For \( X, Y, \) and \( W \) as in (1), if \( X \times W \simeq Y \times W \), then \( G(X) = G(Y) \). If there exists \( k > 0 \) with \( X^k \simeq Y^k \), then \( G(X) = G(Y) \).

Recall that \( X_i \) irreducible means \( X_i \) has no nontrivial retracts. Since \( X_i \) is an H-space, this is equivalent to \( X_i \) having no nontrivial cartesian product factors, i.e. \( X_i \) being indecomposable. In view of the finiteness assumptions on \( X \), it automatically factors into a finite product of indecomposable H-spaces, and hence the important content of Theorem 1(1) is the uniqueness of this factorization. 1(2) is a consequence of this uniqueness. The results can also be stated more categorically as

**THEOREM 2.** Let \( \text{scpfd}H \) be the homotopy category of one-connected H-spaces with reduced integral homology finitely generated over the \( p \)-local integers \( \mathbb{Z}_p \). Then the cartesian product semi-group, \( \text{scpfd}H \), is a free semi-group generated by the irreducible spaces of \( \text{scpfd}H \).

There are similar results for finite co-H-spaces under the wedge product which dualize Theorems 1 and 2 and generalize the results of Freyd [3] for the stable homotopy category. These appear in §3.
In §1, prime spaces are defined and it is shown that prime spaces can be cancelled. The remaining sections then are devoted to proving that irreducible $p$-local $H$-spaces and co-$H$-spaces are prime. This is accomplished by constructing retracts of non-prime spaces of this type in two steps. In §2, it is shown that spaces admitting a pseudoprojection have retracts, and in section three it is shown that flexible spaces that are non-prime admit pseudoprojections. Of course, $H$-spaces and co-$H$-spaces turn out to be flexible, but these are not the only examples.

§1. CANCELLATION IN THE $P$-LOCAL CATEGORY

We work with a fixed prime number $p$ for the remainder of this paper. We establish the $p$-local analogues of the usual homotopy categories of $CW$ complexes, and then prove the needed criteria for cancellation. Recall that $\mathbb{Z}_p$ denotes the integers localized at $p$ and $\mathbb{Z}/p$ the integers mod $p$.

**Definition 1.1.**

1. $scp =$ homotopy category of one-connected $p$-local $CW$ complexes.
2. $scpft =$ homotopy category of one connected $p$-local $CW$ complexes with finitely generated homology over $\mathbb{Z}_p$ in each dimension.
3. $scpfd =$ the subcategory of $scpft$ with finitely generated homology over $\mathbb{Z}_p$, i.e. $\dim H_* < \infty$.
4. $scpfd\pi =$ the subcategory of $scpft$ with finitely generated homotopy over $\mathbb{Z}_p$, i.e. $\dim \pi_* < \infty$.
5. There are the obvious categories obtained by substituting "N" (nilpotent) for "SC" or deleting the "$p$" in the above notation.

**Definition 1.2.** [5], [9], [1]. If $X$ is a nilpotent $CW$ complex, $X_p$ is the $p$-localization of $X$. It has the properties:

1. $\overline{H}_*(X) \otimes \mathbb{Z}_p \cong \overline{H}_*(X_p)$ (i.e. $\overline{H}_*(X_p)$ is $p$-local).
2. $\pi_*(X) \otimes \mathbb{Z}_p \cong \pi_*(X_p)$ (if $\pi_1(X) \neq 0$, this must be suitably interpreted).
3. $(X \times Y)_p \cong X_p \times Y_p$.
4. $(X \vee Y)_p \cong X_p \vee Y_p$, if $X \vee Y$ is nilpotent (which is not often true unless $X$ and $Y$ are simply connected).

Note that if $X$ is in one of the categories of Definition 1.1 without the "$p$", then its $p$-localization is in the analogous category with the "$p$".

**Lemma 1.3.**

1. If $X$ and $Y$ are in $Np$ and $f : X \to Y$ such that $H_*(f, \mathbb{Z}_p)$ is an isomorphism, then $f$ is a homotopy equivalence.
2. If $X$ and $Y$ are in $Npft$ and $f : X \to Y$ such that $H_*(f, \mathbb{Z}/p)$ is an isomorphism, then $f$ is a homotopy equivalence.

**Proof.** Depending on your view of localization, (1) is trivial or slightly disguised version of the generalized Whitehead theorem. (2) follows from (1), and a Bockstein spectral sequence argument using the finite type hypothesis.
Definition 1.4. (1) \( X \) is irreducible if and only if \( X \) has no nontrivial retracts.

(2) \( X \) is completely decomposable if and only if \( X = \Pi X_i \), with the \( X_i \) irreducible.

(3) \( X \) is completely wedge decomposable if and only if \( X = VX_i \), where the \( X_i \) are irreducible.

Note. If \( X \) is an \( H \)-space and \( r: X \rightarrow Y \) is a retract, then \( X \cong Y \times \text{fib}(r) \). If \( X \) is a co-\( H \)-space and \( i: Y \rightarrow X \) is a retract, then \( X \cong Y \vee \text{cof}(i) \). Thus if \( X \) is an \( H \)-space of finite dimension (\( -\text{fd} \) or \( -\text{fdn} \)), then \( X \) is completely decomposable.

There is an analogy between the completely decomposable spaces of \( \text{Npt} \) with the cartesian product semi-group structure and the natural numbers under multiplication. Through this analogy, “\( X \) a retract of \( Y \)” corresponds to “\( a \mid b \)”. Since the prime numbers are characterized by the property, “if for all \( a, b \) such that \( p \mid ab \), then \( p \mid a \) or \( p \mid b \)”, we could define a space \( X \) to be prime if it had the property that whenever \( X \rightarrow A \times B \rightarrow X \) is a retract, then \( X \) is a retract of \( A \) or \( B \). However, we give a more easily verified definition in terms of cohomology and prove that these spaces have the prime property.

Definition 1.5. If \( X \) is in \( \text{Npt} \), then \( X \) is \( H^* \)-prime if and only if for every \( f: X \rightarrow X \) either

(1) \( H^*(f, \mathbb{Z}/p) \) is an isomorphism, or

(2) for every \( n > 0 \), there exists \( N_n \) such that \( (H^n(f, \mathbb{Z}/p))^{N_n} = 0 \), for \( 0 < m \leq n \).

Any \( H^* \)-prime space \( X \) is irreducible, since if \( Y \rightarrow X \rightarrow Y \) were a non-trivial retract, \( Y \) would not satisfy property (1) or (2) above. The reader should compare the next lemma to 4.16 of Cohen [2] and Lemma 1 of Mislin [7].

Lemma 1.6. If \( X, Y, \) and \( W \) are in \( \text{Npt} \) and \( X \) is \( H^* \)-prime, then if \( X \) is a retract of \( Y \times W \) (or \( Y \vee W \)), \( X \) is a retract of \( Y \) or \( W \) by the composite maps.

Proof. If \( i: X \rightarrow W \times Y \) and \( r: W \times Y \rightarrow X \) is the retraction, define \( f_1 \) to be the composition \( X \rightarrow W \times Y \rightarrow W \rightarrow W \times \mathbb{Z}/p \rightarrow X \), and \( f_2 \) in a similar fashion. Then on the indecomposables \( QH^*(X, \mathbb{Z}/p), f_1 * = 1d - f_2 * \). Hence, \( f_1 * f_2 * = f_2 * f_1 * \) and \( (f_1 * + f_2 *)^n - f_1 * p^n + f_2 * p^n = 1d \) for all \( N > 0 \). Obviously, not both \( f_1 * p^n \) and \( f_2 * p^n \) are zero. Since \( X \) is \( H^* \)-prime, at least one is an isomorphism. By (1.3), it is a homotopy equivalence. The proof for \( Y \vee W \) is similar.

Theorem 1.7. (1) If \( X \) is in \( \text{Npt} \) and \( X \) is completely decomposable into a finite product of \( H^* \)-prime spaces, these prime factors are unique up to order.

(2) If \( X \) is in \( \text{Npt} \) and \( X \) is completely wedge decomposable into a finite wedge of \( H^* \)-prime spaces, then these prime factors are unique up to order.

Proof. Let \( f: \Pi X_i \rightarrow \Pi Y_j \) be a homotopy equivalence between a prime decomposition of \( X \) and an irreducible decomposition of \( X \). We adopt the notation of Mislin [8]: \( f(X_s, Y_t) \) is the composition of the \( s \)th inclusion, \( f_s \), and projection onto the \( r \)th factor. By (1.6), for each \( s \), there is a \( t_s \) such that \( f(X_s, Y_{t_s}) \) and \( f^{-1}(Y_{t_s}, X_s) \) is a retraction. Since \( Y_{t_s} \) is irreducible, \( f(X_s, Y_{t_s}) \) is a homotopy equivalence. Then \( \pi_\ast(f^{-1}(\Pi X_s, Y_j, \Pi Y_j, X_i)) \) has two-sided inverse (if \( * = 1 \), use \( H_i \)). Hence it is a homotopy equivalence. By induction on the number
of prime factors, we can assume that it is known that the remaining decompositions are the same. Hence, the prime factors are unique. The wedge case is similar, except one uses $H_n$ instead of $\pi_n$.

§2. RECOGNIZING RETRACTS

The key to applying the results of section one is proving that the irreducible spaces of interest ($H$-spaces and co-$H$-spaces) are actually $H^\ast$-prime. On a case by case basis, the proof that a given space is irreducible seems usually to also prove that it is a prime. As a start toward proving this empirical observation, we characterize reducible spaces by the existence of a self-map with special properties: notice that if $Y \xrightarrow{f} X \xrightarrow{g} Y$ is a retract, then $(ir)^2 = ir$ and hence this relation holds on the induced maps of homotopy functors. We weaken this somewhat to obtain our definition.

Definition 2.1. If $X$ is in $Npft$, then $f : X \to X$ is a pseudoprojection if and only if image $H^\ast (f, \mathbb{Z}_p) = \text{image} (H^\ast (f, \mathbb{Z}_p))^2$ (this is equivalent to $H^\ast (X, \mathbb{Z}_p) = \text{image} H^\ast (f, \mathbb{Z}_p) + \ker H^\ast (f, \mathbb{Z}_p)$). Clearly if $f$ is a pseudoprojection, then any iterate $f^n$ is also a pseudoprojection. The terminology is justified by demonstrating that pseudoprojections give rise to retracts.

Theorem 2.2 If $X$ is in $Npfd$ or $Npfd\pi$, then $X$ has a nontrivial retract if and only if there exists a pseudoprojection $f : X \to X$ with kernel $H^\ast (f, \mathbb{Z}/p) \neq 0$ or $\tilde{H}^\ast (X, \mathbb{Z}/p)$.

Lemma 2.3. If $X$ is in $Npft$ and $f : X \to X$ is a pseudoprojection, then for all $n > 0$, there is a homotopy commutative diagram.

$$
\begin{array}{ccc}
Y_n & \xrightarrow{\pi_n} & X \\
\downarrow{\pi_n} & & \downarrow{f^n} \\
X & \xrightarrow{\pi_n} & X \\
\end{array}
$$

where $Y_n$ is in $Npft$, $H^\ast (y_n, \mathbb{Z}/p)$ is monic for $* < n$ and $H^\ast (\pi_n, \mathbb{Z}/p)$ is onto for $* < n$. In general, $l_n \to \infty$ as $n \to \infty$.

Proof that Theorem 2.2 follows from Lemma 2.3. If $X$ is in $Npfd$ or $Npfd\pi$, we apply Lemma 2.3 with $n > N = \dim X$, and take $Y$ to be the homology or homotopy approximation to $Y_n$. We delete all subscripts and superscripts. To show that $Y \to X \to Y$ is a retract, it suffices to show that $H^\ast (g\pi, \mathbb{Z}/p)$ is monic for $* \leq \dim X$. But ker $\pi^* = \ker f^* = \ker f^{*2}$. Thus if $y = \pi^* x$, and $g^* y = 0$, then $x \in \ker f^{*2}$ and $\pi^* x = y = 0$. Hence $H^\ast (g\pi, \mathbb{Z}/p)$ is an isomorphism, $g\pi$ is a homotopy equivalence, and $Y \to X \to Y$ is a retract.

Proof of Lemma 2.3. The proof by induction. If $n = 0$, the statement is fulfilled with $Y_0 = X$. To facilitate typesetting, we continue to delete the expected propagation of notation. So assume that we have a diagram satisfying the conditions in dimensions less than $n$. The problems of making $g^*$ monic and $\pi^*$ onto in the next dimension are actually the same problem. That is, if $g^*$ is monic and $f^* g^*$ is monic, then $\pi^*$ is onto. The basic situation then, is that given $y \in \ker H^\ast (g, \mathbb{Z}/p)$ we must find a diagram.
that commutes and such that \( i^*: H^*(Y, \mathbb{Z}/p) \to H^*(Y', \mathbb{Z}/p) \) is an isomorphism for \( * < n \), and \( i^* \) in dimension \( n \) is onto with kernel generated by \( y \). There are two cases to consider.

Case I. \( y \) is the reduction of an integral \((\mathbb{Z}_p)\) class \( y' \). Then \( g^*y' = pz \) and \( f^*z = f^*w \) for some \( w \), since \( f \) is a pseudoprojection. If we define \( y'' = y' - p\pi^*w \), then \( y \) is also the reduction of \( y'' \) and \( f^*g^*y'' = 0 \). Hence if \( Y' \) is the total space of the fibration induced by \( y'': Y \to K(\mathbb{Z}_p, n) \) and \( g' \) is a lifting of \( gf \), we obtain a diagram with the desired properties.

Case II. \( y \) is the reduction of a \( \mathbb{Z}/p' \) class \( y_r \), but not of any \( \mathbb{Z}/p'' \) class. Let \( i_r: K(\mathbb{Z}/p', n) \to K(\mathbb{Z}/p, n) \) be the reduction map, and \( \beta_r: K(\mathbb{Z}/p', n) \to K(\mathbb{Z}/p, n + 1) \) be the higher Bockstein operation. Then we are assuming that for any \( y_r \) with \( \tau_r y_r' = y \), then \( \beta_r y_r' \neq 0 \). If \( g^*y_r = z \), then \( \tau_r z = 0 \) since \( g^*y = 0 \). We can assume that \( f^*w = f^* \) since some iterate of \( f \) has this property on \( H^*(X, \mathbb{Z}/p') \). Hence if we define \( y' = y_r - \pi^*z \), then \( \tau_r y' = y \) and \( f^*g^*y' = 0 \). Take \( Y' \) to be the pull-back of \( y' \): \( Y \to K(\mathbb{Z}/p', n) \) and \( g' \) to be a lifting of \( gf \). Since \( \beta_r y' \neq 0 \), \( y \) is effectively killed in \( H^n(Y', \mathbb{Z}/p) \). That is, \( i^* \) is onto with kernel generated by \( y \).

Note that if \( n = 1 \), the only care to be exercised is to take a connected component of fiber produced by killing the 1-dimensional cohomology. That is, if \( X \in \text{Npfl} \), then \( Y \in \text{Npfl} \) at each stage by appeal to the nilpotent fibration lemmas of [5] or [1].

§3. FLEXIBLE SPACES

We wish to investigate a class of spaces admitting pseudoprojections. As motivation, consider the problem of recovering the fact that \( S^3 \times S^3 \) is not irreducible from the information that there is a diagonal map of degree \((1, p)\) on it. This map is not a pseudoprojection, but a map of degree \((1, p)-(0, p)\) is. The essential observation is that for any map on indecomposables in cohomology, there is a self-map realizing a multiple of it, and there is a way of adding this to the original map. More precisely, we make the following definition.

**Definition 3.1.** If \( X \) is in \( \text{Npfl} \) then \( X \) is flexible if and only if

1. \( QH^*(X, \mathbb{Z}_p)/\text{Torsion} \) is a free \( \mathbb{Z}_p \) module for all \( n \);
2. there exists a \( r > 0 \) such that for every \( f: X \to X \) and \( A: QH^*(X, \mathbb{Z}_p)/\text{Torsion} \to QH^*(X, \mathbb{Z}_p)/\text{Torsion} \), there is \( g: X \to X \) with \( H^*(g, \mathbb{Z}/p) = H^*(f, \mathbb{Z}/p) \) and \( g^* = f^* - p^r A \) on \( QH^*(X, \mathbb{Z}_p)/\text{Torsion} \).

Some spaces, such as products of spheres, are flexible by inspection. Most require some proof.
THEOREM 3.2. (1) If $X$ is a rational $H$-space ($X_o$ an $H$-space) and is in $N_{pfd}$ or $N_{pfd\pi}$, then $X$ is flexible if $X$ satisfies (3.1. (i)).

(2) If $X$ is a co-$H$-space in $N_{pfd}$, then $X$ is flexible.

Proof. (2) Since $X$ is a co-$H$-space, there is a map $h: X \to V_{Sp_{n1}}$ and generators $x_i$ of $H^*(X, \mathbb{Z}_p)/\text{Torsion}$ and $s_i$ such that $h^* s_i = p^{r} x_i$ for some $r$. Hence $\pi_*(\text{fib}(h))$ is finite $p$-primary in each dimension.

and some iterate of the $p$th co-power map $\lambda^p$ will kill the obstructions to lifting. That is, there is $h'$ with $h'h \simeq \lambda^{pr}$ for some $r'$. Given $A$, there is $\widetilde{A}: V_{Sp_{n1}} \to V_{Sp_{n1}}$ so that $H^*(\widetilde{A}, \mathbb{Z}_p)/\text{Torsion} = A$ with respect to the basis $s_i$. Now define $g$ as the composition $X \to X \times X \to X \times V_{Sp_{n1}} \times \text{Id} \to X$. Thus $H^*(g, \mathbb{Z}/p) = H^*(f, \mathbb{Z}/p)$ and $g^* = f^* - p'A$ on $H^*(X, \mathbb{Z}_p)/\text{Torsion}$, where $p' = \deg (h^h)^*$. (1) If $X$ is an $H$-space, a proof similar to above can be given. The extension to rational $H$-spaces is due to Zabrodsky (oral). We repeat it here. Let $n = \text{dimension of } X$ and $X_n$, the $n$th Postnikov approximation (thus if $X \in N_{pfd\pi}$, $X = X_n$). There is a map $h: X_n \to K(\mathbb{Z}_p, n)$, the GEM of type $\bar{n} = (n_1, \ldots, n_r)$ = type of $X$, which classifies a set $x_i$ of cohomology classes which reduce to a set of generators of $\mathbb{Q}H^*(X, \mathbb{Z}_p)/\text{Torsion}$. Since $X$ is a rational $H$-space, $h$ is a rational equivalence and $\pi_*(\text{fib}(h))$ is $p$-primary finite. We would like a lifting of $h \times \text{Id}: X_n \times K(\mathbb{Z}_p, \bar{n}) \to K(\mathbb{Z}_p, \bar{n})$ to $X_n$. The obstructions lie in $H^*(X_n \times K(\mathbb{Z}_p, \bar{n}), X_n \times *) = \pi_*(\text{fib}(h)))$ and hence can be killed by successive use of the $p$th power map $\lambda^p$ on $K(\mathbb{Z}_p, \bar{n})$. There are a finite number of these obstructions, so there is an $r > 0$ and a lifting of $h \times \lambda^r$ to $\mu_r: X_n \times K(\mathbb{Z}_p, \bar{n}) \to X_n$ with $\mu_r|X_n \times * = \text{Id}$. $\mu_r$ plays the role of a multiplication. As before, given $A$, there is $\widetilde{A}: K(\mathbb{Z}_p, \bar{n}) \to K(\mathbb{Z}_p, \bar{n})$ realizing $A$. Define $g$ as the composition $X_n \times \text{Id} \times h \to X_n \times K(\mathbb{Z}_p, \bar{n}) \times X_n \times K(\mathbb{Z}_p, \bar{n}) \to X_n \times K(\mathbb{Z}_p, \bar{n})$ if $X_n = X$. If not use for $f$ the map $f^*: X_n \to X_n$ induced by $f: X \to X$ and observe that the composite map $X \to X_n \to X_n$ lifts to $X$.

It seems plausible that a similar construction would hold for rational co-$H$-spaces, but the author prefers not to be responsible for the details. Also, there are some mild generalizations to spaces which are skeletons of $H$-spaces, but the two cases proved above seem the most useful.

Now if $X$ is flexible and not $H^*$-prime, we can produce a pseudoprojection on $X$. One basic fact (used implicitly in the proof of Lemma 2.3) is that if $S$ is a finite set and $C: S \to S$, then some iterate $C^n = C'$ had the property that $C'^2 = C'$. This is applied to Torsion $H^*(X, \mathbb{Z}_p)$ and $(\mathbb{Q}H^*(X, \mathbb{Z}_p)/\text{Torsion} \otimes \mathbb{Z}/p')$.

THEOREM 3.3. If $X \in N_{pfd}$ or $N_{pfd\pi}$ is flexible and not $H^*$-prime, then $X$ admits a nontrivial pseudoprojection.
**Proof.** We are given that there exists \( f : X \to X \) such that \( H^*(f, \mathbb{Z}/p)^N \neq 0 \) for all \( N \). We show the existence of a pseudoprojection in three stages.

**Step 1.** Consider \( f^* \) on \( QH^*(X, \mathbb{Z}_p)/\text{Torsion} \otimes \mathbb{Z}/p' \), where the \( r \) is given by the flexibility condition on \( X \). Denote \( f^* \) by \( B \) and the indecomposables by \( V \). \( V \) is then a free module over \( \mathbb{Z}_p \) of finite rank (if \( X \) is in \( \text{Npfd}_\pi \), do the construction only up to the dimension of \( X \)). Thus we can assume that \( B : V \otimes \mathbb{Z}/p' \to V \otimes \mathbb{Z}/p' \) has the property that \( B^2 = B \). The object is to find a basis \( z_1, z_2, \ldots, y_1, \ldots, y_s \) of \( V \) and a transformation \( B' \) such that \( B - B' = 0 \) mod \( p' \), \( B'z_i = 0 \) for all \( i \), and \( B' : V/(z_1, \ldots) \to V/(z_1, \ldots) \) is an isomorphism. Let \( x_i \in V \) such that \( x_i \neq 0 \) mod \( p \), but \( Bx_i = 0 \) mod \( p \). Then \( z_1 = x_1 - Bx_1 \) extends to a basis of \( V \) and \( Bz_1 = p'w_1 \), since \( B^2 = B \) mod \( p' \). Define \( T_1 \) to be the projection onto \( z_1 \) and \( B_1 = B - (p'w_1)T_1 \). Then \( B_1z_1 = 0 \). This process can be repeated until the \( \text{mod } p \) kernel is exhausted. Since \( X \) is flexible, and \( B' - f^* = p'A \), there is \( g : X \to X \) with \( g^* = B' \) on \( QH^*/\text{Torsion} \). We call \( g \) by \( f \) in the following.

**Step 2.** If image \( f^* = \text{image } f^{*2} \) on \( QH^*/\text{Torsion} \) and on \( H^*/\text{Torsion} \) up to dimension \( n \), then in dimension \( n \), \( H(X, \mathbb{Z}_p)/\text{Torsion} = \text{image } f^{*2} + \ker f^{*2} \). Hence if \( X \) has dimension \( n \) we wind up using \( f^{2n} \) for the new \( f \).

**Step 3.** By step 2, we can assume that image \( f^{*2} = \text{image } f^* \) on \( H^*(X, \mathbb{Z}_p)/\text{Torsion} \). Since torsion \( H^*(X, \mathbb{Z}_p) \) is finite, we can assume that \( f^{*2} = f^* \) on torsion. Then if \( y \in H^*(X, \mathbb{Z}_p), y = f^*w + z + t \), where \( f^*z = t_1 \), and \( t \) are torsion. But then we can write \( y = f^*(w + t_1 + t) + (z - f^*t_1) + (t - f^*t) \). The last two terms are in the kernel of \( f^* \). Thus \( f \) is a pseudoprojection. It is only distantly related to the original \( f \) given by the nonprimeness of \( X \), but its kernel \( H^*(\mathbb{Z}/p) \) agrees with the “stable kernel” of the original \( f \), and hence \( f \) is a nontrivial pseudoprojection.

**Corollary 3.4.** If \( X \in \text{Npfd} \) or \( \text{Npfd}_\pi \) is flexible and irreducible, then \( X \) is \( H^* \)-prime.

**Corollary 3.5.** (1) If \( X \) is an irreducible \( H_\omega \)-space in \( \text{Npfd} \) or \( \text{Npfd}_\pi \), then \( X \) is \( H^* \)-prime.

(2) If \( X \) is an irreducible co-\( H \)-space in \( \text{Npfd} \), then \( X \) is \( H^* \)-prime.

**Corollary 3.6.** (1) If \( X \) is an \( H \)-space in \( \text{Npfd} \) or \( \text{Npfd}_\pi \), then \( X \) has a unique decomposition as a product of irreducible \( H \)-spaces.

(2) If \( X \) is a co-\( H \)-space in \( \text{Npfd} \), then \( X \) has unique decomposition as a wedge of irreducible co-\( H \)-spaces.

**Corollary 3.7.** (1) If \( X, Y \) and \( W \) are \( H \)-spaces in \( \text{Npfd} \) or \( \text{Npfd}_\pi \) and \( X \times Y \simeq X \times W \), then \( Y \simeq W \). If \( X^k \simeq Y^k \) for some \( k > 0 \), then \( X \simeq Y \).

(2) If \( X, Y, \) and \( W \) are co-\( H \)-spaces in \( \text{Npfd} \) and \( X \vee Y \simeq X \vee W \), then \( Y \simeq W \). If \( V_k X \simeq V_k Y \) for some \( k > 0 \), then \( X \simeq Y \).

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