Magnetostrictive/electrostrictive fracture of the piezomagnetic and piezoelectric layers in a multiferroic composite: Anti–plane case

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ABSTRACT

The main purpose of the present work is to study the influences of magnetostriction, electrostriction and piezomagnetic/piezoelectric stiffening on the fracture behavior of a layered multiferroic composite. For comparison, it is assumed that there is a crack, parallel to the interface, in each layer. Methods of cosine transform and Cauchy singular integral equations are used to solve the crack problem. Numerical results of the stress intensity factor (SIF) are provided and the computational accuracy is demonstrated. Discussion on the numerical results indicates that the multiferroic composite consisting of cobalt ferrite and barium titanate layers are more prone to fracture under electric loading than under magnetic loading. In the case of magnetostriction, to increase the shear modulus of the piezomagnetic layer would raise the SIF; but to increase that of the piezoelectric layer would reduce the SIF; in the case of electrostriction, inverse results are obtained. Piezomagnetic stiffening can affect the SIF when the composite is under electrostriction; piezoelectric stiffening can influence the SIF if the composite is under magnetostriction. In addition, it is also revealed that two parallel equal cracks may shield each other even if an interface exists between them.

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1. Introduction

Smart composites consisting of piezomagnetic and piezoelectric phases may possess magnetoelectric (ME) coupling effect at room temperature, which does not necessarily exist in single-phase piezomagnetic or piezoelectric materials (Huang and Kuo, 1997). The ME coupling phenomenon has brought a broad application prospect for these composites in smart devices such as sensors, detectors, actuators, transducers and so on. In the past, most researchers have been concerned about the sintered bulk composites made from piezomagnetic and piezoelectric ingredients, which have been called “magnetoelectroelastic composites” (Wang and Mai, 2007; Li and Lee, 2008) or “piezomagnetic/piezoelectric composites” (Zhou et al., 2008). Due to their intrinsic brittleness, the fracture problems of magnetoelectroelastic composites have received much attention since the beginning of their engineering applications.

Magnetoelectroelastic composites with strong ME coupling effect can be used to manufacture highly sensitive smart devices, which are needed in industrial situations involving precise measurement and control. However, recent investigations indicated that the ME coupling of sintered bulk magnetoelectroelastic composites is often quite low at room temperature, restricting their applications in many cases. Actually, layered composites composed of alternative piezomagnetic and piezoelectric layers or fibre composites consisting of piezomagnetic/piezoelectric fibre and substrate may exhibit at normal temperature excellent ME coupling much stronger than that of sintered bulk composites (Bichurin et al., 2001, 2003; Zhang et al., 2009). These layered or fibred composites are often called “multiferroic composites” (Li and Lee, 2010a).

Smart devices made of multiferroic composites can serve under not only electrical but also magnetical loadings. When loaded magnetically or electrically, multiferroic composites may have peculiar mechanical behaviors much different from those of sintered bulk piezomagnetic/piezoelectric composites (Li and Lee, 2010b). Under electrical loading, the piezoelectric layers will deform initiatively, but the piezomagnetic layers will deform passively, indicating that the latter actually serves as constraints to the former. Conversely, when multiferroic composites are loaded magnetically, spontaneous deformation would occur in the piezomagnetic layers, and the piezoelectric layers would play the role of constraint instead. Therefore, to investigate the mechanical behaviors induced by magnetic/electric loadings has much significance in the applications of multiferroic composites. Bichurin et al. (2001, 2003) developed a phenomenological theory to calculate the ME coupling
coefficients in layered multiferroic composites. Nan et al. (2005) studied the ME effect in piezoelectric/magnetostrictive composites by a physics-based Green’s function method. Wang et al. (2008) derived the solutions for two-dimensional Green’s functions in anisotropic multiferroic biomaterials with a viscous interface by the method of unified Stroh formalism. The numerical simulation techniques (Liu et al., 2004; Blackburn et al., 2008) have also been used to investigate the multiferroic ME behavior.

Now, fracture analysis on multiferroic composites is still a new field needing investigation, and only very few work has been carried out up to now. Liu et al. (2008) first studied the fracture toughness of a fiber reinforced multiferroic composite by a micro-mechanical energy approach. Li and Lee (2010a and 2010b) investigated the influences of piezomagnetic/piezoelectric stiffening and magnetic/electric loading on the stress intensity factor of an interfacial crack in multiferroic composites. Besides the interface, each layer of the multiferroic composite may also fracture under the action of magnetic/electric loading. In the present work, we continue to study the magnetostrictive and electrostrictive fracture problems of a bi-layered multiferroic composite, in each layer of which, there is a crack parallel to the interface. The crack problem is solved by the standard method of Fourier cosine transform and Cauchy singular integral equation. By perturbing material properties, the fracture behavior induced by magnetostriction is compared with that produced by electrostriction. The influences of piezomagnetic/piezoelectric stiffening on the stress intensity factor are also displayed in different loading cases. Last but not the least, it deserves noting that the difference between the literature (Li and Lee, 2010a) and the present work is that the interfacial fracture problems are studied by Li and Lee (2010a and 2010b), but the fracture problems of the piezomagnetic and the piezoelectric layers are investigated by the present work instead.

2. Problem formulation

Shown in Fig. 1 is a multiferroic composite consisting of a piezomagnetic layer with thickness $h_1$ and another piezoelectric layer with thickness $h_2$, in each of which there is a crack parallel to the interface. The distances between the cracks and the interface are $d_1$ and $d_2$, and the half-lengths of the cracks are $a_1$ and $a_2$. Herein and hereafter, subscripts/superscripts $1$ and $2$ stand for the quantities of the piezomagnetic and piezoelectric layers, respectively. A rectangular coordinate system is established with the rightward $x$-axis along the bottom and the upward $y$-axis through the crack centers. Both layers have the $6$ mm material symmetry and are potted along the z-direction.

In the current work, it is assumed that the multiferroic composite is loaded by magnetic/electric field in the $xoy$ plane and meanwhile constrained on the upper and lower surfaces. So, only the anti-plane deformation would be coupled with the in-plane magnetic/electric field (Li and Lee, 2009a, 2010b). Under anti-plane deformation, the basic equations for the piezomagnetic and piezoelectric layers are

$$\tau_{ik}^{(1)} = C_{ik}^{(1)}e_{ik}^{(1)} - \delta_{ij}h_{15}H_k - \delta_{ik}e_{15}E_k, \quad (j = 1, 2; \ k = x,y),$$  

$$B_k = h_{15}^{(1)}e_{15}^{(1)} + \mu_1 H_k, \quad (k = x,y),$$  

$$D_k = e_{15}^{(1)}e_{15}^{(1)} + e_{11} E_k, \quad (k = x,y),$$  

$$\tau_{ik}^{(2)} = C_{ik}^{(2)}e_{ik}^{(2)} - \delta_{ij}h_{15}H_k - \delta_{ik}e_{15}E_k, \quad (j = 1, 2; \ k = x,y),$$  

$$H_k = -\phi_k, \quad (k = x,y),$$  

$$E_k = -\phi_k, \quad (k = x,y),$$  

$$\tau_{xk} + \tau_{yk} = 0, \quad (j = 1, 2),$$  

$$B_{xx} + B_{yy} = 0,$$  

$$D_{xx} + D_{yy} = 0,$$

where $w$, $\phi$ and $\psi$ are the mechanical displacement, magnetic potential and electric potential; $\tau$, $B$ and $D$ the stress, magnetic induction and electric displacement; $\gamma$, $H$ and $E$ the strain, magnetic field and electric field; $c_{44}, h_{15}, \mu_1, e_{15}$ and $e_{11}$ the elastic constant, piezomagnetic coefficient, magnetic permeability, piezoelectric coefficient and dielectric coefficient. $\delta_{ij}$ is the Kronecker delta, which is 1 when its two subscripts are identical and 0 otherwise. The comma in the subscript denotes the derivation with respect to its subsequent coordinate.

By using Eqs. (2) and (1), one can transform Eq. (3) into (Li and Lee, 2010b)

$$\nabla^2 w = 0, \quad (j = 1, 2); \quad \nabla^2 \phi = 0; \quad \nabla^2 \psi = 0,$$

where $\nabla^2$ is the two-dimensional Laplacian operator.

According to the principle of superposition, the present crack problem can be regarded as the superposition of two sub-problems: (I) the composite is un-cracked and subjected to the applied loadings; (II) the structure is cracked and only loaded on the crack surfaces by equivalent traction.

Assume that the composite is loaded by in-plane magnetic field $H_0$ or electric field $E_0$, normal to the interface and surfaces, and the induced anti-plane constraining traction on the surfaces is $\tau_0$. Then, the boundary and continuity conditions for the first subproblem have the same form as Eqs. (14)–(20) of Li and Lee (2010b). Because the loading and material properties do not vary in the x-direction, the governing equations are greatly simplified for the first sub-problem. Through simple magnetoelastic analysis (Li and Lee, 2010b), the equivalent traction on the crack surfaces are obtained as

$$\tau_e = q_m h_{15} H_0 \quad \text{(magnetic loading case)},$$  

$$\tau_e = q_e e_{15} E_0 \quad \text{(electric loading case)},$$

where $q_m$ and $q_e$ are two dimensionless constants given in Appendix, which are expressed by the dimensionless piezoelectric and piezomagnetic stiffening factors, $k_s$ and $k_m$ (Li and Lee, 2010b).

For the second sub-problem, the boundary conditions on the surfaces and interface are

$$\tau_{ik}^{(1)}(x, h_1 + h_2) = 0, \quad B_k(x, h_1 + h_2) = 0,$$  

$$\tau_{ik}^{(2)}(x, 0) = 0, \quad D_k(x, 0) = 0,$$  

$$B_k(x, h_2) = 0, \quad D_k(x, h_2) = 0,$$  

$$\tau_{ik}^{(2)}(x, h_2) = \tau_{ik}^{(1)}(x, h_2), \quad w_2(x, h_2) = w_1(x, h_2).$$

It is reasonable to regard the anti-plane cracks as magnetically/electrically permeable. Then, the mixed boundary value conditions for the two cracks of the second sub-problem take the form

![Fig. 1. Two parallel cracks in a bi-layered multiferroic composite.](image-url)
where the integral kernel functions, $P_j (j = 1, 2, 3, 4)$, are listed in Appendix. Considering the asymptotic values in Eq. (A.20), one can recast $P_1 (s_1 x_1)$ and $P_2 (s_2 x_2)$ into (Wang et al., 2009)

\[
\begin{align*}
\frac{1}{\pi} \int_{-1}^1 \bar{g}_1 (s_1) \left[ \frac{1}{|s_1 - x_1|} + \bar{R}_{11} (\bar{s}_1, x_1) \right] d\bar{s}_1 + \frac{1}{\pi} \int_{-1}^1 \bar{g}_2 (s_2) \bar{R}_{12} (\bar{s}_2, x_1) d\bar{s}_2 &= \frac{2a}{c_4} \\
\frac{1}{\pi} \int_{-1}^1 \bar{g}_1 (s_1) \bar{R}_{21} (\bar{s}_1, x_2) d\bar{s}_1 + \frac{1}{\pi} \int_{-1}^1 \bar{g}_2 (s_2) \left[ \frac{1}{|s_2 - x_2|} + \bar{R}_{22} (s_2, x_2) \right] d\bar{s}_2 &= \frac{2a}{c_4} 
\end{align*}
\]

(21)

where $\bar{s}_j = x_j / a_j$, $\bar{s}_j = s_j / a_j (j = 1, 2$), $\bar{g}_1 (s_1) = g_1 (a_1 \bar{s}_1)$, $\bar{g}_2 (s_2) = g_2 (a_2 \bar{s}_2)$, $\bar{R}_i (i = 1, 2, j = 1, 2)$ are listed in Appendix.

The solution of Eq. (21) takes the form

\[
\bar{g}_j (s_j) = 2 \tau_s \frac{f_i (\bar{s}_j)}{\sqrt{1 - s_j^2}}, \quad (j = 1, 2),
\]

(22)

where $f_i (\bar{s}_j) (j = 1, 2)$ are unknown functions being continuous and bounded in the interval $-1 < \bar{s}_j < 1 (j = 1, 2)$ and nonzero at the end points $\bar{s}_j = \pm 1 (j = 1, 2)$. Here, $2 \tau_s$ is introduced into the solution just for convenience.

Discretize $\bar{s}_i (r = 1, 2)$ and $s_i (r = 1, 2)$ as the roots of Chebyshev polynomials of the first and second kinds (see Appendix), respectively. Then, one can transform Eqs. (21) and (17b) into algebraic equations (Theocaris and Ioakimas, 1977)

\[
\begin{align*}
\sum_{j=0}^{m} \beta_j \left\{ \frac{1}{|s_j - x_1|} + \bar{R}_{11} (\bar{s}_j, x_1) \right\} f_1 (\bar{s}_j) + \bar{R}_{12} (\bar{s}_2, x_1) f_2 (\bar{s}_2) &= \frac{\pi}{c_4} \\
\sum_{j=0}^{m} \alpha_j \left[ \bar{R}_{21} (\bar{s}_1, \bar{s}_2) f_1 (\bar{s}_1) + \frac{1}{|s_2 - x_2|} + \bar{R}_{22} (s_2, x_2) \right] f_2 (s_2) &= \frac{\pi}{c_4}
\end{align*}
\]

(23)

where $k = 1, 2, \ldots, m$, $m$ is the node number of quadrature (see Eq. (A.26)), $\beta_0 = \alpha_m = 1 / 2$ and $\beta_1 = \ldots = \alpha_{m-1} = 1$.

Eq. (22) indicates that the auxiliary functions have the conventional square-root singularity at the crack tips. For permeable cracks, magnetic and electric fields have no singularity at the crack tips. Thus, it is sufficient to have the stress intensity factor (SIF) as the fracture parameter. Due to the symmetry of the present problem, only the SIFs of the right crack tips need considering here, which are defined by

\[
\begin{align*}
K^{(1)}_{11} (a_1) &= \lim_{x \rightarrow a_1} \frac{1}{2 \pi} \int_{-1}^1 \bar{g}_1 (s_1) \left[ \frac{1}{|x - s_1|} + \bar{R}_{11} (\bar{s}_1, x_1) \right] d\bar{s}_1, \\
K^{(2)}_{11} (a_2) &= \lim_{x \rightarrow a_2} \frac{1}{2 \pi} \int_{-1}^1 \bar{g}_1 (s_1) \left[ \frac{1}{|x - s_1|} + \bar{R}_{11} (\bar{s}_1, x_1) \right] d\bar{s}_1
\end{align*}
\]

(25)
The singular parts of the stresses are
\[
\lim_{x \to x_i^+} \tau_{ij}^{(1)}(x, h_2 + d_1) = \lim_{y_1 \to y_i^+} \lim_{y_2 \to y_i^+} \int_1 \frac{g(y_1^+)}{\sqrt{y_1^+}} \, dy_1 \]  
\lim_{x \to x_i^-} \tau_{ij}^{(2)}(x, h_2 - d_2) = \lim_{y_1 \to y_i^-} \lim_{y_2 \to y_i^-} \int_1 \frac{g(y_2^-)}{\sqrt{y_2^-}} \, dy_2
\]  
\tag{26}
\]
Substituting Eq. (22) into Eq. (26), and the resulting equations into Eq. (25), one obtains (Li and Lee, 2009c)
\[
K_{ii}^{(j)}(ai) = -q_i c_{ij}^{(1)}(1) \varepsilon_{ij} E_0 \sqrt{\pi a_i}, \quad (j = 1, 2)
\]  
\tag{27}
\]
for the magnetic loading case, and
\[
K_{ii}^{(j)}(ai) = -q_i c_{ij}^{(2)}(1) \varepsilon_{ij} E_0 \sqrt{\pi a_i}, \quad (j = 1, 2)
\]  
\tag{28}
\]
for the electric loading case.

From the constitutive Eq. (1a), it is readily to see that
\[-h_{ij} H_0 \sqrt{\pi a_i} \text{ and } -\varepsilon_{ij} E_0 \sqrt{\pi a_i} \]  
have the dimension of SIF. Therefore, one can define the normalized SIFs as below
\[
\tilde{K}_{ii}^{(j)}(ai) = q_i c_{ij}^{(1)}(1), \quad (j = 1, 2)
\]  
\tag{29}
\]
for the magnetic loading case, and
\[
\tilde{K}_{ii}^{(j)}(ai) = q_i c_{ij}^{(2)}(1), \quad (j = 1, 2)
\]  
\tag{30}
\]
for the electric loading case.

4. Numerical results and discussion

In multiferroic composites, CoFe$_2$O$_4$ is often used as the piezomagnetic phase and BaTiO$_3$ the piezoelectric phase. Their material constants are (Wang and Pan, 2007)
\[
\text{CoFe}_2\text{O}_4 : \quad c_{ij}^{(1)} = 4.53 \times 10^{10} \text{ N/m}^2; \quad h_{15} = 550 \text{ N/(Am);} \quad \mu_{11} = 5.9 \times 10^{-4} \text{ Ns}^2/\text{C}^2.
\]  
\tag{31}
\]
\[
\text{BaTiO}_3 : \quad c_{ij}^{(2)} = 4.3 \times 10^{10} \text{ N/m}^2; \quad \varepsilon_{11} = 116 \text{ C/m}^2; \quad e_{11} = 1.12 \times 10^{-8} \text{ C}^2/\text{N m}^2.
\]  
\tag{32}
\]
In this section, numerical computation has been performed with respect to the cobalt ferrite and barium titanate composite characterized by Eqs. (31) and (32).

4.1. Computational accuracy

The computational accuracy of the SIFs mainly depends on two factors. The first is the calculation of the dimensionless kernel functions $R_{ij}(i = 1, 2; j = 1, 2)$, which is the main difficulty in the whole solving process because it involves infinite integrals on the semiinfinite interval $[0, +\infty)$. During computation, the semiinfinite interval is actually truncated to be $[0, N]$, where the upper limit $N$ is determined according to the convergence behavior of the dimensionless nonsingular kernel functions. Computations indicate that the convergence speed of $R_{ij}(i = 1, 2; j = 1, 2)$ is principally affected by the distance between the crack and interface. Generally, if the distance is large, the computation converges very quickly. For example, when $d_1 = d_2 = 10 \text{ mm}$, Fig. 2(a) shows that $N = 500$ is enough to make $\tilde{R}_{ij}$ converge well (at a precision of $1.0 \times 10^{-6}$). However, if the distance is small, the computation would converge quite slowly. When $d_1 = d_2 = 1 \text{ mm}$, Fig. 2(b) shows that $N = 3000$ is needed to ensure the convergence of $\tilde{R}_{ij}$ (at the precision of $1.0 \times 10^{-6}$). In the computation, the convergence accuracy of $\tilde{R}_{ij}$ is generally fixed at $1.0 \times 10^{-6}$, and through trial calculation the value of $N$ is changed adaptively with the distance between the cracks and interface. The second factor affecting the precision of the SIFs is the choice of the node number $m$. Numerical computation reveals that $m = 20$ is generally well enough to ensure an accuracy of $1.0 \times 10^{-6}$ for the SIFs, which is sufficient for most engineering applications.

4.2. Parametric studies

In Figs. 3–5, the material constants in Eqs. (31) and (32) are perturbed by $\pm 5\%$, respectively, to demonstrate the influences of material stiffness and piezomagnetic/piezoelectric stiffening on the SIFs under magnetostriction (i.e., deformation induced by magnetic loading) or electrostriction (i.e., deformation produced by electric loading). It needs noting that for each line in these figures only one material constant is perturbed. For example, only $c_{44}^{(2)}$ is perturbed for the lines labeled by "$c_{44}^{(2)}$" in Fig. 3(a).

4.2.1. Effect of stiffness perturbation

Illustrated in Fig. 3 are the variations of SIFs versus the perturbations of shear moduli. Fig. 3(a) indicates that in the case of magnetostriction the SIFs increase with the increasing $c_{44}^{(2)}$ but decrease with the increasing $c_{44}^{(1)}$. Here, the former variation is easy to understand because higher stiffness often leads to larger SIFs according to the common sense of fracture mechanics; comparatively, the latter variation seems a little "unexpected". However, it could be explained by analyzing the different roles of the two layers in the deformation. When the composite is loaded by magnetic field, the piezomagnetic layer deforms spontaneously, but the piezoelectric layer deforms...
that to increase passively. That is to say, the latter actually serves as a constraint to the former. Therefore, to increase $(2)$ will certainly strengthen such a constraint effect, which will consequently lower down the SIFs.

Fig. 3(b) shows the variations of SIFs versus the perturbations of shear moduli in the case of electrostriction. Contrarily, the piezomagnetic layer would instead act as a constraint to the piezoelectric layer this time. As expected, the SIFs increase with the increasing $(44)$, but decrease with the increasing $(44)$.

4.2.2. Effect of piezomagnetic stiffening

Shown in Fig. 4 are the variations of SIFs versus the perturbations of piezomagnetic coefficient and magnetic permeability. It is found that when the composite is under magnetostriction the perturbations of $c_{44}$ would lower down the SIFs but the variation of $c_{11}$ would enhance them. Again, if the composite is under electrostriction, the perturbations of both piezoelectric and dielectric coefficients would not change the SIFs.

Combining Fig. 5 together with the definition of piezoelectric stiffening factor $k_e$ (Li and Lee, 2010b), one can also readily infer that piezoelectric stiffening can affect the SIFs when the composite is under magnetostriction rather than electrostriction.

4.2.4. Effect of magnetic/electric loadings

Comparing the cases of magnetic loading with those of electric loading as illustrated in Figs. 3–5, one can readily find that with other conditions unchanged the SIFs induced by magnetostriction is always much smaller than that by electrostriction. Therefore, layered multiferroic devices composed of the described cobalt ferrite and barium titanate layers are more prone to fracture under electric loading than under magnetic loading.

4.2.5. Effect of crack distance

The effect of crack distance on the SIFs is also displayed in Figs. 3–5. It is observed that the SIFs are always larger when $d_1 = d_2 = 10$ mm than when $d_1 = d_2 = 0.5$ mm. Such an effect of SIF
reduction is generally called the shielding effect between parallel cracks (Ratwani and Gupta, 1974; Zhou and Wang, 2004), which means that the SIFs of two parallel cracks of equal length would reduce as the crack distance decreases. The present work also reveals that the shielding effect still exist even if there is an interface between the parallel cracks. This is consistent with the findings of Ratwani and Gupta (1974), implying that the interaction between parallel cracks is stronger than that between a crack and a parallel interface.

5. Conclusions

Investigated in the present work is the parallel-crack problem in a bi-layered multiferroic composite. The composite is under the action of magnetic/electric field and mechanical constraining traction. Effects of the magnetic/electric loading and piezomagnetic/piezoelectric stiffening on the SIFs are specially studied. Some interesting conclusions are drawn with respect to the described cobalt ferrite and barium titanate composite. This multiferroic composite is more prone to fracture under electric loading than under magnetic loading. Under magnetostriction, the piezoelectric layer acts as a constraint to the piezomagnetic layer; but the situation is reversed under electrostriction. Piezomagnetic/piezoelectric stiffening may affect the SIFs when the piezomagnetic/piezoelectric layer deforms passively. In addition, it is also found that the interaction between parallel cracks is stronger than that between a crack and a parallel interface.
\[
Q_{10} = \frac{c_{44}^{(1)}}{2} \left[ \mu_{13}^{(1)} (e^{(1)}_{\nu} - e^{(2)}_{\nu}) + \epsilon_{0}^{(1)} (e^{(2)}_{\nu} - e^{(\nu)}) \right] + h_{15}^{(1)} \left[ e^{(1)}_{\nu} - e^{(2)}_{\nu} \epsilon_{0}^{(1)} \right],
\]
(A.15)

\[
Q_{11} = \frac{c_{44}^{(2)}}{1 - e^{(2)}_{\nu}} \left[ e^{(1)}_{\nu} - e^{(2)}_{\nu} \epsilon_{0}^{(1)} \right],
\]
(A.16)

\[
Q_{12} = c_{66}^{(2)} (1 - e^{(2)}_{\nu}) \left[ e^{(1)}_{\nu} - e^{(2)}_{\nu} \epsilon_{0}^{(1)} \right] + \frac{1}{d^{(2)}_{\nu}} \left[ e^{(1)}_{\nu} - e^{(2)}_{\nu} \epsilon_{0}^{(1)} \right],
\]
(A.17)

\[
P_{1}(s_1, x_1) = \int_{0}^{s_1} \left( Q_{10} + Q_{10} Q_{3} \right) \sin(\xi_1) \cos(\xi_1) d\xi_1,
\]
(A.18)

\[
P_{2}(s_2, x_2) = \int_{0}^{s_2} \left( Q_{20} + Q_{20} Q_{4} \right) \sin(\xi_2) \cos(\xi_2) d\xi_2,
\]
(A.19)

\[
\lim_{s_1 \to 0} \left( Q_{10} + Q_{10} Q_{3} \right) = -\frac{1}{2} c_{44}^{(1)}; \quad \lim_{s_2 \to 0} \left( Q_{20} + Q_{20} Q_{4} \right) = 0
\]
(A.20)

\[
T_{1}(s_1, x_1) = \int_{0}^{s_1} \left( Q_{60} + Q_{60} Q_{3} - \frac{1}{2} c_{44}^{(1)} \right) \sin(\xi_1) \cos(\xi_1) d\xi_1,
\]
(A.21)

\[
R_{11}(s_1, x_1) = \frac{2 T_{1}(s_1, x_1)}{c_{44}^{(1)}}, \quad R_{12}(s_2, x_1) = -\frac{2 P_{2}(s_2, x_1)}{c_{44}^{(2)}},
\]
(A.22)

\[
R_{21}(s_1, x_2) = \frac{2 P_{2}(s_1, x_2)}{c_{44}^{(2)}}, \quad R_{22}(s_2, x_2) = -\frac{2 T_{2}(s_2, x_2)}{c_{44}^{(2)}},
\]
(A.23)

\[
\tilde{R}_{11}(s_1, x_1) = a_{1} R_{11}(a_{1} s_1, a_{1} x_1); \quad \tilde{R}_{12}(s_2, x_1) = a_{2} R_{12}(a_{2} s_2, a_{2} x_1),
\]
(A.24)

\[
\tilde{R}_{21}(s_2, x_2) = a_{1} R_{21}(a_{2} s_2, a_{2} x_2); \quad \tilde{R}_{22}(s_1, x_2) = a_{2} R_{22}(a_{1} s_2, a_{1} x_2),
\]
(A.25)

\[
\tilde{s}_{j} = \cos(j \pi / m), \quad (r = 1, 2; j = 0, 1, \ldots, m)
\]
\[
\tilde{s}_{k} = \cos((2k - 1) \pi / (2m)), \quad (r = 1, 2; k = 1, 2, \ldots, m)
\]
(A.26)