

Contents lists available at [SciVerse ScienceDirect](http://SciVerse.Sciencedirect.com)

# Discrete Applied Mathematics

journal homepage: [www.elsevier.com/locate/dam](http://www.elsevier.com/locate/dam)

## Noise-resilient group testing: Limitations and constructions

Mahdi Cheraghchi<sup>\*,1</sup>

Computer Science Department, Carnegie Mellon University, Pittsburgh PA, USA

### ARTICLE INFO

#### Article history:

Received 15 July 2011

Received in revised form 16 July 2012

Accepted 29 July 2012

Available online 24 August 2012

#### Keywords:

Group testing

Randomness condensers

Extractors

List decoding

### ABSTRACT

We study combinatorial group testing schemes for learning  $d$ -sparse Boolean vectors using highly unreliable disjunctive measurements. We consider an adversarial noise model that only limits the number of false observations, and show that any noise-resilient scheme in this model can only approximately reconstruct the sparse vector. On the positive side, we take this barrier to our advantage and show that approximate reconstruction (within a satisfactory degree of approximation) allows us to break the information theoretic lower bound of  $\Omega(d^2 \log n)$  that is known for exact reconstruction of  $d$ -sparse vectors of length  $n$  via non-adaptive measurements, by a multiplicative factor  $\tilde{\Omega}(d)$ .

Specifically, we give simple randomized constructions of non-adaptive measurement schemes, with  $m = O(d \log n)$  measurements, that allow efficient reconstruction of  $d$ -sparse vectors up to  $O(d)$  false positives even in the presence of  $\delta m$  false positives and  $O(m/d)$  false negatives within the measurement outcomes, for any constant  $\delta < 1$ . We show that, information theoretically, none of these parameters can be substantially improved without dramatically affecting the others. Furthermore, we obtain several explicit constructions, in particular one matching the randomized trade-off but using  $m = O(d^{1+o(1)} \log n)$  measurements. We also obtain explicit constructions that allow fast reconstruction in time  $\text{poly}(m)$ , which would be sublinear in  $n$  for sufficiently sparse vectors. The main tool used in our construction is the list-decoding view of randomness condensers and extractors.

An immediate consequence of our result is an adaptive scheme that runs in only two non-adaptive rounds and exactly reconstructs any  $d$ -sparse vector using a total  $O(d \log n)$  measurements, a task that would be impossible in one round and fairly easy in  $O(\log(n/d))$  or  $d$  rounds.

© 2012 Elsevier B.V. All rights reserved.

### 1. Introduction

Group testing is an area in applied combinatorics that deals with the following problem: Suppose that in a large population of individuals, it is suspected that a small number of the individuals (known as *defectives*, or *positives*) are affected by a condition that can be detected by carrying out a particular test (for example, a disease that can be diagnosed by testing blood samples). Moreover suppose that a *pooling strategy* is permissible, namely, that it is possible to perform a test on a chosen group of individuals, in which case the outcome of the test would be positive if at least one of the individuals in the group possesses the condition (for example, performing a test on a mixture of blood samples results in positive if at least one of the samples is positive). The trivial strategy would be to test each individual separately, which takes as many tests as the population size. The basic question in group testing is: how can we do better?

\* Tel.: +1 512 586 5075; fax: +1 412 268 5574.

E-mail addresses: [cheraghchi@gmail.com](mailto:cheraghchi@gmail.com), [cheraghchi@cmu.edu](mailto:cheraghchi@cmu.edu).

<sup>1</sup> A preliminary summary of this work appears (under the same title) in proceedings of the 17th International Symposium on Fundamentals of Computation Theory (FCT 2009), September 2009 [7].

This question is believed to be first posed by Dorfman [14] during the screening process of draftees in World War II. In this scenario, blood samples are drawn from a large number of people which are tested for a particular disease. If a number of samples are pooled in a group, on which the test is applied, the outcome would be positive if at least one of the samples in the group carries a particular antigen that certifies the disease. Since then, group testing has been applied for a wide range of purposes, from testing for defective items (e.g., defective light bulbs or resistors) as a part of industrial quality assurance [53] to DNA sequencing [45] and DNA library screening in molecular biology (see, e.g., [42,51,39,40,6] and the references therein), and less obvious applications such as multiaccess communication [58], data compression [32], pattern matching [10], streaming algorithms [11], software testing [3], and compressed sensing [12], to name a few. Moreover, over the decades, a vast amount of tools and techniques has been developed for various settings of the problem. We refer the interested reader to the books by Du and Hwang [15,16] for a detailed account of the major developments in this area.

More formally, the basic goal in group testing is to reconstruct a  $d$ -sparse<sup>2</sup> Boolean vector  $x \in \{0, 1\}^n$ , for a known integer parameter  $d > 0$ , from a set of observations. Each observation is the outcome of a measurement that outputs the bitwise “or” (disjunction) of a prescribed subset of the coordinates in  $x$ . Hence, a measurement can be seen as a binary vector in  $\{0, 1\}^n$  which is the characteristic vector of the subset of the coordinates being combined together. More generally, a set of  $m$  measurements can be seen as an  $m \times n$  binary matrix (that we call the *measurement matrix*) whose rows define the individual measurements.

In this work we study group testing in presence of highly unreliable measurements that can produce false outcomes. We will mainly focus on situations where up to a constant fraction of the measurement outcomes can be incorrect. Moreover, we will mainly restrict our attention to *non-adaptive* measurements; the case in which the measurement matrix is fully determined before the observation outcomes are known. Non-adaptive measurements are particularly important for applications as they allow the tests to be performed independently and in parallel, which saves significant time and cost.

On the negative side, we show that when the measurements are allowed to be highly noisy, the original vector  $x$  cannot be uniquely reconstructed. Thus in this case it would be inevitable to resort to approximate reconstructions, i.e., producing a sparse vector  $\hat{x}$  that is close to the original vector in Hamming distance. The reconstruction can err by producing *false positives* (i.e., a position at which  $\hat{x}$  is 1 but  $x$  is 0), or *false negatives* (where  $\hat{x}$  is 0 but  $x$  is 1). In particular, our result shows that if a constant fraction of the measurements can go wrong, the reconstruction might be different from the original vector in  $\Omega(d)$  positions, irrespective of the number of measurements. For most applications this might be an unsatisfactory situation, as even a close estimate of the set of positives might not reveal whether any particular individual is defective or not, and in certain scenarios (such as an epidemic disease or industrial quality assurance) it is unacceptable to miss any affected individuals. This motivates us to focus on approximate reconstructions with *one-sided* error. Namely, we will require that the support of  $\hat{x}$  contains the support of  $x$  and be possibly larger by up to  $O(d)$  positions. It can be argued that, for most applications, such a scheme is as good as exact reconstruction, as it allows one to significantly narrow-down the set of defectives to up to  $O(d)$  *candidate positives*. In particular, as observed in [36], one can use a *second stage* if necessary and individually test the resulting set of candidates to identify the exact set of positives, hence resulting in a so-called *trivial two-stage* group testing algorithm. Next, we will show that in any scheme that produces no or few false negatives in the reconstruction, only up to  $O(1/d)$  fraction of false negatives (i.e., observation of a 0 instead of 1) in the measurements can be tolerated, while there is no such restriction on the amount of tolerable false positives. Thus, one-sided approximate reconstruction breaks down the symmetry between false positives and false negatives in our error model.

On the positive side, we give a general construction for noise-resilient measurement matrices that guarantees approximate reconstruction up to  $O(d)$  false positives. Our main result is a general reduction from the noise-resilient group testing problem to construction of well-studied combinatorial objects known as *randomness condensers* that play an important role in theoretical computer science. Different qualities of the underlying condenser correspond to different qualities of the resulting group testing scheme, as we describe later. Using the state of the art constructions of randomness condensers, we obtain different instantiations of our framework with incomparable properties, as summarized in Table 1. In particular, the resulting randomized constructions (obtained from optimal lossless condensers and extractors) can be set to tolerate (with overwhelming probability) *any* constant fraction ( $<1$ ) of false positives and an  $\Omega(1/d)$  fraction of false negatives, and they are able to produce an accurate reconstruction up to  $O(d)$  false positives (where the positive constant behind  $O(\cdot)$  can be made arbitrarily small), which is the best trade-off one can hope for, all using only  $O(d \log n)$  measurements. This almost matches the information-theoretic lower bound  $\Omega(d \log(n/d))$  shown by simple counting. We will also show explicit (deterministic) constructions that can approach the optimal trade-off, and finally, those that are equipped with fully efficient reconstruction algorithms with running time polynomial in the number of measurements.

*Related work.* There is a large body of work in the group testing literature that is related to the present work; here we briefly discuss a few with the highest relevance. The exact group testing problem in the noiseless scenario is handled by what is known as *superimposed coding* (see [19,37]) or the closely related concepts of *cover-free families* or *disjunct matrices*.<sup>3</sup>

A  $d$ -superimposed code is a collection of binary vectors with the property that from the bitwise “or” of up to  $d$  words in the family one can uniquely identify the comprising vectors. A  $d$ -cover-free family is a collection of subsets of a universe, none of which is contained in any union of up to  $d$  of the other subsets. A  $d$ -disjunct matrix is a binary matrix whose

<sup>2</sup> We define a  $d$ -sparse vector as a vector whose number of nonzero coefficients is at most  $d$ .

<sup>3</sup> These notions are naturally extended to the noisy setting, e.g., in [41].

**Table 1**

A summary of constructions in this paper. The parameters  $\alpha \in [0, 1]$  and  $\delta \in (0, 1]$  are arbitrary constants,  $m$  is the number of measurements,  $e_0$  (resp.,  $e_1$ ) the number of tolerable false positives (resp., negatives) in the measurements, and  $e'_0$  is the number of false positives in the reconstruction. The fifth column shows whether the construction is deterministic (Det) or randomized (Rnd), and the last column shows the running time of the reconstruction algorithm.

$m$	$e_0$	$e_1$	$e'_0$	Det/Rnd	Rec. time
$O(d \log n)$	$\alpha m$	$\Omega(m/d)$	$O(d)$	Rnd	$O(mn)$
$O(d \log n)$	$\Omega(m)$	$\Omega(m/d)$	$\delta d$	Rnd	$O(mn)$
$O(d^{1+o(1)} \log n)$	$\alpha m$	$\Omega(m/d)$	$O(d)$	Det	$O(mn)$
$d \cdot \text{quasipoly}(\log n)$	$\Omega(m)$	$\Omega(m/d)$	$\delta d$	Det	$O(mn)$
$d \cdot \text{quasipoly}(\log n)$	$\alpha m$	$\Omega(m/d)$	$O(d)$	Det	$\text{poly}(m)$
$\text{poly}(d)\text{poly}(\log n)$	$\text{poly}(d)\text{poly}(\log n)$	$\Omega(e_0/d)$	$\delta d$	Det	$\text{poly}(m)$

columns correspond to characteristic vectors of a  $d$ -cover-free family. These notions turn out to precisely characterize the combinatorial structure needed for (worst-case) noiseless group testing.

It is known that, even for the noiseless case, exact reconstruction of  $d$ -sparse vectors (when  $d$  is not too large) requires at least  $\Omega(d^2 \log n / \log d)$  measurements (several proofs of this fact are known, e.g., [20,50,23]). On the other hand, the best known upper bound on the optimal number of measurements in this case is  $O(d^2 \log n)$ , which can be obtained by a standard probabilistic argument (cf. [15]). An important class of superimposed codes is constructed from combinatorial designs, among which we mention the construction based on MDS codes given by Kautz and Singleton [35], which, in the group testing notation, achieves  $O(d^2 \log^2 n)$  measurements.<sup>4</sup> The non-asymptotic bounds on the construction based on MDS codes were improved by D'yachkov et al. [17,18]. A nearly optimal construction of  $d$ -disjunct matrices with  $O(d^2 \log n)$  rows (using a derandomized construction of codes on the Gilbert–Varshamov bound) was obtained by Porat and Rothschild [46]. They also use combinatorial designs based on error-correcting codes with large distance as their main technical tool<sup>5</sup>. More recently (independently of the initial publication of our work [7]), Indyk, Ngo, and Rudra gave a randomized construction of  $d$ -disjunct matrices that also achieves the nearly-optimal  $O(d^2 \log n)$  number of measurements [34] for noiseless group testing. Similar to the present work, their technique is based on list-decodable codes and their construction is equipped with an efficient reconstruction algorithm (with polynomial running time in the number of measurements). They also obtain an explicit construction when the sparsity parameter  $d$  is small (namely,  $d = O(\log n / \log \log n)$ ). This result was later improved in [43], where in particular, the authors remove the restriction on  $d$ .

Approximate reconstruction of sparse vectors up to a small number of false positives (that is one focus of this work) has been studied as a major ingredient of trivial two-stage schemes [36,40,2,13,22,6]. In particular, a generalization of superimposed codes, known as *selectors*, was introduced in [13] which, roughly speaking, allows for identification of the sparse vector up to a prescribed number of false positives in the noiseless model. The authors of [13] give a non-constructive result showing that there are such (non-adaptive) schemes that keep the number of false positives at  $O(d)$  using  $O(d \log(n/d))$  measurements, matching the optimal “counting bound”. A probabilistic construction of asymptotically optimal selectors (resp., a related notion of *resolvable matrices*) is given in [6] (resp., [22]), and [33,9] give slightly sub-optimal “explicit” constructions based on certain expander graphs obtained from dispersers.<sup>6</sup>

A slightly weaker notion than selectors, known in the literature as *list-decoding superimposed codes*, has been studied earlier by D'yachkov and Rykov [19] and Rashad [48]. List-decoding superimposed codes also suffice for, and in fact exactly capture the identification problem of sparse vectors up to a number of false positives in the noiseless model. Among other applications, these codes have been used by D'yachkov et al. in the context of ALOHA medium access protocol [21]. The notion of error-resilient measurement matrices that we study in this work (Definition 1) is more general than list-decoding superimposed codes (as well as selectors) in that it considers the possibility of measurement noise and also presence of false negatives in the reconstruction. However, the special case where the measurements are noise-free and no false negatives are allowed in the reconstruction can be seen to be essentially equivalent to the notion of list-decoding superimposed codes. Therefore, the known upper and lower bounds on list-decoding superimposed codes (e.g., [48]) can be translated to similar bounds for this special case of noise-resilient measurements matrices as well.

To give a concise comparison of the present work with those listed above, we mention some of the qualities of the group testing schemes that we will aim to attain:

1. Low number of measurements.
2. Arbitrarily good degree of approximation.
3. Maximum possible noise tolerance.

<sup>4</sup> Interestingly, this classical construction can be regarded as a special instantiation of our framework where a “bounded degree univariate polynomial” is used in place of the underlying randomness condenser. However, the analysis and the properties of the resulting group testing schemes substantially differ for the two cases, and in particular, the MDS-based construction owes its properties essentially to the large distance of the underlying code.

<sup>5</sup> This construction is *weakly explicit*, in the sense that we later define in Definition 1.

<sup>6</sup> The notion of selectors is useful in a noiseless setting. However, as remarked in [6], it can be naturally extended to include a “noise” parameter, and the probabilistic constructions of selectors can be naturally extended to this case. Nonetheless, this generalization does not distinguish between false positives and negatives and the explicit constructions of selectors [33,9] cannot be used in a (highly) noisy setting.

4. Efficient, deterministic construction: As typically the sparsity  $d$  is very small compared to  $n$ , a measurement matrix must be ideally *fully explicitly constructible* in the sense that each column of the  $m \times n$  measurement matrix should be computable in deterministic time  $\text{poly}(m, \log n)$  (e.g., while the constructions in [13,22,6,33,9] are all polynomial-time computable in  $n$ , they are not fully explicit in this sense).
5. Fully efficient reconstruction algorithm: For a similar reason, the length of the observation vector is typically far smaller than  $n$ ; thus, it is desirable to have a reconstruction algorithm that identifies the support of the sparse vector in time polynomial in the number of measurements (which might be exponentially smaller than  $n$ ).

While the works that we mentioned focus on few of the criteria listed above (e.g., none of the above-mentioned schemes for approximate group testing are equipped with a fully efficient reconstruction algorithm), our approach can potentially attain *all* at the same time. As we will see later, using the best known constructions of condensers we will have to settle to sub-optimal results in one or more of the aspects above. Nevertheless, the fact that any improvement in the construction of condensers would readily translate to improved group testing schemes (and also the rapid growth of derandomization theory) justifies the significance of the construction given in this work.

The remainder of the paper is organized as follows. We will continue in Section 2 with some preliminaries on the basic notions that we will use. In Section 3, we show our negative results on the possible trade-offs between the amount of tolerable measurement error and the proximity of the reconstruction. We introduce our general construction of measurement matrices in Section 4.1, and in Section 4.2 show several possible instantiations that achieve the trade-offs listed in Table 1. Finally, in Section 4.3, we discuss the notion of list-recoverable codes and the way it is related to our construction.

## 2. Preliminaries

Let  $x$  and  $y$  be binary vectors in  $\{0, 1\}^n$ . The Hamming distance between  $x$  and  $y$ , denoted by  $\text{dist}(x, y)$ , is defined as the number of coordinate positions where  $x$  and  $y$  differ. For non-negative integers  $e_0$  and  $e_1$ , we say that the ordered pair  $(x, y)$  are  $(e_0, e_1)$ -close (or  $x$  is  $(e_0, e_1)$ -close to  $y$ ) if  $y$  can be obtained from  $x$  by flipping at most  $e_0$  bits from 0 to 1 and at most  $e_1$  bits from 1 to 0. Hence, such  $x$  and  $y$  will be  $(e_0 + e_1)$ -close in Hamming-distance. Further,  $(x, y)$  are called  $(e_0, e_1)$ -far if they are not  $(e_0, e_1)$ -close. Note that if  $x$  and  $y$  are seen as characteristic vectors of subsets  $X$  and  $Y$  of  $[n]$ , respectively,<sup>7</sup> they are  $(|Y \setminus X|, |X \setminus Y|)$ -close. Furthermore,  $(x, y)$  are  $(e_0, e_1)$ -close iff  $(y, x)$  are  $(e_1, e_0)$ -close.

A group of  $m$  non-adaptive measurements for binary vectors of length  $n$  can be seen as an  $m \times n$  matrix (that we call the *measurement matrix*) whose  $(i, j)$ th entry is 1 if and only if the  $j$ th coordinate of the vector is present in the disjunction defining the  $i$ th measurement. For a measurement matrix  $M$ , we denote by  $M[x]$  the outcome of the measurements defined by  $M$  on a binary vector  $x$ , that is, the bitwise “or” of those columns of  $M$  chosen by the support of  $x$ . For example, for the measurement matrix

$$M := \begin{pmatrix} \mathbf{0} & \mathbf{0} & 1 & \mathbf{1} & 0 & 1 & 1 & 0 \\ \mathbf{1} & \mathbf{0} & 1 & \mathbf{0} & 0 & 1 & 0 & 1 \\ \mathbf{0} & \mathbf{1} & 0 & \mathbf{1} & 0 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & 1 & 0 & 1 & 1 \\ \mathbf{1} & \mathbf{0} & 1 & \mathbf{0} & 1 & 1 & 1 & 0 \end{pmatrix}$$

and Boolean vector  $x := (1, 1, 0, 1, 0, 0, 0, 0)$ , we have  $M[x] = (1, 1, 1, 0, 1)$ , which is the bit-wise “or” of the columns shown in boldface.

As motivated by our negative results, for the specific setting of the group testing problem that we are considering in this work, it is necessary to give an *asymmetric* treatment that distinguishes between inaccuracies due to false positives and false negatives. Thus, we will work with a notion of error-tolerating measurement matrices that directly and conveniently captures this requirement, as given below:

**Definition 1.** Let  $m, n, d, e_0, e_1, e'_0, e'_1$  be integers. An  $m \times n$  measurement matrix  $A$  is called  $(e_0, e_1, e'_0, e'_1)$ -correcting for  $d$ -sparse vectors if, for every  $y \in \{0, 1\}^m$  there exists  $z \in \{0, 1\}^n$  (called a *valid decoding* of  $y$ ) such that for every  $x \in \{0, 1\}^n$ , whenever  $(x, z)$  are  $(e'_0, e'_1)$ -far,  $(A[x], y)$  are  $(e_0, e_1)$ -far. The matrix  $A$  is called *fully explicit* (or simply explicit) if each column of the matrix can be computed in time<sup>8</sup>  $\text{poly}(m, \log n)$ , and *weakly explicit* if it can be computed in time  $\text{poly}(m, n)$ .

Intuitively, the definition states that two measurements are allowed to be confused only if they are produced from close vectors. In particular, an  $(e_0, e_1, e'_0, e'_1)$ -correcting matrix gives a group testing scheme that reconstructs the sparse vector up to  $e'_0$  false positives and  $e'_1$  false negatives even in the presence of  $e_0$  false positives and  $e_1$  false negatives in the measurement outcome.

Under this notation, unique decoding would be possible using an  $(e_0, e_1, 0, 0)$ -correcting matrix if the amount of measurement errors is bounded by at most  $e_0$  false positives and  $e_1$  false negatives. However, when  $e'_0 + e'_1$  is positive,

<sup>7</sup> We use the shorthand  $[n]$  for the set  $\{1, 2, \dots, n\}$ .

<sup>8</sup> We will use this convention since typically  $m \ll n$ . In particular, when the sparsity parameter  $d$  is small,  $n$  can be exponentially larger than  $m$ .

decoding may result in a bounded amount of ambiguity, namely, up to  $e'_0$  false positives and  $e'_1$  false negatives in the decoded sequence.

The special case of  $(0, 0, 0, 0)$ -correcting matrices is equivalent to what is known in the combinatorics literature as *d-superimposed codes* or *d-separable matrices* and is closely related (in fact, equivalent up to an additive constant in the parameter  $d$ ) to the notions of *d-cover-free families* and *d-disjunct matrices* (as discussed in the introduction; cf. [15] for precise definitions). Also,  $(0, 0, e'_0, 0)$ -correcting matrices are related to the notion of *selectors* in [13], *resolvable matrices* in [22], and *list-decoding superimposed codes* in [19].

The basic combinatorial tools that we use in this work are the notions of randomness condensers and extractors. Here we briefly review the essential definitions related to these objects. A detailed treatment of these notions can be found in the standard theoretical computer science literature, and in particular, the book by Arora and Barak [1].

The *min-entropy* of a distribution  $\mathcal{X}$  over a finite support  $S$  is given by

$$H_\infty(\mathcal{X}) := \min_{x \in S} \{-\log \Pr_{\mathcal{X}}(x)\},$$

where  $\Pr_{\mathcal{X}}(x)$  is the probability that  $\mathcal{X}$  assigns to  $x$ , and the logarithm is to base 2.

The *statistical distance* of two distributions  $\mathcal{X}$  and  $\mathcal{Y}$  defined over the same finite space  $S$  is given by

$$\frac{1}{2} \sum_{s \in S} |\Pr_{\mathcal{X}}(s) - \Pr_{\mathcal{Y}}(s)|,$$

which is half the  $\ell_1$  distance of the two distributions when regarded as vectors of probabilities over  $S$ . Two distributions  $\mathcal{X}$  and  $\mathcal{Y}$  are said to be  $\epsilon$ -close if their statistical distance is at most  $\epsilon$ .

We will use the shorthand  $\mathcal{U}_n$  for the uniform distribution on  $\{0, 1\}^n$ , and  $X \sim \mathcal{X}$  for a random variable  $X$  drawn from a distribution  $\mathcal{X}$ . The notions of randomness condenser and extractor are defined as follows.

**Definition 2.** A function  $f: \{0, 1\}^n \times \{0, 1\}^t \rightarrow \{0, 1\}^\ell$  is a *strong  $k \rightarrow_\epsilon k'$  condenser* (or simply a  $k \rightarrow_\epsilon k'$  condenser) if for every distribution  $\mathcal{X}$  on  $\{0, 1\}^n$  with min-entropy at least  $k$ , random variable  $X \sim \mathcal{X}$  and a seed  $Y \sim \mathcal{U}_t$ , the distribution of  $(Y, f(X, Y))$  is  $\epsilon$ -close to a distribution  $(\mathcal{U}_t, \mathcal{Z})$  with min-entropy at least  $t + k'$ . The parameters  $k, \epsilon, k - k'$ , and  $\ell - k'$  are called the *entropy requirement*, the *error*, the *entropy loss* and the *overhead* of the condenser, respectively. A condenser with zero entropy loss is called a  $(k, \epsilon)$ -*lossless condenser*, and a condenser with zero overhead is called a (strong)  $(k, \epsilon)$ -*extractor*. A condenser is *explicit* if it is polynomial-time computable.

### 3. Negative results

In coding theory, it is possible to construct codes that can tolerate up to a constant fraction of adversarially chosen errors and still guarantee unique decoding. Hence it is natural to ask whether a similar possibility exists in group testing, namely, whether there is a measurement matrix that is robust against a constant fraction of adversarial errors and still recovers the measured vector exactly. Below we show that this is not possible.<sup>9</sup>

**Lemma 3.** Suppose that an  $m \times n$  measurement matrix  $M$  is  $(e_0, e_1, e'_0, e'_1)$ -resilient for  $d$ -sparse vectors. Then  $(\max\{e_0, e_1\} + 1)/(e'_0 + e'_1 + 1) \leq m/d$ .

**Proof.** We use similar arguments as those used in [4,26] in the context of black-box hardness amplification in NP: Define a partial ordering  $<$  between binary vectors using bit-wise comparisons (with  $0 < 1$ ). Let  $t := d/(e'_0 + e'_1 + 1)$  be an integer,<sup>10</sup> and consider any monotonically increasing sequence of vectors  $x_0 < \dots < x_t$  in  $\{0, 1\}^n$  where  $x_i$  has weight  $i(e'_0 + e'_1 + 1)$ . Thus,  $x_0$  and  $x_t$  will have weights zero and  $d$ , respectively. Note that we must also have  $M[x_0] < \dots < M[x_t]$  due to monotonicity of the “or” function.

A fact that is directly deduced from Definition 1 is that, for every  $x, x' \in \{0, 1\}^n$ , if  $(M[x], M[x'])$  are  $(e_0, e_1)$ -close, then  $x$  and  $x'$  must be  $(e'_0 + e'_1, e'_0 + e'_1)$ -close. This can be seen by setting  $y := M[x']$  in the definition, for which there exists a valid decoding  $z \in \{0, 1\}^n$ . As  $(M[x], y)$  are  $(e_0, e_1)$ -close, the definition implies that  $(x, z)$  must be  $(e'_0, e'_1)$ -close. Moreover,  $(M[x'], y)$  are  $(0, 0)$ -close and thus,  $(e_0, e_1)$ -close, which implies that  $(z, x')$  must be  $(e'_1, e'_0)$ -close. Thus by the triangle inequality,  $(x, x')$  must be  $(e'_0 + e'_1, e'_0 + e'_1)$ -close.

Now, observe that for all  $i$ ,  $(x_i, x_{i+1})$  are  $(e'_0 + e'_1, e'_0 + e'_1)$ -far, and hence, their encodings must be  $(e_0, e_1)$ -far, by the fact we just mentioned. In particular this implies that  $M[x_t]$  must have weight at least  $t(e_0 + 1)$ , which must be trivially upper bounded by  $m$ . Hence it follows that  $(e_0 + 1)/(e'_0 + e'_1 + 1) \leq m/d$ . Similarly we can also show that  $(e_1 + 1)/(e'_0 + e'_1 + 1) \leq m/d$ .  $\square$

The above lemma gives a trade-off between the tolerable error in the measurements versus the reconstruction error. In particular, for unique decoding to be possible (i.e.,  $e'_0 = e'_1 = 0$ ) one can only guarantee resiliency against up to  $O(1/d)$  fraction of errors in the measurement. On the other hand, tolerance against a constant fraction of errors (i.e.,  $e_0 = \Omega(m)$ )

<sup>9</sup> We remark that the negative results in this section hold for both adaptive and non-adaptive measurements.

<sup>10</sup> For the sake of simplicity in this presentation we ignore the fact that certain fractions might in general give non-integer values. However, it should be clear that this will cause no loss of generality.

or  $e_1 = \Omega(m)$ ) would make an ambiguity of order  $\Omega(d)$  in the decoding inevitable, irrespective of the number of measurements.

As discussed in the introduction, for most applications it is desirable to have a one-sided error in reconstruction, in which case the support of the reconstruction outcome  $\hat{x}$  is required to contain the support of the original vector  $x$  being measured, and be possibly larger by up to  $O(d)$  positions. Moreover, such schemes can be used in *trivial two-stage* schemes as defined in [36].

The trade-off given by the following lemma only focuses on false negatives and is thus useful for trivial two-stage schemes:

**Lemma 4.** *Suppose that an  $m \times n$  measurement matrix  $M$  is  $(e_0, e_1, e'_0, e'_1)$ -resilient for  $d$ -sparse vectors. Then for every  $\epsilon > 0$ , either*

$$e_1 < \frac{(e'_1 + 1)m}{\epsilon d}$$

or

$$e'_0 \geq \frac{(1 - \epsilon)(n - d + 1)}{(e'_1 + 1)^2}.$$

**Proof.** Let  $x \in \{0, 1\}^n$  be chosen uniformly at random among vectors of weight  $d$ . Randomly flip  $e'_1 + 1$  of the bits on the support of  $x$  to 0, and denote the resulting vector by  $x'$ . Using the partial ordering  $<$  in the proof of the last lemma, it is obvious that  $x' < x$ , and hence,  $M[x'] < M[x]$ . Let  $b$  denote any disjunction of a number of coordinates in  $x$  and  $b'$  the same disjunction in  $x'$ . We must have

$$\Pr[b' = 0 | b = 1] \leq \frac{e'_1 + 1}{d},$$

as for  $b$  to be 1 at least one of the variables on the support of  $x$  must be present in the disjunction and one particular such variable must necessarily be flipped to bring the value of  $b'$  down to zero. Using this, the expected Hamming distance between  $M[x]$  and  $M[x']$  can be bounded as follows:

$$\mathbb{E}[\text{dist}(M[x], M[x'])] = \sum_{i \in [m]} \mathbb{1}(M[x]_i = 1 \wedge M[x']_i = 0) \leq \frac{e'_1 + 1}{d} \cdot m,$$

where the expectation is over the randomness of  $x$  and the bit flips,  $\text{dist}(\cdot, \cdot)$  denotes the Hamming distance between two vectors, and  $\mathbb{1}(\cdot)$  denotes an indicator predicate.

Fix a particular choice of  $x'$  that keeps the expectation at most  $(e'_1 + 1)m/d$ . Now the randomness is over the possibilities of  $x$ , that is, flipping up to  $e'_1 + 1$  zero coordinates of  $x'$  randomly. Denote by  $\mathcal{X}$  the set of possibilities of  $x$  for which  $M[x]$  and  $M[x']$  are  $\frac{(e'_1 + 1)m}{\epsilon d}$ -close, and by  $\mathcal{B}$  the set of all vectors that are monotonically larger than  $x'$  and are  $(e'_1 + 1)$ -close to it. Obviously,  $\mathcal{X} \subseteq \mathcal{B}$ , and, by Markov's inequality, we know that  $|\mathcal{X}| \geq (1 - \epsilon)|\mathcal{B}|$ .

Let  $z$  be any valid decoding of  $M[x']$ . Thus,  $(x', z)$  must be  $(e'_0, e'_1)$ -close. Now assume that  $e_1 \geq \frac{(e'_1 + 1)m}{\epsilon d}$  and consider any  $x \in \mathcal{X}$ . Hence,  $(M[x], M[x'])$  are  $(e_0, e_1)$ -close and  $(x, z)$  must be  $(e'_0, e'_1)$ -close by Definition 1. Regard  $x, x', z$  as the characteristic vectors of sets  $X, X', Z \subseteq [n]$ , respectively, where  $X' \subseteq X$ . We know that  $|X \setminus Z| \leq e'_1$  and  $|X \setminus X'| = e'_1 + 1$ . Therefore,

$$|(X \setminus X') \cap Z| = |X \setminus X'| - |X \setminus Z| + |X' \setminus Z| > 0, \tag{1}$$

and  $z$  must take at least one nonzero coordinate from  $\text{supp}(x) \setminus \text{supp}(x')$ .

Now we construct an  $(e'_1 + 1)$ -hypergraph<sup>11</sup>  $H$  as follows: The vertex set is  $[n] \setminus \text{supp}(x')$ , and for every  $x \in \mathcal{X}$ , we put a hyperedge containing  $\text{supp}(x) \setminus \text{supp}(x')$ . The density of this hypergraph is at least  $1 - \epsilon$ , by the fact that  $|\mathcal{X}| \geq (1 - \epsilon)|\mathcal{B}|$ . Now Lemma 22 implies that  $H$  has a matching of size at least

$$t := \frac{(1 - \epsilon)(n - d + 1)}{(e'_1 + 1)^2}.$$

As by (1),  $\text{supp}(z)$  must contain at least one element from the vertices in each hyperedge of this matching, we conclude that  $|\text{supp}(z) \setminus \text{supp}(x')| \geq t$ , and that  $e'_0 \geq t$ .  $\square$

The lemma above shows that if one is willing to keep the number  $e'_1$  of false negatives in the reconstruction at the zero level (or bounded by a constant), only an up to  $O(1/d)$  fraction of false negatives in the measurements can be tolerated (regardless of the number of measurements), unless the number  $e'_0$  of false positives in the reconstruction grows to an enormous amount (namely,  $\Omega(n)$  when  $n - d = \Omega(n)$ ) which is certainly undesirable.

<sup>11</sup> See Appendix for definitions.

Recall that, as mentioned in the introduction, exact reconstruction of  $d$ -sparse vectors of length  $n$ , even in a noise-free setting, requires at least  $\Omega(d^2 \log_d n)$  non-adaptive measurements. However, it turns out that there is no such restriction when an approximate reconstruction is sought for, except for the following bound which can be shown using simple counting and holds for adaptive noiseless schemes as well:

**Lemma 5.** *Let  $M$  be an  $m \times n$  measurement matrix that is  $(0, 0, e'_0, e'_1)$ -resilient for  $d$ -sparse vectors. Then,*

$$m \geq d \log(n/d) - d - e'_0 - 3e'_1 - e'_1 \log(n - d - e'_0) - 1.$$

**Proof.** The proof is a simple counting argument. For integers  $a \geq b \geq 0$ , we use the notation  $V(a, b)$  for the volume of a Hamming ball of radius  $b$  in  $\{0, 1\}^a$ . It is given by

$$V(a, b) = \sum_{i=0}^b \binom{a}{i} \leq 2^a.$$

The following bounds are elementary estimates:

$$b \log(a/b) \leq \log V(a, b) \leq b \log a + 3b + 1.$$

Also, denote by  $V'(a, b, e_0, e_1)$  the number of vectors in  $\{0, 1\}^a$  that are  $(e_0, e_1)$ -close to a fixed  $b$ -sparse vector. Obviously,  $V'(a, b, e_0, e_1) \leq V(b, e_0)V(a-b, e_1)$ . Now consider any (without loss of generality, deterministic) reconstruction algorithm  $D$  and let  $X$  denote the set of all vectors in  $\{0, 1\}^n$  that it returns for some noiseless encoding; that is,

$$X := \{x \in \{0, 1\}^n \mid \exists y \in \mathcal{B}, x = D(A[y])\},$$

where  $\mathcal{B}$  is the set of  $d$ -sparse vectors in  $\{0, 1\}^n$ . Notice that all vectors in  $X$  must be  $(d + e'_0)$ -sparse, as they have to be close to the corresponding “correct” decoding. For each vector  $x \in X$  and  $y \in \mathcal{B}$ , we say that  $x$  is *matching* to  $y$  if  $(y, x)$  are  $(e'_0, e'_1)$ -close. A vector  $x \in X$  can be matching to at most  $v := V'(n, d + e'_0, e'_0, e'_1)$  vectors in  $\mathcal{B}$ , and we upper bound  $\log v$  as follows:

$$\log v \leq \log V(n - d - e'_0, e'_1) + \log V(d + e'_0, e'_0) \leq e'_1 \log(n - d - e'_0) + 3e'_1 + d + e'_0 + 1.$$

Moreover, every  $y \in \mathcal{B}$  must have at least one matching vector in  $X$ , namely,  $D(M[y])$ . This means that  $|X| \geq |\mathcal{B}|/v$ , and that

$$\log |X| \geq \log |\mathcal{B}| - \log v \geq d \log(n/d) - d - e'_0 - e'_1 \log(n - d - e'_0) - 3e'_1 - 1.$$

Finally, we observe that the number of measurements has to be at least  $|X|$  to allow  $D$  output all the vectors in  $X$ .  $\square$

According to the lemma, even in the noiseless scenario, any reconstruction method that returns an approximation of the sparse vector up to  $e'_0 = O(d)$  false positives and without false negatives will require  $\Omega(d \log(n/d))$  measurements (when  $d \ll n$ ). As we will show in the next section, an upper bound of  $O(d \log n)$  is in fact attainable even in a highly noisy setting using only non-adaptive measurements. This in particular implies an asymptotically optimal trivial two-stage group testing scheme.

#### 4. A noise-resilient construction

In this section we introduce our general construction and design measurement matrices for testing  $d$ -sparse vectors in  $\{0, 1\}^n$ . The matrices can be seen as adjacency matrices of certain unbalanced bipartite graphs constructed from good randomness condensers. The main technique that we use to show the desired properties is the *list-decoding view* of randomness condensers, extractors, and expanders, developed over the recent years starting from the work of Ta-Shma and Zuckerman on *extractor codes* [55] and followed by Guruswami et al. [30] and Vadhan [57].

##### 4.1. Construction from condensers

We start by introducing the terms and tools that we will use in our construction and its analysis.

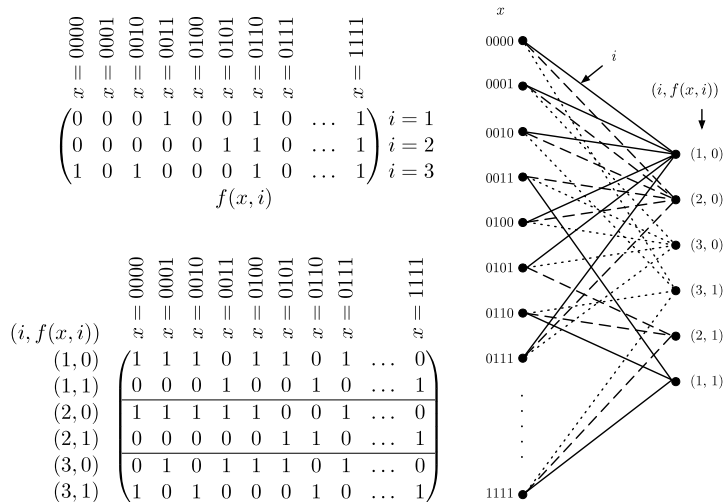
**Definition 6 (Mixtures, Agreement, and Agreement List).** Let  $\Sigma$  be a finite set. A *mixture* over  $\Sigma^n$  is an  $n$ -tuple  $S := (S_1, \dots, S_n)$  such that every  $S_i, i \in [n]$ , is a subset of  $\Sigma$ .

The *agreement* of  $w := (w_1, \dots, w_n) \in \Sigma^n$  with  $S$ , denoted by  $\text{Agr}(w, S)$ , is the quantity

$$\frac{1}{n} |\{i \in [n]: w_i \in S_i\}|.$$

Moreover, we define the quantities

$$\text{wgt}(S) := \sum_{i \in [n]} |S_i|$$



**Fig. 1.** A function  $f: \{0, 1\}^4 \times \{3\} \rightarrow \{0, 1\}$  with its truth table (top left), codeword graph of the induced code (right), and the adjacency matrix of the graph (bottom left). Solid, dashed and dotted edges in the graph respectively correspond to the choices  $i = 1, i = 2$ , and  $i = 3$  of the second argument.

and

$$\rho(S) := \text{wgt}(S) / (n|\Sigma|),$$

where the latter is the expected agreement of a random vector with  $S$ .

For a code  $\mathcal{C} \subseteq \Sigma^n$  and  $\alpha \in (0, 1]$ , the  $\alpha$ -agreement list of  $\mathcal{C}$  with respect to  $S$ , denoted by  $\text{LIST}_{\mathcal{C}}(S, \alpha)$ , is defined as the set<sup>12</sup>

$$\text{LIST}_{\mathcal{C}}(S, \alpha) := \{c \in \mathcal{C} : \text{Agr}(c, S) > \alpha\}.$$

As an example, consider a mixture  $S := (S_1, \dots, S_8)$  over  $[4]^8$  where  $S_1 := \emptyset, S_2 := \{1, 3\}, S_3 := \{1, 2\}, S_4 := \{1, 4\}, S_5 := \{1\}, S_6 := \{3\}, S_7 := \{4\}, S_8 := \{1, 2, 3, 4\}$ . For this example, we have

$$\text{Agr}((1, 3, 2, 3, 4, 3, 4, 4), S) = 5/8,$$

and  $\rho(S) = 13/32$ .

**Definition 7 (Induced Code).** Let  $f: \Gamma \times \Omega \rightarrow \Sigma$  be a function mapping a finite set  $\Gamma \times \Omega$  to a finite set  $\Sigma$ . For  $x \in \Gamma$ , we use the shorthand  $f(x)$  to denote the vector  $y := (y_i)_{i \in \Omega}, y_i := f(x, i)$ , whose coordinates are indexed by the elements of  $\Omega$  in a fixed order. The code induced by  $f$ , denoted by  $\mathcal{C}(f)$  is the set

$$\{f(x) : x \in \Gamma\}.$$

The induced code has a natural encoding function given by  $x \mapsto f(x)$ .

**Definition 8 (Codeword Graph).** Let  $\mathcal{C} \subseteq \Sigma^n, |\Sigma| = q$ , be a  $q$ -ary code. The codeword graph of  $\mathcal{C}$  is a bipartite graph with left vertex set  $\mathcal{C}$  and right vertex set  $n \times \Sigma$ , such that for every  $x = (x_1, \dots, x_n) \in \mathcal{C}$ , there is an edge between  $x$  on the left and  $(1, x_1), \dots, (n, x_n)$  on the right. The adjacency matrix of the codeword graph is an  $n|\Sigma| \times |\mathcal{C}|$  binary matrix, with columns indexed by the codewords of  $\mathcal{C}$ , whose  $(i, j)$ th entry is 1 if and only if there is an edge between the  $i$ th right vertex and the  $j$ th left vertex.

A simple example of a function with its truth table, codeword graph of the induced code along with its adjacency matrix is given in Fig. 1.

The following theorem is a straightforward generalization of a result in [55] that is also shown in [30] (we have included a proof for completeness):

**Theorem 9.** Let  $f: \{0, 1\}^{\bar{n}} \times \{0, 1\}^t \rightarrow \{0, 1\}^{\bar{\ell}}$  be a strong  $k \rightarrow_{\epsilon} k'$  condenser, and  $\mathcal{C} \subseteq \Sigma^{2^t}$  be its induced code, where  $\Sigma := \{0, 1\}^{\bar{\ell}}$ . Then for any mixture  $S$  over  $\Sigma^{2^t}$  we have

$$|\text{LIST}_{\mathcal{C}}(S, \rho(S)2^{\bar{\ell}-k'} + \epsilon)| < 2^k.$$

**Proof.** Index the coordinates of  $S$  by the elements of  $\{0, 1\}^t$  and denote the  $i$ th coordinate by  $S_i$ . Let  $Y$  be any random variable with min-entropy at least  $t + k'$  distributed on  $\{0, 1\}^{t+\bar{\ell}}$ . Define an information-theoretic test  $T: \{0, 1\}^{\bar{\ell}} \times \{0, 1\}^t \rightarrow \{0, 1\}$

<sup>12</sup> When  $\alpha = 1$ , we consider codewords with full agreement with the mixture.



as follows:  $T(x, i) = 1$  if and only if  $x \in S_i$ . Observe that

$$\Pr[T(Y) = 1] \leq \text{wgt}(S)2^{-(t+k')} = \rho(S)2^{\bar{\ell}-k'},$$

and that for every vector  $w \in (\{0, 1\}^{\bar{\ell}})^{2^t}$ ,

$$\Pr_{i \sim \mathcal{U}_t}[T(w_i, i) = 1] = \text{Agr}(w, S).$$

Now, let the random variable  $X = (X_1, \dots, X_{2^t})$  be uniformly distributed on the codewords in  $\text{LIST}_{\mathcal{C}}(S, \rho(S)2^{\bar{\ell}-k'} + \epsilon)$  and  $Z \sim \mathcal{U}_t$ . From Definition 6 we know that

$$\Pr_{X,Z}[T(X_Z, Z) = 1] > \rho(S)2^{\bar{\ell}-k'} + \epsilon.$$

As the choice of  $Y$  was arbitrary, this implies that  $T$  is able to distinguish between the distribution of  $(Z, X)$  and any distribution on  $\{0, 1\}^{t+\bar{\ell}}$  with min-entropy at least  $t + k'$ , with bias greater than  $\epsilon$ , which by the definition of condensers implies that the min-entropy of  $X$  must be less than  $k$ , or

$$|\text{LIST}_{\mathcal{C}}(S, \rho(S)2^{\bar{\ell}-k'} + \epsilon)| < 2^k. \quad \square$$

Now using the above tools, we are ready to describe and analyze our construction of error-resilient measurement matrices. We first state a general result without specifying the parameters of the condenser, and then instantiate the construction with various choices of the condenser, resulting in matrices with different properties.

**Theorem 10.** *Let  $f: \{0, 1\}^{\bar{n}} \times \{0, 1\}^t \rightarrow \{0, 1\}^{\bar{\ell}}$  be a strong  $k \rightarrow_{\epsilon} k'$  condenser, and  $\mathcal{C}$  be its induced code. Suppose that the parameters  $p, v, \gamma > 0$  are chosen so that*

$$(p + \gamma)2^{\bar{\ell}-k'} + v/\gamma < 1 - \epsilon,$$

and  $d := \gamma 2^{\bar{\ell}}$ . Then the adjacency matrix of the codeword graph of  $\mathcal{C}$  (which has  $m := 2^{t+\bar{\ell}}$  rows and  $n := 2^{\bar{n}}$  columns) is a  $(pm, (v/d)m, 2^k - d, 0)$ -resilient measurement matrix for  $d$ -sparse vectors.

Moreover, there is a reconstruction algorithm that outputs a valid decoding  $\hat{x} \in \{0, 1\}^{\bar{n}}$  of the  $d$ -sparse vector being measured, with the property that  $\text{supp}(\hat{x}) = \text{LIST}_{\mathcal{C}}(S, \alpha)$  for appropriate choices of  $S, \alpha$  determined by the algorithm.

**Proof.** Define  $L := 2^{\bar{\ell}}$  and  $T := 2^t$ . Let  $M$  be the adjacency matrix of the codeword graph of  $\mathcal{C}$ . It immediately follows from the construction that the number of rows of  $M$  (denoted by  $m$ ) is equal to  $TL$ . Moreover, notice that the Hamming weight of each column of  $M$  is exactly  $T$ .

Let  $x \in \{0, 1\}^n$  and denote by  $y \in \{0, 1\}^m$  its encoding, i.e.,  $y := M[x]$ , and by  $\hat{y} \in \{0, 1\}^m$  a received word, or a noisy version of  $y$ . That is,  $\hat{y}$  is any vector in  $\{0, 1\}^m$  that is  $(pm, (v/d)m)$ -close to  $y$ .

The encoding of  $x$  can be schematically viewed as follows: The coefficients of  $x$  are assigned to the left vertices of the codeword graph and the encoded bit on each right vertex is the bitwise “or” of the values of its neighbors.

The coordinates of  $x$  can be seen in one-to-one correspondence with the codewords of  $\mathcal{C}$ . Let  $X \subseteq \mathcal{C}$  be the set of codewords corresponding to the support of  $x$ . The coordinates of the noisy encoding  $\hat{y}$  are indexed by the elements of  $[T] \times [L]$  and thus,  $\hat{y}$  naturally defines a mixture  $S = (S_1, \dots, S_T)$  over  $[L]^T$ , where  $S_j$  contains  $j$  if and only if  $\hat{y}$  at position  $(i, j)$  is 1.

Observe that  $\rho(S)$  is the relative Hamming weight (denoted below by  $\delta(\cdot)$ ) of  $\hat{y}$ ; thus, we have

$$\rho(S) = \delta(\hat{y}) \leq \delta(y) + p \leq d/L + p = \gamma + p,$$

where the last inequality comes from the fact that the relative weight of each column of  $M$  is exactly  $1/L$  and that  $x$  is  $d$ -sparse.

Furthermore, from the assumption we know that the number of false negatives in the measurement is at most  $vTL/d = vT/\gamma$ . Therefore, any codeword in  $X$  must have agreement at least  $1 - v/\gamma$  with  $S$ . This is because  $S$  is indeed constructed from a mixture of the elements in  $X$ , modulo false positives (that do not decrease the agreement) and at most  $vT/\gamma$  false negatives each of which can reduce the agreement by at most  $1/T$ .

Accordingly, we consider a decoder which simply outputs a binary vector  $\hat{x}$  supported on the coordinates corresponding to those codewords of  $\mathcal{C}$  that have agreement larger than  $1 - v/\gamma$  with  $S$ . Clearly, the running time of the decoder is linear in the size of the measurement matrix.

By the discussion above,  $\hat{x}$  must include the support of  $x$ . Moreover, Theorem 9 applies for our choice of parameters, implying that  $\hat{x}$  must have weight less than  $2^k$ .  $\square$

#### 4.2. Instantiations

Now we instantiate the general result given by Theorem 10 with various choices of the underlying condenser and compare the obtained parameters. First, we consider two extreme cases, namely, a non-explicit optimal condenser with zero overhead (i.e., extractor) and then a non-explicit optimal condenser with zero loss (i.e., lossless condenser) and then consider how known explicit constructions can approach the obtained bounds. A summary of the results is given in Table 1.

#### 4.2.1. Optimal extractors

Radhakrishnan and Ta-Shma [47] showed that non-constructively, for every choice of the parameters  $k, \tilde{n}, \epsilon$ , there is a strong  $(k, \epsilon)$ -extractor with input length  $\tilde{n}$ , seed length  $t = \log(\tilde{n} - k) + 2 \log(1/\epsilon) + O(1)$  and output length  $\tilde{\ell} = k - 2 \log(1/\epsilon) - O(1)$ . Moreover, their proof shows that the bound is achieved by a uniformly random function, and is essentially the best one can hope for [47] (up to additive absolute constants). Plugging this result in [Theorem 10](#), we obtain a non-explicit measurement matrix from a simple, randomized construction that achieves the desired trade-off with high probability:

**Corollary 11.** *For every choice of constants  $p \in [0, 1)$  and  $\nu \in [0, \nu_0)$ ,  $\nu_0 := (\sqrt{5 - 4p} - 1)^3/8$ , and positive integers  $d$  and  $n \geq d$ , there is an  $m \times n$  measurement matrix, where  $m = O(d \log n)$ , that is  $(pm, (\nu/d)m, O(d), 0)$ -resilient for  $d$ -sparse vectors of length  $n$  and allows for a reconstruction algorithm with running time  $O(mn)$ .*

**Proof.** For simplicity we assume that  $n = 2^{\tilde{n}}$  for a positive integer  $\tilde{n}$ . However, it should be clear that this restriction will cause no loss of generality and can be eliminated with a slight change in the constants behind the asymptotic notations.

We instantiate the parameters of [Theorem 10](#) with an optimal strong extractor. If  $\nu = 0$ , we choose  $\gamma, \epsilon$  as small constants such that  $\gamma + \epsilon < 1 - p$ . Otherwise, we choose  $\gamma := \sqrt[3]{\nu}$ , which makes  $\nu/\gamma = \sqrt[3]{\nu^2}$ , and  $\epsilon < 1 - p - \sqrt[3]{\nu} - \sqrt[3]{\nu^2}$ . (One can easily see that the right hand side of the latter inequality is positive for  $\nu < \nu_0$ ). Hence, the condition  $p + \nu/\gamma < 1 - \epsilon - \gamma$  required by [Theorem 10](#) is satisfied.

Let  $r = 2 \log(1/\epsilon) + O(1) = O(1)$  be the entropy loss of the extractor for error  $\epsilon$ , and set up the extractor for min-entropy  $k = \log d + \log(1/\gamma) + r$ , which means that  $K := 2^k = O(d)$  and  $L := 2^{\tilde{\ell}} = d/\gamma = O(d)$ . Now we can apply [Theorem 10](#) and conclude that the measurement matrix is  $(pm, (\nu/d)m, O(d), 0)$ -resilient. The seed length required by the extractor is  $t \leq \log \tilde{n} + 2 \log(1/\epsilon) + O(1)$ , which gives  $T := 2^t = O(\log n)$ . Therefore, the number of measurements becomes  $m = TL = O(d \log n)$ .

From [Theorem 10](#), the running time of the reconstruction algorithm is determined by the time it takes to generate some agreement list  $\text{LIST}_e(S, \alpha)$ , where  $\mathcal{C}$  is the code whose codewords are encoded by columns of the measurement matrix. Clearly, regardless of the choice of the parameters  $S$  and  $\alpha$ , this process can be done in linear time in the size of the measurement matrix (by enumerating all the codewords), which is  $O(mn)$ .  $\square$

#### 4.2.2. Optimal lossless condensers

The probabilistic construction of Radhakrishnan and Ta-Shma can be extended to the case of lossless condensers and one can show that a uniformly random function is with high probability a strong lossless  $(k, \epsilon)$ -condenser with input length  $\tilde{n}$ , seed length  $t = \log \tilde{n} + \log(1/\epsilon) + O(1)$  and output length  $\tilde{\ell} = k + \log(1/\epsilon) + O(1)$ , and this trade-off is essentially optimal [5].

Now we instantiate [Theorem 10](#) with an optimal strong lossless condenser and obtain the following corollary.

**Corollary 12.** *For positive integers  $n \geq d$  and every constant  $\delta > 0$  there is an  $m \times n$  measurement matrix, where  $m = O(d \log n)$ , that is  $(\Omega(m), \Omega(1/d)m, \delta d, 0)$ -resilient for  $d$ -sparse vectors of length  $n$  and allows for a reconstruction algorithm with running time  $O(mn)$ .*

**Proof.** We will use the notation of [Theorem 10](#) and apply it using an optimal strong lossless condenser. This time, we set up the condenser with error  $\epsilon := \frac{1}{2}\delta/(1 + \delta)$  and min-entropy  $k$  such that  $K := 2^k = d/(1 - 2\epsilon)$ . As the error is a constant, the overhead and hence  $2^{\tilde{\ell}-k}$  will also be a constant. The seed length is  $t = \log(\tilde{n}/\epsilon) + O(1)$ , which makes  $T := 2^t = O(\log n)$ . As  $L := 2^{\tilde{\ell}} = O(d)$ , the number of measurements becomes  $m = TL = O(d \log n)$ , as desired.

Moreover, note that our choice of  $K$  implies that  $K - d = \delta d$ . Thus we only need to choose  $p$  and  $\nu$  appropriately to satisfy the condition

$$(p + \gamma)L/K + \nu/\gamma < 1 - \epsilon, \quad (2)$$

where  $\gamma = d/L = K/(L(1 + \delta))$  is a constant, as required by the lemma. Substituting for  $\gamma$  in (2) and after simple manipulations, we get the condition

$$pL/K + \nu(L/K)(1 + \delta) < \frac{\delta}{2(1 + \delta)},$$

which can be satisfied by choosing  $p$  and  $\nu$  to be appropriate positive constants.

The claim about the running time follows by the same argument as in the proof of [Corollary 11](#).  $\square$

Both results obtained in [Corollaries 11](#) and [12](#) almost match the lower bound of [Lemma 5](#) for the number of measurements. However, we note the following distinction between the two results: Instantiating the general construction of [Theorem 10](#) with an extractor gives us a sharp control over the fraction of tolerable errors, and in particular, we can obtain a measurement matrix that is robust against *any* constant fraction (bounded from 1) of false positives. However, the number of potential false positives in the reconstruction will be bounded by some constant fraction of the sparsity of the vector that cannot be made arbitrarily close to zero.

On the other hand, using a lossless condenser enables us to bring down the number of false positives in the reconstruction to an arbitrarily small fraction of  $d$  (which is, in light of Lemma 3, the best we can hope for), though it does not give as good a control on the fraction of tolerable errors as in the extractor case, though we still obtain resilience against the same order of errors.

4.2.3. Applying the Guruswami–Umans–Vadhan’s extractor

While Corollaries 11 and 12 give probabilistic constructions of noise-resilient measurement matrices, certain applications require a fully explicit matrix that is guaranteed to work. To that end, we need to instantiate Theorem 10 with an explicit condenser. First, we use a nearly-optimal explicit extractor due to Guruswami et al. [30], that currently gives the best trade-off for the range of parameters needed for our application. The parameters achieved by this extractor is quoted in the theorem below.

**Theorem 13** ([30]). *For all positive integers  $\tilde{n} \geq k$  and all  $\epsilon > 0$ , there is an explicit strong  $(k, \epsilon)$ -extractor  $\text{Ext}: \{0, 1\}^{\tilde{n}} \times \{0, 1\}^t \rightarrow \{0, 1\}^{\tilde{\ell}}$  with  $\tilde{\ell} = k - 2 \log(1/\epsilon) - O(1)$  and  $t = \log \tilde{n} + O(\log k \cdot \log(k/\epsilon))$ .  $\square$*

Using this extractor, we obtain a similar trade-off as in Corollary 11, except for a higher number of measurements which would be bounded by  $O(2^{O(\log^2 \log d)} d \log n) = O(d^{1+o(1)} \log n)$ .

**Corollary 14.** *For every choice of constants  $p \in [0, 1)$  and  $\nu \in [0, \nu_0)$ ,  $\nu_0 := (\sqrt{5 - 4p} - 1)^3/8$ , and positive integers  $d$  and  $n \geq d$ , there is a fully explicit  $m \times n$  measurement matrix, where*

$$m = O(2^{O(\log^2 \log d)} d \log n) = O(d^{1+o(1)} \log n),$$

that is  $(pm, (\nu/d)m, O(d), 0)$ -resilient for  $d$ -sparse vectors of length  $n$  and allows for a reconstruction algorithm with running time  $O(mn)$ .  $\square$

4.2.4. Applying “zig-zag” lossless condenser

An important explicit construction of lossless condensers that has an almost optimal output length is due to Capalbo et al. [5]. This construction borrows the notion of “zig-zag products” that is a combinatorial tool for construction of expander graphs as a major ingredient of the condenser. The following theorem quotes a setting of this construction that is most useful for our application.

**Theorem 15** ([5]). *For every  $k \leq \tilde{n} \in \mathbb{N}$ ,  $\epsilon > 0$  there is an explicit lossless  $(\tilde{n}, k)$ -condenser<sup>13</sup> with seed length  $t = O(\log^3(\tilde{n}/\epsilon))$  and output length  $\tilde{\ell} = k + \log(1/\epsilon) + O(1)$ .  $\square$*

Combining Theorem 10 with the above condenser, we obtain a similar result as in Corollary 12, except that the number of measurements now becomes  $d2^{\log^3(\log n)} = d \cdot \text{quasipoly}(\log n)$ .

**Corollary 16.** *For positive integers  $n \geq d$  and every constant  $\delta > 0$  there is a fully explicit  $m \times n$  measurement matrix, where*

$$m = d2^{\log^3(\log n)} = d \cdot \text{quasipoly}(\log n),$$

that is  $(\Omega(m), \Omega(1/d)m, \delta d, 0)$ -resilient for  $d$ -sparse vectors of length  $n$  and allows for a reconstruction algorithm with running time  $O(mn)$ .  $\square$

4.2.5. Measurements allowing sublinear time reconstruction

Recall the reconstruction algorithm of Theorem 10 is nothing but a list-decoding algorithm for the underlying code. That is, the running time of the algorithm is given by the time it takes to generate an agreement list  $\text{LIST}_{\mathcal{C}}(S, \alpha)$ , where  $\mathcal{C}$  is the code defined by condenser, for appropriate choices of  $S$  and  $\alpha$ . Naively, this task can be performed in linear time in the size of the measurement matrix (as discussed in the proof of Corollary 11). However, for very sparse vectors (i.e.,  $d \ll n$ ), it might be of practical importance to have a reconstruction algorithm that runs in sublinear time in  $n$ , the length of the vector, and ideally, polynomial in the number of measurements, which is merely  $\text{poly}(\log n, d)$  if the number of measurements is optimal.

As shown in [55], if the code  $\mathcal{C}$  in Theorem 9 is obtained from a strong extractor constructed from a black-box pseudorandom generator (PRG), it is possible to compute the agreement list (which is guaranteed by the theorem to be small) more efficiently than a simple exhaustive search over all possible codewords. In particular, in this case they show that  $\text{LIST}_{\mathcal{C}}(S, \rho(S) + \epsilon)$  can be computed in time  $\text{poly}(2^t, 2^{\tilde{\ell}}, 2^k, 1/\epsilon)$  (where  $t, \tilde{\ell}, k, \epsilon$  are respectively the seed length, output length, entropy requirement, and error of the extractor), which can be much smaller than  $2^{\tilde{n}}$  ( $\tilde{n}$  being the input length of the extractor).

<sup>13</sup> Though not explicitly mentioned in [5], these condensers satisfy the “strong” definition of condensers as in Definition 2.

Currently two constructions of extractors from black-box PRGs are known: Trevisan's extractor [56] (as well as its improvement in [49]) and Shaltiel–Umans' extractor [52]. However, the latter can only extract a sub-constant fraction of the min-entropy and is not suitable for our needs, albeit it requires a considerably shorter seed than Trevisan's extractor. Thus, here we only consider an improvement of Trevisan's extractor given by Raz et al., quoted below.

**Theorem 17** ([49]). *For every  $\tilde{n}, k, \tilde{\ell} \in \mathbb{N}$ , ( $\tilde{\ell} \leq k \leq \tilde{n}$ ) and  $\epsilon > 0$ , there is an explicit strong  $(k, \epsilon)$ -extractor  $\text{Tre}: \{0, 1\}^{\tilde{n}} \times \{0, 1\}^t \rightarrow \{0, 1\}^{\tilde{\ell}}$  with  $t = O(\log^2(\tilde{n}/\epsilon) \cdot \log(1/\alpha))$ , where  $\alpha := k/(\tilde{\ell} - 1) - 1$  must be less than  $1/2$ .  $\square$*

Using this extractor in Theorem 10, we obtain a measurement matrix for which the reconstruction is possible in polynomial time in the number of measurements; however, as the seed length required by this extractor is larger than that of the extractor in Theorem 13, we now require a higher number of measurements than before. Specifically, using Trevisan's extractor, we get the following result (the proof is essentially the same as Corollary 11 but using the parameters of Theorem 17 and the efficient list-decoding algorithm developed for this extractor in [55]).

**Corollary 18.** *For every choice of constants  $p \in [0, 1)$  and  $\nu \in [0, \nu_0)$ ,  $\nu_0 := (\sqrt{5 - 4p} - 1)^3/8$ , and positive integers  $d$  and  $n \geq d$ , there is a fully explicit  $m \times n$  measurement matrix  $M$  that is  $(pm, (\nu/d)m, O(d), 0)$ -resilient for  $d$ -sparse vectors of length  $n$ , where*

$$m = O(d2^{\log^3 \log n}) = d \cdot \text{quasipoly}(\log n).$$

Furthermore,  $M$  allows for a reconstruction algorithm with running time  $\text{poly}(m)$ , which would be sublinear in  $n$  for  $d = O(n^c)$  and a suitably small constant  $c > 0$ .  $\square$

On the condenser side, we observe that a family of lossless (and lossy) condensers due to Guruswami et al. also allow efficient list-decoding. The parameters of their lossless condenser can be set up as follows.

**Theorem 19** ([30]). *For all constants  $\alpha \in (0, 1)$  and every  $k \leq \tilde{n} \in \mathbb{N}$ ,  $\epsilon > 0$  there is an explicit lossless  $(k, \epsilon)$ -condenser with seed length  $t = (1 + 1/\alpha) \log(\tilde{n}k/\epsilon) + O(1)$  and output length  $\tilde{\ell} = t + (1 + \alpha)k$ . Moreover, the condenser admits efficient list decoding.  $\square$*

The code induced by the above condenser is precisely a list-decodable code due to Parvaresh and Vardy [44] (though with an unusual set-up of the parameters). Thus, the efficient list decoding algorithm of the condenser is merely the list-decoding algorithm for this code.<sup>14</sup> Combined with Theorem 10, we can show that codeword graphs of Parvaresh–Vardy codes correspond to good measurement matrices that allow sublinear time reconstruction, but with incomparable parameters to what we obtained from Trevisan's extractor (the proof is similar to Corollary 12):

**Corollary 20.** *For positive integers  $n \geq d$  and any constants  $\delta, \alpha > 0$  there is an  $m \times n$  measurement matrix, where*

$$m = O(d^{3+\alpha+2/\alpha} (\log n)^{2+2/\alpha}),$$

that is  $(\Omega(e), \Omega(e/d), \delta d, 0)$ -resilient for  $d$ -sparse vectors of length  $n$ , where

$$e := (\log n)^{1+1/\alpha} d^{2+1/\alpha}.$$

Moreover, the matrix allows for a reconstruction algorithm with running time  $\text{poly}(m)$ .  $\square$

We remark that we could also use a lossless condenser due to Ta-Shma et al. [54] which is based on Trevisan's extractor and also allows efficient list recovery, but it achieves inferior parameters compared to Corollary 20.

### 4.3. Connection with list-recoverability

Extractor codes that we used in Theorem 10 are instances of *soft-decision decodable codes*<sup>15</sup> that provide high list-decodability in “extremely noisy” scenarios. In fact it is not hard to see that good extractors or condensers are required for our construction to carry through, as Theorem 9 can be shown to hold, up to some loss in parameters, in the reverse direction as well (as already shown by Ta-Shma and Zuckerman [55, Theorem 1] for the case of extractors).

However, for designing measurement matrices for the noiseless (or low-noise) case, it is possible to resort to the slightly weaker notion of *list recoverable codes*. Formally, a code  $\mathcal{C}$  of block length  $\tilde{n}$  over an alphabet  $\Sigma$  is called  $(\alpha, d, \tilde{\ell})$ -list recoverable if for every mixture  $S$  over  $\Sigma^{\tilde{n}}$  consisting of sets of size at most  $d$  each, we have  $|\text{LIST}_e(S, \alpha)| \leq \tilde{\ell}$ . A simple argument essentially repeating the proof of Theorem 10 shows that the adjacency matrix of the codeword graph of such a code with rate  $R$  gives a  $(\log n)|\Sigma|/R \times n$  measurement matrix<sup>16</sup> for  $d$ -sparse vectors in the noiseless case with at most  $\tilde{\ell} - d$  false positives in the reconstruction.

<sup>14</sup> For similar reasons, any construction of measurement matrices based on codeword graphs of algebraic codes that are equipped efficient soft-decision decoding (including the original Reed–Solomon based construction of Kautz and Singleton [35]) allow sublinear time reconstruction.

<sup>15</sup> To be precise, here we are dealing with a special case of soft-decision decoding with binary weights.

<sup>16</sup> For codes over large alphabets, the factor  $|\Sigma|$  in the number of rows can be improved using *concatenation* with a suitable *inner* measurement matrix.

Ideally, a list-recoverable code with  $\alpha = 1$ , alphabet size  $O(d)$ , rate  $\Omega(1/\log d)$ , and list size  $\tilde{\ell} = O(d)$  would give an  $O(d \log n) \times n$  matrix for  $d$ -sparse vectors, which is almost optimal (furthermore, the recovery would be possible in sublinear time if  $\mathcal{C}$  is equipped with efficient list recovery). Existence of such codes is easy to see by a random coding argument. However, no explicit construction matching these bounds is so far known.

Two natural choices of codes with good list-recoverability properties are Reed–Solomon and Algebraic–Geometric codes, which in fact provide list recovery with short list size (cf. [24, Chapter 6]). However, list-recovery guarantees of these codes do not benefit much from the gap allowed between the list size  $\tilde{\ell}$  and sparsity parameter  $d$ . Specifically, Reed–Solomon codes can achieve  $\tilde{\ell} = O(d)$  at rate  $R = \Omega(1/d)$  with alphabet size  $|\Sigma| = O(\tilde{n}) = O((\log n)/R)$ . On the other hand, known constructions of Algebraic–Geometric codes attain  $\tilde{\ell} = O(d)$  at rate  $R = \Omega(1/d)$  and alphabet size  $|\Sigma| = O(d^2)$ . As a result, the final number of rows of the measurement matrix would be comparable to what obtained in the unique decoding case (i.e., where no false positives or negatives are allowed in the reconstruction); namely,  $O(d^2 \log^2 n)$  measurements for Reed–Solomon codes and  $O(d^3 \log n)$  measurements for Algebraic–Geometric codes (cf. [8, Section 4.3.2]).

On the other hand, it is shown in [28,29] that *folded Reed–Solomon Codes* are list-recoverable even at a constant rate, which makes them more favorable than standard Reed–Solomon codes in this regard. Unfortunately, for our applications, these codes suffer from large alphabet and list sizes.<sup>17</sup> Namely, they achieve constant rate, list size  $\tilde{\ell} = d^{O(\log \tilde{n})}$  and alphabet size  $|\Sigma| = d^{O(\log \tilde{n})}$ . Therefore, the final number of measurements attained by these codes, as well as the amount of false positives in the reconstruction, is seen to be worse compared to the standard Reed–Solomon codes.

More recently, using the idea of *subspace-evasive sets*, Guruswami [25] constructed sub-codes of folded Reed–Solomon codes that offer improved list size guarantees. This result shows that using a Monte Carlo construction of such codes (using subspace-evasive sets generated by a randomized construction), one can get an optimal list size  $\tilde{\ell} = O(d)$ , and a constant rate. However, the resulting alphabet size is  $|\Sigma| = O(d\tilde{n})^d$ , which is still too large for our applications. Besides, the best known fully explicit construction of such codes resulting from explicit subspace-evasive sets due to Dvir and Lovett [38] attains comparable bounds but an exponentially larger list size; i.e.,  $\tilde{\ell} = d^{O(d)}$ .

The alphabet size can be reduced to  $2^{O(d)}$  if folded Algebraic–Geometric codes are used instead of folded Reed–Solomon codes [31]. We see that, using Algebraic–Geometric codes combined with randomly constructed subspace-evasive sets, both the alphabet size and the list size become independent of the block length, and indeed a list size  $\tilde{\ell} = O(d)$  can be achieved. However, the alphabet size remains exponentially large in  $d$ . Again, although compared to standard Algebraic–Geometric codes, the use of folding and subspace-evasive sets improves the rate to a constant, this comes at the cost of a severe adverse effect on the alphabet size, which overall substantially increases the number of measurements.

Finally, we also point out a construction of  $(\alpha, d, d)$  list-recoverable codes (allowing list recovery in time  $O(\tilde{n}d)$ ) in [27] with polynomially small rate but alphabet size exponentially large in  $d$ , from which they obtain superimposed codes. Thus this code allows for exact recovery of sparse vectors (and in particular, results in a disjunct matrix) but, again, is not favorable in terms of the number of measurements.

**Acknowledgments**

The author is thankful to Amin Shokrollahi for introducing him to the group testing problem and for his comments on an earlier draft of this paper. He also thanks Venkatesan Guruswami for several illuminating discussions that led to considerable improvement of the results presented in this work, and Chris Umans for his feedback on earlier drafts that improved the presentation.

Part of work was done while the author was with the School of Computer and Communication Sciences, Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland. The research was supported in part by the Swiss NSF grants 200020-115983/1 and PBELP2-133367, and the ERC Advanced investigator grant 228021 of A. Shokrollahi.

**Appendix. Technical details of the proof of Lemma 4**

For a positive integer  $c > 1$ , define a  $c$ -hypergraph as a tuple  $(V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of hyperedges such that every  $e \in E$  is a subset of  $V$  of size  $c$ . The degree of a vertex  $v \in V$ , denoted by  $\deg(v)$ , is the size of the set  $\{e \in E : v \in e\}$ . Note that  $|E| \leq \binom{|V|}{c}$  and  $\deg(v) \leq \binom{|V|}{c-1}$ . The *density* of the hypergraph is given by  $|E| / \binom{|V|}{c}$ . A *vertex cover* on the hypergraph is a subset of vertices that contains at least one vertex from every hyperedge. A *matching* is a set of pairwise disjoint hyperedges. It is well known that any dense hypergraph must have a large matching. Below we reconstruct a proof of this claim.

**Proposition 21.** *Let  $H$  be a  $c$ -hypergraph such that every vertex cover of  $H$  has size at least  $k$ . Then  $H$  has a matching of size at least  $k/c$ .*

<sup>17</sup> As shown in [30], by a careful setup of the parameters, folded Reed–Solomon codes can be used to construct lossless condensers, which eliminates the list size problem. They nevertheless result in inferior parameters compared to Parvaresh–Vardy codes used in Corollary 20, since the alphabet size is adversely affected.

**Proof.** Let  $M$  be a maximal matching of  $H$ , i.e., a matching that cannot be extended by adding further hyperedges. Let  $C$  be the set of all vertices that participate in hyperedges of  $M$ . Then  $C$  has to be a vertex cover, as otherwise one could add an uncovered hyperedge to  $M$  and violate maximality of  $M$ . Hence,  $c|M| = |C| \geq k$ , and the claim follows.  $\square$

**Lemma 22.** Let  $H = (V, E)$  be a  $c$ -hypergraph with density at least  $\epsilon > 0$ . Then  $H$  has a matching of size at least  $\frac{\epsilon}{2}(|V| - c + 1)$ .

**Proof.** For every subset  $S \subseteq V$  of size  $c$ , denote by  $\mathbb{1}(S)$  the indicator value of  $S$  being in  $E$ . Let  $C$  be any vertex cover of  $H$ . Denote by  $\mathcal{S}$  the set of all subsets of  $V$  of size  $c$ . Then we have

$$\epsilon \binom{|V|}{c} \leq \sum_{S \in \mathcal{S}} \mathbb{1}(S) \leq \sum_{v \in C} \deg(v) \leq |C| \binom{|V|}{c-1}.$$

Hence,  $|C| \geq \epsilon(n - c + 1)/c$ , and the claim follows using Proposition 21.  $\square$

## References

- [1] S. Arora, B. Barak, Computational Complexity: A Modern Approach, Cambridge University Press, 2009.
- [2] T. Berger, J.W. Mandell, P. Subrahmanya, Maximally efficient two-stage group testing, Biometrics 56 (2000) 833–840.
- [3] A. Blass, Y. Gurevich, Pairwise testing, Bulletin of the EATCS 78 (2002) 100–132.
- [4] J. Buresh-Oppenheim, V. Kabanets, R. Santhanam, Uniform hardness amplification in NP via monotone codes, 2006, ECCV Technical Report TR06-154.
- [5] M. Capalbo, O. Reingold, S. Vadhan, A. Wigderson, Randomness conductors and constant-degree expansion beyond the degree/2 barrier, 2002, in: Proceedings of the 34th Annual ACM Symposium on Theory of Computing, STOC, pp. 659–668.
- [6] Y. Cheng, D.-Z. Du, New constructions of one- and two-stage pooling designs, Journal of Computational Biology 15 (2) (2008) 195–205.
- [7] M. Cheraghchi, Noise-resilient group testing: limitations and constructions, in: Proceedings of the 17th International Symposium on Fundamentals of Computation Theory, FCT, in: Lecture Notes in Computer Science, vol. 5699, 2009, pp. 62–73.
- [8] M. Cheraghchi, Applications of derandomization theory in coding, Ph.D. Thesis, Switzerland, 2010. Available online at [http://eccv.hpi-web.de/static/books/Applications\\_of\\_Derandomization\\_Theory\\_in\\_Coding/](http://eccv.hpi-web.de/static/books/Applications_of_Derandomization_Theory_in_Coding/).
- [9] B.S. Chlebus, D.R. Kowalski, Almost optimal explicit selectors, in: Proceedings of the 13th International Symposium on Fundamentals of Computation Theory, FCT, in: Lecture Notes in Computer Science, vol. 3623, 2005, pp. 270–280.
- [10] R. Clifford, K. Efremenko, E. Porat, A. Rothschild, Pattern matching with don't cares and few errors, Journal of Computer and System Sciences 76 (2) (2010) 115–124.
- [11] G. Cormode, S. Muthukrishnan, What's hot and what's not: tracking most frequent items dynamically, ACM Transactions on Database Systems 30 (1) (2005) 249–278.
- [12] G. Cormode, S. Muthukrishnan, Combinatorial algorithms for compressed sensing, Proceedings of Information Sciences and Systems (2006) 198–201.
- [13] A. De Bonis, L. Gasieniec, U. Vaccaro, Optimal two-stage algorithms for group testing problems, SIAM Journal on Computing 34 (5) (2005) 1253–1270.
- [14] R. Dorfman, The detection of defective members of large populations, Annals of Mathematical Statistics 14 (1943) 436–440.
- [15] D.-Z. Du, F.-K. Hwang, Combinatorial Group Testing and its Applications, second ed., World Scientific, 2000.
- [16] D.-Z. Du, F.-K. Hwang, Pooling Designs and Nonadaptive Group Testing, World Scientific, 2006.
- [17] A.G. D'yachkov, A.J. an Macula, V.V. Rykov, New applications and results of superimposed code theory arising from the potentialities of molecular biology, Numbers, Information and Complexity (2000) 265–282.
- [18] A.G. D'yachkov, A.J. an Macula, V.V. Rykov, New constructions of superimposed codes, IEEE Transactions on Information Theory 46 (1) (2000) 284–290.
- [19] A.G. D'yachkov, V.V. Rykov, A survey of superimposed code theory, Problems of Control and Information Theory 12 (4) (1983) 229–242.
- [20] A.G. D'yachkov, V.V. Rykov, Bounds of the length of disjunct codes, Problems of Control and Information Theory 11 (1982) 7–13.
- [21] A.G. D'yachkov, V.V. Rykov, Superimposed codes for multiple accessing of the OR-channel, 1998, in: Proceedings of IEEE International Symposium on Information Theory, ISIT, p. 404.
- [22] D. Eppstein, M.T. Goodrich, D.S. Hirschberg, Improved combinatorial group testing algorithms for real-world problem sizes, SIAM Journal on Computing 36 (5) (2007) 1360–1375.
- [23] Z. Füredi, On  $r$ -cover-free families, Journal of Combinatorial Theory, Series A 73 (1996) 172–173.
- [24] V. Guruswami, List decoding of error-correcting codes, Ph.D. Thesis, 2001.
- [25] V. Guruswami, Linear-algebraic list decoding of folded Reed–Solomon codes, 2011, in: Proceedings of the 26th IEEE Conference on Computational Complexity, CCC.
- [26] V. Guruswami, P. Gopalan, Hardness amplification within NP against deterministic algorithms, Journal of Computer and System Sciences 77 (1) (2011) 107–121.
- [27] V. Guruswami, P. Indyk, Linear-time list decoding in error-free settings, in: Proceedings of the 31st International Colloquium on Automata, Languages and Programming, ICALP, in: Lecture Notes in Computer Science, vol. 3142, 2004, pp. 1–5.
- [28] V. Guruswami, A. Rudra, Concatenated codes can achieve list decoding capacity, 2008, in: Proceedings of the 19th Symposium on Discrete Algorithms, SODA.
- [29] V. Guruswami, A. Rudra, Explicit codes achieving list decoding capacity: error-correction with optimal redundancy, IEEE Transactions on Information Theory 54 (1) (2008) 135–150.
- [30] V. Guruswami, C. Umans, S. Vadhan, Unbalanced expanders and randomness extractors from Parvaresh–Vardy codes, Journal of the ACM 56 (4) (2009).
- [31] V. Guruswami, C. Xing, Folded codes from function field towers and improved optimal rate list decoding, 2012, in: Proceedings of the 44th Annual ACM Symposium on Theory of Computing, STOC.
- [32] E.-S. Hong, R.E. Ladner, Group testing for image compression, 2000, Data Compression Conference, pp. 3–12.
- [33] P. Indyk, Explicit constructions of selectors with applications, 2002, in: Proceedings of the 13th Symposium on Discrete Algorithms, SODA.
- [34] P. Indyk, H.-Q. Ngo, A. Rudra, Efficiently decodable non-adaptive group testing, 2010, in: Proceedings of the 21st Annual ACM–SIAM Symposium on Discrete Algorithms, SODA.
- [35] W.H. Kautz, R.C. Singleton, Nonrandom binary superimposed codes, IEEE Transactions on Information Theory 10 (1964) 363–377.
- [36] E. Knill, Lower bounds for identifying subset members with subset queries, 1995, in: Proceedings of the 6th Symposium on Discrete Algorithms, SODA, pp. 369–377.
- [37] E. Knill, W.J. Bruno, D.C. Torney, Non-adaptive group testing in the presence of errors, Discrete Applied Mathematics 88 (1–3) (1998) 261–290.
- [38] S. Lovett, Z. Dvir, Subspace evasive sets, 2012, in: Proceedings of the 44th Annual ACM Symposium on Theory of Computing, STOC.
- [39] A.J. Macula, Probabilistic nonadaptive group testing in the presence of errors and DNA library screening, Annals of Combinatorics 3 (1) (1999) 61–69.
- [40] A.J. Macula, Probabilistic nonadaptive and two-stage group testing with relatively small pools and DNA library screening, Journal of Combinatorial Optimization 2 (1999) 385–397.
- [41] A.J. Macula, Error correcting nonadaptive group testing with  $d^c$ -disjunct matrices, Discrete Applied Mathematics 80 (2, 3) (1997) 217–222.

- [42] H.-Q. Ngo, D.-Z. Du, A survey on combinatorial group testing algorithms with applications to DNA library screening, DIMACS Series on Discrete Mathematics and Theoretical Computer Science 55 (2000) 171–182.
- [43] H. Ngo, E. Porat, A. Rudra, Efficiently decodable error-correcting list disjoint matrices and applications, in: Proceedings of the 38th International Colloquium on Automata, Languages and Programming, ICALP, in: Lecture Notes in Computer Science, vol. 6755, 2011, pp. 557–568.
- [44] F. Parvaresh, A. Vardy, Correcting errors beyond the Guruswami–Sudan radius in polynomial time, 2005, in: Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, FOCS, pp. 285–294.
- [45] P. Pevzner, R.J. Lipshutz, Towards DNA sequencing chips, in: Proceedings of MFCS, in: Lecture Notes in Computer Science, vol. 841, 1994, pp. 143–158.
- [46] E. Porat, A. Rothschild, Explicit non-adaptive combinatorial group testing schemes, IEEE Transactions on Information Theory 57 (12) (2011) 7982–7989.
- [47] J. Radhakrishnan, A. Ta-Shma, Tight bounds for depth-two superconcentrators, 1997, in: Proceedings of the 38th Annual IEEE Symposium on Foundations of Computer Science, FOCS, pp. 585–594.
- [48] A.M. Rashad, Random coding bounds on the rate for list-decoding superimposed codes, Problems of Control and Information Theory 19 (2) (1990) 141–149.
- [49] R. Raz, O. Reingold, S. Vadhan, Extracting all the randomness and reducing the error in Trevisan’s extractor, Journal of Computer and System Sciences 65 (1) (2002) 97–128.
- [50] M. Ruzsinkó, On the upper bound of the size of the  $r$ -cover-free families, Journal of Combinatorial Theory, Series A 66 (1994) 302–310.
- [51] A. Schliep, D. Torney, S. Rahmann, Group testing with DNA chips: generating designs and decoding experiments, Proceedings of Computational Systems Bioinformatics (2003).
- [52] R. Shaltiel, C. Umans, Simple extractors for all min-entropies and a new pseudorandom generator, Journal of the ACM 52 (2) (2005) 172–216.
- [53] M. Sobel, P.A. Groll, Group-testing to eliminate efficiently all defectives in a binomial sample, Bell Systems Technical Journal 38 (1959) 1179–1252.
- [54] A. Ta-Shma, C. Umans, D. Zuckerman, Lossless condensers, unbalanced expanders, and extractors, 2001, in: Proceedings of the 33rd Annual ACM Symposium on Theory of Computing, STOC, pp. 143–152.
- [55] A. Ta-Shma, D. Zuckerman, Extractor codes, IEEE Transactions on Information Theory 50 (12) (2004) 3015–3025.
- [56] L. Trevisan, Extractors and pseudorandom generators, Journal of the ACM 48 (4) (2001) 860–879.
- [57] S. Vadhan, The unified theory of pseudorandomness, 2010, in: Proceedings of the International Congress of Mathematicians.
- [58] J. Wolf, Born-again group testing: multiaccess communications, IEEE Transactions on Information Theory 31 (1985) 185–191.