Inversion of $e$-simple block matrices

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Abstract

We recall previous results on inverting matrices the digraph of which is $e$-simple, i.e. such that every edge is contained in at most one simple cycle. We present and analyze a finite algorithm for the inversion. Applications to $M$-matrices are included.

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1. Introduction

In 1963, the author defined [2,3] a useful notion of an $e$-simple directed graph which can be considered as bridge between the theory of branching continued fractions and certain special classes of matrices (even over a noncommutative ring). The presented theory substantially generalizes the well-known relationship between tridiagonal matrices and (usual) continued fractions as well as results on inversion of unipathic matrices [5].
Let us recall the basic definition. A finite directed graph (digraph) $\vec{D}$ is called $e$-simple if every edge is contained in at most one (simple, i.e. without repeating vertices) cycle. We refer the reader to the book [4] for elementary definitions.

It is immediate that the class of $e$-simple digraphs generalizes the class of unipathic digraphs [4,5] (in which there is at most one path from every vertex to any other vertex). Clearly, the digraph with three vertices 1, 2, 3 and edges (1, 2), (1, 3) and (2, 3) is $e$-simple but not unipathic.

If the set $V$ of $n$ vertices of an $e$-simple digraph $\vec{D} = (V, \vec{E})$ is $N = \{1, 2, \ldots, n\}$, we can assign to $\vec{D}$ an $n \times n$ matrix $A(\vec{D})$. We usually consider the case that the diagonal entry $A_{ii}$ of this matrix is itself a matrix (square of order $n_i$), the off-diagonal entry, for $(i, k) \in \vec{E}$, is a $n_i \times n_k$ matrix $A_{ik}$, and a zero matrix of this dimension if $(i, k)$ is not an edge in $\vec{D}$. Thus, $A(\vec{D})$ is then a usual square block matrix. We call it an $e$-simple block matrix.

In these terms, let us state the main results of the paper [2].

**Theorem A.** Let $A = (A_{ik})$ be an $e$-simple block matrix, $\vec{D} = (N, \vec{E})$ its $e$-simple digraph. Let there exist a solution $C_{ii}$, $C_{ik}$ $(i, k \in N, (i, k) \in \vec{E})$, $C_{ik}$ blocks of the same size as $A_{ii}$, $C_{ik}$ of the same size as $A_{ik}$, of the system (for $k \in N, (i, k) \in \vec{E}$)

$$C_{kk}^{-1} + \sum_{i, (i, k) \in \vec{E}} \left[(C_{kk} - C_{kj}^{-1} C_{j1}^{-1} C_{j2}^{-1} \cdots C_{js}^{-1} C_{is} C_{ik})^{-1} - C_{kk}^{-1}\right] = A_{kk},$$

$$-C_{ii}^{-1} C_{ik} \left(C_{kk} - C_{kj}^{-1} C_{j1}^{-1} C_{j2}^{-1} \cdots C_{js}^{-1} C_{is} C_{ik}\right)^{-1} = A_{ik},$$

where $(k, j_1, \ldots, j_s, i, k)$ is the unique cycle in $\vec{D}$ containing $(i, k)$. [If there is no such cycle, the second summand in the round bracket is missing.]

Then, $A$ is nonsingular and $A^{-1} = (B_{ik})$ with

$$B_{ii} = C_{ii}, \quad B_{ik} = \sum C_{ij} C_{j1}^{-1} C_{j2}^{-1} \cdots C_{js}^{-1} C_{js} C_{jk} \quad \text{if} \quad i \neq k,$$

where the sum is over all simple paths $(i, j_1, \ldots, j_s, k)$ from $i$ to $k$ in $\vec{D}$.

A certain converse was also proved in [2] which enables to find the solution if only some invertibility conditions are fulfilled. For this purpose, assign to every edge $(i, j)$ of $\vec{D}$ its relevant branch $B(i, j)$ as the set of vertices $k \neq j$ for which there exists a path in $\vec{D}$ from $k$ to $j$ not containing $(i, j)$. The cardinality of $B(i, j)$ will be called the height of the edge $(i, j)$. The relevant branches corresponding to edges with positive height will be called nontrivial.

In addition, we call extended relevant branch assigned to an edge $(i, j) \in \vec{E}$ the set of vertices $\overline{\vec{B}}(i, j) = B(i, j) \cup \{j\}$.

**Theorem B.** Construct by induction with respect to the heights of edges $(i, j) \in \vec{E}$ a set of square matrices $R_{(i)j}$ (of the same order as $A_{jj}$) by setting
\[ R_{(i)j} = A_{jj} \quad (4) \]

if the height of \((i, j)\) is zero, and

\[ R_{(i)j} = A_{jj} - \sum_{k \neq i, (k, j) \in \vec{E}} (-A_{jj})R_{(j)j}^{-1}(-A_{j1})R_{(j1)j2}^{-1} \cdots (-A_{js k})R_{(js k)}^{-1}(-A_{j1}) \quad (5) \]

if this height is positive and \((j, j_1, \ldots, j_s, k, j)\) is the cycle in \(\vec{D}\) containing the edge \((k, j)\).

Further, let analogously

\[ R_j = A_{jj} - \sum_{(k, j) \in \vec{E}} (-A_{jj})R_{(j)j}^{-1}(-A_{j1})R_{(j1)j2}^{-1} \cdots (-A_{js k})R_{(js k)}^{-1}(-A_{j1}). \quad (6) \]

Then, if all inverse matrices in the formulae exist,

\[ C_{ii} = R_i^{-1}. \quad (7) \]

\[ C_{ij} = -R_i^{-1}A_{ij}R_{(i)j}^{-1} \quad \text{if } (i, j) \in \vec{E} \quad (8) \]

is the solution of (1) and (2).

**Remark C.** The algorithm in Theorem B can be performed since all edges appearing on the right-hand side of (5) have smaller height than that of \((i, j)\).

2. Results

It follows from (3) that \(B_{ik} = C_{ik}\) if \((i, k) \in \vec{E}\). Thus, if all matrices occurring in (5) and (6) are invertible, i.e. in the generic case, the following holds:

**Corollary 2.1.** If all matrices occurring in (5) and (6) are invertible, then the inverse matrix \(A^{-1}\) is completely determined by the diagonal blocks and by the blocks corresponding to edges of \(\vec{E}\).

In addition, if \((i, k) \in \vec{E}\), the block \(B_{ik}\) of \(A^{-1}\) has the same rank as the block \(A_{ik}\) of \(A\).

Our main task will be to prove an extension of Theorems A and B.

**Theorem 2.2.** The algorithm in Theorem B can be performed if the matrix \(A\) as well as all principal submatrices of \(A\) of order one less than that of \(A\) and those corresponding to relevant branches and extended relevant branches of \(\vec{D}\) are invertible;
in the case of the trivial (i.e., void) relevant branch $B(i, j)$, the block $A_{ij}$ has to be invertible.

Before giving the proof, we state two lemmas and three examples.

**Lemma 2.3.** Let $\tilde{D}$ be a strongly connected digraph with loops, $\tilde{D}_0$ the digraph obtained from $\tilde{D}$ by removing all loops. Then, the following are equivalent:

1. $\tilde{D}_0$ is $e$-simple with at least one edge.
2. $\tilde{D}_0$ can be obtained recurrently from a cycle by joining at each step one more cycle in one vertex.

**Proof.** It is obvious that (2) implies (1) since every edge of the digraph has appeared at some step by adding a cycle and this cycle is the only one containing that edge.

To show that (1) implies (2), observe first that the cycles in an $e$-simple graph $\tilde{D}_0$ have the property that any two of them can have at most one common vertex. We use induction with respect to the number of cycles in $\tilde{D}_0$. If $\tilde{D}_0$ has just one cycle, we are finished. Otherwise, let us say that a simple path in $\tilde{D}_0$ is incident with a cycle of $\tilde{D}_0$ if it contains at least one edge of the cycle. Let $(v_1, \ldots, v_n)$ be a path in $\tilde{D}_0$ incident with the largest number of cycles. It follows that the cycle $C$ containing the edge $(v_1, v_2)$ (which exists) contains a single vertex in common with the digraph $\tilde{D}_1$ obtained from $\tilde{D}_0$ by deleting all edges in $C$. Since $\tilde{D}_1$ is again $e$-simple with a smaller number of cycles, it has the property (2) by the induction hypothesis. Thus, $\tilde{D}_0$ is obtained by adding $C$ to $\tilde{D}_1$ and the proof is complete. □

**Corollary 2.4.** Let a strongly connected $e$-simple digraph have $n$ vertices and $m$ edges. Then, it has $m - n + 1$ cycles.

**Lemma 2.5.** Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a block matrix which has both diagonal blocks $A_{11}$ and $A_{22}$ invertible. Then, the following are equivalent:

1. $A$ is invertible.
2. $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is invertible.
3. $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is invertible.

In this case,

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$
Proof. The first part follows from the Schur complement formula. The second is easily checked if we multiply the last matrix by $A$. □

Remark 2.6. The off-diagonal blocks of $A^{-1}$ can alternatively be expressed using the identities

$$
(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} = A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1},
$$

$$
A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}.
$$

To better understand Theorem B, let us present three examples.

Example 2.7. Let $A$ be the block tridiagonal matrix

$$
A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix}.
$$

(Thus, $\tilde{D}$ is the digraph on three vertices 1, 2, 3 and edges (1, 2), (2, 1), (2, 3), (3, 2), not counting loops.)

Using Lemma 2.5 by partitioning $A$ into two blocks, one shows easily:

If $A_{11}$, $A_{33}$, $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, and $\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}$ are all invertible, then the following are equivalent:

1. $A$ is invertible.
2. $A_{11} - A_{12}(A_{22} - A_{23}A_{33}^{-1}A_{32})^{-1}A_{21}$ is invertible.
3. $A_{22} - A_{21}A_{11}^{-1}A_{12} - A_{23}A_{33}^{-1}A_{32}$ is invertible.
4. $A_{33} - A_{32}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{23}$ is invertible.

Now, let us use Theorem B to solve the system (1) and (2), i.e. the system

$$
\begin{align*}
(C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1} &= A_{11}, \\
(C_{33} - C_{32}C_{22}^{-1}C_{23})^{-1} &= A_{33}, \\
(C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1} + (C_{22} - C_{23}C_{33}^{-1}C_{32})^{-1} - C_{22}^{-1} &= A_{22}, \\
-C_{11}^{-1}C_{12}(C_{22} - C_{21}C_{11}^{-1}C_{12})^{-1} &= A_{12}, \\
-C_{22}^{-1}C_{21}(C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1} &= A_{21}, \\
-C_{33}^{-1}C_{32}(C_{22} - C_{23}C_{33}^{-1}C_{32})^{-1} &= A_{32}, \\
-C_{22}^{-1}C_{23}(C_{33} - C_{32}C_{22}^{-1}C_{23})^{-1} &= A_{23}.
\end{align*}
$$
Since \( B(2, 1) \) and \( B(2, 3) \) are void, \( B(1, 2) = \{3\}, B(3, 2) = \{1\} \), we obtain recurrently
\[
R_{(1)2} = A_{11}, \quad R_{(2)3} = A_{33}, \\
R_{(1)2} = A_{22} - (-A_{23})R_{(2)3}^{-1}(-A_{32}), \quad R_{(3)2} = A_{22} - (-A_{21})R_{(2)3}^{-1}(-A_{12}),
\]
i.e.,
\[
R_{(1)2} = A_{22} - A_{23}A_{33}^{-1}A_{32}, \quad R_{(3)2} = A_{22} - A_{21}A_{11}^{-1}A_{12}.
\]
Further,
\[
R_1 = A_{11} - (-A_{12})R_{(1)2}^{-1}(-A_{21}),
\]
i.e.,
\[
R_1 = A_{11} - A_{12} \left( A_{22} - A_{23}A_{33}^{-1}A_{32} \right)^{-1} A_{21}.
\]
Analogously,
\[
R_3 = A_{33} - A_{32} \left( A_{22} - A_{21}A_{11}^{-1}A_{12} \right)^{-1} A_{23},
\]
and finally,
\[
R_2 = A_{22} - (-A_{21})R_{(1)2}^{-1}(-A_{12}) - (-A_{23})R_{(2)3}^{-1}(-A_{32}),
\]
i.e.,
\[
R_2 = A_{22} - A_{21}A_{11}^{-1}A_{12} - A_{23}A_{33}^{-1}A_{32}.
\]
By (7) and (8),
\[
C_{11} = \left( A_{11} - A_{12} \left( A_{22} - A_{23}A_{33}^{-1}A_{32} \right)^{-1} A_{21} \right)^{-1}, \\
C_{22} = \left( A_{22} - A_{21}A_{11}^{-1}A_{12} - A_{23}A_{33}^{-1}A_{32} \right)^{-1}, \\
C_{33} = \left( A_{33} - A_{32} \left( A_{22} - A_{21}A_{11}^{-1}A_{12} \right)^{-1} A_{23} \right)^{-1}, \\
C_{12} = - \left( A_{11} - A_{12} \left( A_{22} - A_{23}A_{33}^{-1}A_{32} \right)^{-1} A_{21} \right)^{-1} \\
\times A_{12} \left( A_{22} - A_{23}A_{33}^{-1}A_{32} \right)^{-1}, \\
C_{32} = - \left( A_{33} - A_{32} \left( A_{22} - A_{21}A_{11}^{-1}A_{12} \right)^{-1} A_{23} \right)^{-1} \\
\times A_{32} \left( A_{22} - A_{21}A_{11}^{-1}A_{12} \right)^{-1}, \\
C_{21} = - \left( A_{22} - A_{21}A_{11}^{-1}A_{12} - A_{23}A_{33}^{-1}A_{32} \right)^{-1} A_{21}A_{11}^{-1}, \\
C_{23} = - \left( A_{22} - A_{21}A_{11}^{-1}A_{12} - A_{23}A_{33}^{-1}A_{32} \right)^{-1} A_{23}A_{33}^{-1}.
\]
It is easy to check that this is indeed a solution of the system.

**Remark 2.8.** Observe that $A_{22}$ is not supposed to be invertible.

**Example 2.9.** Let $A$ be the block matrix

$$A = \begin{pmatrix}
A_{11} & A_{12} & 0 & 0 \\
0 & A_{22} & A_{23} & 0 \\
0 & 0 & A_{33} & A_{34} \\
A_{41} & 0 & 0 & A_{44}
\end{pmatrix}$$

whose digraph is the 4-cycle (with loops). The corresponding systems (1) and (2) have the form

$$\begin{align*}
(C_{11} - C_{12}C_{22}^{-1}C_{23}C_{33}^{-1}C_{34}C_{44}^{-1}C_{41})^{-1} &= A_{11}, \\
(C_{22} - C_{23}C_{33}^{-1}C_{34}C_{44}^{-1}C_{41}C_{11}^{-1}C_{12})^{-1} &= A_{22}, \\
(C_{33} - C_{34}C_{44}^{-1}C_{41}C_{11}^{-1}C_{12}C_{22}^{-1}C_{23})^{-1} &= A_{33}, \\
(C_{44} - C_{41}C_{11}^{-1}C_{12}C_{22}^{-1}C_{23}C_{33}^{-1}C_{34})^{-1} &= A_{44}, \\
-C_{11}^{-1}C_{12} \left(C_{22} - C_{23}C_{33}^{-1}C_{34}C_{44}^{-1}C_{41}C_{11}^{-1}C_{12}\right)^{-1} &= A_{12}, \\
-C_{22}^{-1}C_{23} \left(C_{33} - C_{34}C_{44}^{-1}C_{41}C_{11}^{-1}C_{12}C_{22}^{-1}C_{23}\right)^{-1} &= A_{23}, \\
-C_{33}^{-1}C_{34} \left(C_{44} - C_{41}C_{11}^{-1}C_{12}C_{22}^{-1}C_{23}C_{33}^{-1}C_{34}\right)^{-1} &= A_{34}, \\
-C_{44}^{-1}C_{41} \left(C_{11} - C_{12}C_{22}^{-1}C_{23}C_{33}^{-1}C_{34}C_{44}^{-1}C_{41}\right)^{-1} &= A_{41}.
\end{align*}$$

Observe that all edges in $\vec{D}$ have height zero so that, as claimed in Theorem B,

$$R(i),i_{i+1} = A_{i+1,i_{i+1}}, \quad i = 1, 2, 3, \quad R(4)_{1} = A_{11},$$

$$R_{i} = A_{ii} - A_{i,i+1}A_{i+1,i+1}^{-1} \cdots A_{i-1,i}, \quad i = 1, 2, 3, 4,$$ 

where indices are taken mod 4.

Thus, using the same convention, by (7) and (8),

$$C_{ii} = \left(A_{ii} - A_{i,i+1}A_{i+1,i+1}^{-1} \cdots A_{i-1,i}\right)^{-1}, \quad i = 1, 2, 3, 4,$$

$$C_{i,i+1} = - \left(A_{ii} - A_{i,i+1}A_{i+1,i+1}^{-1} \cdots A_{i-1,i}\right)^{-1}A_{i,i+1}A_{i+1,i+1}^{-1}, \quad i = 1, 2, 3, 4.$$

It is again easy to check that this is indeed a solution of the mentioned system. The conditions under which the solution exists are invertibility of all matrices $A_{ii}$ and
nonsingularity of $A$ itself. This last condition is, as in Lemma 2.3, equivalent with invertibility of each of the matrices $A_{ii} - A_{i,i+1}^{-1} A_{i+1,i+1} ^{-1} \cdots A_{i-1,i}^{-1}$, $i = 1, 2, 3, 4$. The inverse of $A$ is then

$$
\begin{pmatrix}
C_{11} & C_{12} & C_{12} C_{22}^{-1} C_{23} & C_{12} C_{22}^{-1} C_{23} C_{33}^{-1} C_{34} \\
C_{33} C_{34}^{-1} C_{41} & C_{22} & C_{23} & C_{23} C_{33}^{-1} C_{34} \\
C_{34} C_{41}^{-1} C_{12} & C_{34} C_{41}^{-1} C_{12} C_{13}^{-1} C_{14} & C_{41} C_{11}^{-1} C_{12} C_{22}^{-1} C_{23} & C_{41} C_{11}^{-1} C_{12} C_{22}^{-1} C_{23} C_{44}^{-1} C_{44} \\
C_{41} & C_{12} C_{22}^{-1} C_{23} & C_{12} C_{22}^{-1} C_{23} C_{33}^{-1} C_{34} & C_{12} C_{22}^{-1} C_{23} C_{33}^{-1} C_{34} C_{44}^{-1} C_{44}
\end{pmatrix}.
$$

Example 2.10. Let $A$ be the block matrix

$$
A = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & 0 & 0 \\
A_{31} & 0 & A_{33} & 0 \\
A_{41} & 0 & 0 & A_{44}
\end{pmatrix}
$$

whose digraph is the trifoil (with loops). The corresponding systems (1) and (2) have the form

$$
\begin{align*}
(C_{11} - C_{12} C_{22}^{-1} C_{21})^{-1} &+ (C_{11} - C_{13} C_{33}^{-1} C_{31})^{-1} \\
(C_{22} - C_{21} C_{11}^{-1} C_{12})^{-1} &- 2C_{11}^{-1} = A_{11}, \\
(C_{33} - C_{31} C_{11}^{-1} C_{13})^{-1} & = A_{22}, \\
(C_{44} - C_{41} C_{11}^{-1} C_{14})^{-1} & = A_{44}, \\
-C_{11}^{-1} C_{12} (C_{22} - C_{21} C_{11}^{-1} C_{12})^{-1} & = A_{12}, \\
-C_{11}^{-1} C_{13} (C_{33} - C_{31} C_{11}^{-1} C_{13})^{-1} & = A_{13}, \\
-C_{11}^{-1} C_{14} (C_{44} - C_{41} C_{11}^{-1} C_{14})^{-1} & = A_{14}, \\
-C_{22}^{-1} C_{21} (C_{11} - C_{12} C_{22}^{-1} C_{21})^{-1} & = A_{21}, \\
-C_{33}^{-1} C_{31} (C_{11} - C_{13} C_{33}^{-1} C_{31})^{-1} & = A_{31}, \\
-C_{44}^{-1} C_{41} (C_{11} - C_{14} C_{44}^{-1} C_{41})^{-1} & = A_{41}.
\end{align*}
$$

By Theorem B, the solution is as follows, observing that $B(1, k) = \emptyset$ for $k = 2, 3, 4$, $B(2, 1) = \{3, 4\}$, etc.
\begin{align*}
R_{(1)2} &= A_{22}, \quad R_{(1)3} = A_{33}, \quad R_{(1)4} = A_{44}, \\
\text{and further, after simplification,} \\
R_{(2)1} &= A_{11} - A_{13}A_{33}^{-1}A_{31} - A_{14}A_{44}^{-1}A_{41}, \\
R_{(3)1} &= A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{14}A_{44}^{-1}A_{41}, \\
R_{(4)1} &= A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{13}A_{33}^{-1}A_{31}, \\
R_1 &= A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{13}A_{33}^{-1}A_{31} - A_{14}A_{44}^{-1}A_{41}, \\
R_2 &= A_{22} - A_{21} \left( A_{11} - A_{13}A_{33}^{-1}A_{31} - A_{14}A_{44}^{-1}A_{41} \right)^{-1} A_{12}, \\
R_3 &= A_{33} - A_{31} \left( A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{14}A_{44}^{-1}A_{41} \right)^{-1} A_{13}, \\
R_4 &= A_{44} - A_{41} \left( A_{11} - A_{12}A_{22}^{-1}A_{21} - A_{13}A_{33}^{-1}A_{31} \right)^{-1} A_{14}.
\end{align*}

Then, (7) and (8) yield the solution. The inverse of \( A \) is then

\[
A^{-1} = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{21} & C_{22} & C_{21}C_{11}^{-1}C_{13} & C_{21}C_{11}^{-1}C_{14} \\
C_{31} & C_{32}C_{11}^{-1}C_{12} & C_{33} & C_{31}C_{11}^{-1}C_{14} \\
C_{41} & C_{42}C_{11}^{-1}C_{12} & C_{43} & C_{44}
\end{pmatrix}.
\]

Let us now return to the proof of Theorem 2.2. Observe that it suffices to prove for the Schur complements

\[
R_j = A / A(N \setminus \{j\}) \quad \text{for all } j \in N,
\]

and for all edges \((i, j) \in \vec{E}\), either

\[
R_{(i)j} = A_{jj} \quad \text{if } B(i, j) \text{ is void, or}
\]

\[
R_{(i)j} = A( B(i, j) \cup \{j\} ) / A( B(i, j) )
\]

otherwise.

By (4) and (7), conditions (10) and (9) follow. To prove (11), we use induction with respect to the number \( c(\vec{D}) \) of cycles in \( \vec{D} \). If \( c(\vec{D}) = 1 \), a simple generalization of Example 2.9 shows that (11) holds in a trivial manner.

Now, let \( c(\vec{D}) > 1 \). By Lemma 2.3, there is a cycle \( \vec{C} \) in \( \vec{D} \) which was constructed as the last, and this cycle contains a single vertex which also belongs to the \( e \)-simple digraph \( \vec{D}_0 \) obtained from \( \vec{D} \) by deleting all vertices and edges of \( \vec{C} \) except \( j \).

Denote by \( W \) the set of vertices in \( \vec{C} \) excluding \( j \). Without loss of generality, we can assume that \( W = \{1, \ldots, s\} \), \( \vec{C} \) being the cycle \( (1, 2, \ldots, s, j, 1) \).

The matrix \( A \) can then be written as

\[
A = \begin{pmatrix}
B_{11} & B_{12} & 0 \\
B_{21} & B_{22} & B_{23} \\
0 & B_{32} & B_{33}
\end{pmatrix},
\]
where

\[
B_{11} = \begin{pmatrix}
A_{11} & A_{12} & 0 & \cdots & 0 \\
0 & A_{22} & A_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_{ss}
\end{pmatrix},
\]

(12)

\[
B_{21} = (A_{s1} \ 0 \ \cdots \ 0), \quad B_{12} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}, \quad B_{22} = A_{jj}.
\]

(13)

The Schur complement \(A/A(W)\) is then the matrix

\[
\hat{A} = \begin{pmatrix}
B_{22} & B_{23} \\
B_{32} & B_{33}
\end{pmatrix} - \begin{pmatrix}
0 \\
B_{21} & 0
\end{pmatrix} B_{11}^{-1} \begin{pmatrix}
0 \\
B_{12}
\end{pmatrix},
\]

i.e.,

\[
\hat{A} = \begin{pmatrix}
B_{22} - B_{21} B_{11}^{-1} B_{12} & B_{23} \\
B_{32} & B_{33}
\end{pmatrix}.
\]

Clearly, \(\hat{A}\) is the block matrix of \(\vec{D}_0\). It is now easily shown that by (12) and (13),

\[
B_{22} - B_{21} B_{11}^{-1} B_{12} = A_{jj} - (-A_{j1})A_{11}^{-1}(-A_{12})A_{22}^{-1} \cdots A_{ss}^{-1}(-A_{sj})
\]

(14)

since the upper right corner entry of \(B_{11}^{-1}\) is \(A_{11}^{-1}(-A_{12})A_{22}^{-1} \cdots (-A_{s-1,s})A_{ss}^{-1}\).

However, (14) is exactly the matrix \(R_{(i)j}\) for matrix \(A\).

Let \((p, q)\) be an edge in \(\vec{D}_0\). Then, \(B(p, q) = \hat{B}(p, q) \cup W\) if \(j \in \hat{B}(p, q)\), \(B(p, q) = \hat{B}(p, q)\) otherwise. By the induction hypothesis for \(\vec{D}_0\), (11) holds, i.e., in a clear notation,

\[
\hat{R}_{(p)q} = \hat{A}(\hat{B}(p, q) \cup \{q\})/\hat{A}(\hat{B}(p, q)).
\]

Since \(\hat{A} = A/A(W)\), we obtain by the Crabtree–Haynsworth formula [1]

\[
\hat{A}(\hat{B}(p, q) \cup \{q\})/\hat{A}(\hat{B}(p, q)) = A(B(p, q) \cup \{q\})/A(B(p, q)).
\]

If \(j \in \hat{B}(p, q)\), we obtain (11) since \(R_{(p)q} = \hat{R}_{(p)q}\) in this case.

If \(j \notin \hat{B}(p, q)\), (11) is also true since the matrices \(B_{ik}\) and \(A_{ik}\) for \(i, k \) exceeding 1 coincide.

As we shall see, the assumptions in Theorem 2.2 are always fulfilled when \(A\) is an \(M\)-matrix.

**Theorem 2.11.** Let \(A = (A_{ik})\) be a symmetrically partitioned \(M\)-matrix whose block digraph is \(e\)-simple. Then, the inverse \(A^{-1}\) has the form (\(B_{ik}\)) from (3) where the matrices \(C_{ik}\) are nonnegative and can be obtained by the algorithm in Theorem B. In this algorithm, the matrices \(R_i\) as well as \(R_{(i)j}\) are \(M\)-matrices.
Proof. Follows immediately from Theorem B, formula (11) and the well-known fact that all Schur complements (with respect to principal submatrices) in a (nonsingular) $M$-matrix are also $M$-matrices. □

Remark 2.12. Observe that the algorithm in Theorem B is a generalization of the continued fraction expansion of the ratio of the determinant of the tridiagonal matrix and the determinant obtained by deleting the first row and the first column for $R_1$ if $\vec{D}$ is the path $(1, 2, \ldots, n)$ together with the path $(n, n-1, \ldots, 1)$.

Remark 2.13. It may be interesting to notice that the algorithm in Theorem B is finite but different from the elimination algorithm. As was already observed in [3], in the case of a matrix with the $e$-simple structure one can always find a sequence of pivots for which in the elimination procedure all intermediate matrices have the $e$-simple structure as well.

References