Generalized Rees Rings and
Arithmetical Graded Rings

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0. Introduction

In this paper we study graded rings with an arithmetical ideal theory for the graded ideals, e.g., Gr-Dedekind and Gr-principal ideal rings. If these rings are positively graded rings, then the structure of Gr-Dedekind and of Gr-principal ideal rings is easily investigated and it is much like the structure of the ungraded equivalents. For arbitrary \( \mathbb{Z} \)-gradations, however, the new classes of rings introduced here have an interesting structure relating to the class group of the part of degree zero. The main results in Section 2 determine the structure of Gr-Dedekind rings. First, if \( R \) is a Gr-Dedekind ring such that \( RR_1 = R \), then \( R \) is a generalised Rees ring (and vice versa). These are obtained as follows: let \( I \) be a fractional ideal of a Dedekind domain \( R_0 \), consider the graded ring \( \bar{R}_0(I) = \sum_{n \in \mathbb{Z}} I^nX^n \) which is a graded subring of \( K_0[X, X^{-1}] \), \( K_0 \) being the field of fractions of \( R_0 \). Note that classically, the Rees ring of an ideal \( I \) of a domain \( R \) was defined to be the graded ring \( \bar{R}(I) = R \oplus I \oplus \cdots \oplus I^n \oplus \cdots \approx R + IX + I^2X^2 + \cdots + I^nX^n + \cdots \). Now, for an arbitrary Gr-Dedekind ring \( R \), the part of degree 0, \( R_0 \) say, is a Dedekind ring and in Theorem 2.10 we establish that there is an \( e \in \mathbb{N} \) such that \( R^{(e)} \) is a generalized Rees ring, where \( R^{(e)} \) is the graded ring defined by \( (R^{(e)})_k = R_{ek} \). The closing paragraphs of Section 2 deal with the study of the class groups of Gr-Dedekind rings \( R \) and the relations between these and the class groups of \( R_0 \) the part of degree 0.

In particular, we pay attention to some connections between the structure of class groups and the ramification in \( R \) of prime ideals of \( R_0 \), containing a certain ideal \( \delta(R) \), called the discriminator of \( R \), i.e., \( \delta(R) = R_{-1}R_1 \). In [5], we give some applications of these ideas, in particular the constructions of a "graded" Zeta-function etc.

Another application is to the theory of the Brauer group of a commutative ring, where arithmetical graded rings play a very peculiar role. We hope to present this material in a forthcoming paper.
1. Preliminaries

Throughout this paper $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is a graded commutative ring without zero divisors. $K$ will be its field of fractions and $K^e$ stands for the graded field of fractions. Recall from [3] that a graded field is a graded ring such that each nonzero homogeneous element is invertible and that a commutative graded field has the form $k[X, X^{-1}]$, where the part of degree 0, $k$ say, is a field and $X$ a variable of degree $e > 0$. We write $h(R)$ for the set of homogenous elements of $R$ while $h(R)^*$ stands for the set of nonzero elements of $h(R)$. A graded fractional ideal of $R$ is graded $R$-submodule $I$ of $K^e$ such that there exists a $d \neq 0$ in $R$ such that $dI \subseteq R$ (obviously, $d \in h^*(R)$ may be assumed here). A fractional ideal $I$ of $R$ is said to be invertible if there exists a fractional ideal $J \subseteq K$ such that $IJ = R$, we say that $J = I^{-1}$. If $I \subseteq K^e$ is a nonzero graded fractional ideal, then $I^{-1}$ is graded and $I^{-1} \subseteq K^e$.

A graded domain $R$ is said to be a graded principal ideal ring if every graded ideal is principal. A graded domain is a Gr Dedekind ring if every graded ideal of $R$ is a projective module.

**Lemma 1.1.** For a graded domain $R$ the following assertions are equivalent:

1. $R$ is a Gr-Dedekind ring.
2. Every graded ideal of $R$ is invertible.
3. $R$ is Noetherian, integrally closed in $K$ and every nonzero graded prime ideal is a maximal graded ideal of $R$.
4. Every graded ideal is in a unique way (up to ordering) a product of graded prime ideals.
5. The graded fractional ideals of $R$ form a multiplicative Abelian group.
6. $R$ is Noetherian and $R_M$ is a principal ideal ring for each maximal graded ideal $M$ of $R$.
7. $R$ is Noetherian and $R_P$ is a principal ideal ring for every graded prime ideal $P$ of $R$.
8. $R$ is Noetherian and $Q^*_h(R)$ is a Gr-principal ideal ring for every maximal graded ideal $M$.

**Proof.** The proof is fairly easy; similar to the ungraded case.

**Corollaries 1.2.**

1.2.1 A Gr-principal ideal ring is a Gr-Dedekind ring.
If $R$ is a Gr-Dedekind ring, then the class group of $R$, denoted by $C(R)$, is isomorphic to the graded class group $C_g(R)$. Henceforth we will only use the notation $C(R)$ even when $C_g(R)$ is meant. For definitions and details on $C(R)$ and $C_g(R)$ we may refer to [1].

In a Gr-Dedekind ring every graded fractional ideal can be generated by two homogeneous elements, one chosen arbitrarily in the ideal.

Since we can easily dispose of positively graded Gr-Dedekind rings, cf. Remark 2.7, we assume that the gradation of $R$ is not positive here.

**Lemma 1.3**. If $R$ is a Gr-Dedekind ring, then there is an $e \in \mathbb{N}$ such that $R = \bigoplus_{n \in \mathbb{Z}} R_n$ with $R_n \neq 0$ for every $n \in \mathbb{Z}$.

**Proof**. Choose $d, e > 0$ minimal such that $R_{-d} \neq 0$, $R_{e} \neq 0$. Write $RR_e = P_1 \cap \cdots \cap P_n$ with $P_i$, $i = 1, \ldots, n$, graded prime ideals of $R$. If $d > e$, then $0 = (RR_e)_0 = (P_1)_0 \cap \cdots \cap (P_n)_0$; hence $(P_i)_0 = 0$ for some $i \in \{1, \ldots, n\}$. Then it is clear that $P_i \cap R_{kd} = 0$ for all $k \in \mathbb{Z}$ and therefore each homogeneous element of $P_i$ has to be nilpotent (of order $d$), i.e., $P_i = 0$. The case $e > d$ may be dealt with similarly, thus $e = d$ follows. Consider $R_{ne+k}$ with $0 < k < e$, then $R_{ne}R_{ne+k} = 0$ with $R_{ne} \neq 0$ yields $R_{ne+k} = 0$.

**Corollary 1.4**. Without loss of generality we may assume that $R = \bigoplus_{n \in \mathbb{Z}} R_n$ with $R_n \neq 0$ for all $n \in \mathbb{Z}$.

**Lemma 1.5**. If $R$ is Gr-Dedekind ring, then $R_0$ is a Dedekind ring.

**Proof**. That $R_0$ is Noetherian is easily seen. If $I$ is an ideal of $R_0$, generated by $e_1, \ldots, e_s$ say, then the fact that $RI$ is projective yields that $RI$ is a direct summand of a graded free $R$-module $I$, cf. [2, Chap. 1]. Then $I_0$ is a free $R_0$-module and $(RI)_0 = I$ is a direct summand of $L_0$, i.e., $I$ is a projective $R_0$-module.

**2. Structure Theorems**

**Theorem 2.1**. Let $R$ be a Gr-Dedekind ring such that $RR_1 = R$; then for each graded prime ideal $P$ of $R$, $P = RP_0$.

**Proof**. Decompose $RP_0$ into a product of graded prime ideals $RP_0 = P_1^{r_1} \cdots P_m^{r_m}$. Since for $n \in \mathbb{N}$, $n \neq 1$, and any graded prime ideal $P$ there exists $t \in \mathbb{Z}$ such that $P_t \notin P^n$, the assumption $P_0 \subset P^n$ entails $RR_{-t}, RP_t \subset P^n$, i.e., $RR_{-t} \subset P$. Being a graded field, $R/P$ may be written as $R_0/P_0[X, X^{-1}]$ for some invertible $X$ of $R/P$. Obviously, $R_1 \notin P$, hence $(R/P)_1 \neq 0$ and consequently $\deg X = 1$, $X^{-t} \in (R/P)_{-t}$ follows.
(R/P)_\ell \neq 0 \text{ and consequently } \deg X = 1, X^{-\ell} \in (R/P)_{-\ell} \text{ follows. Now } (R/P)_\ell \neq 0 \text{ contradicts } R_{-\ell} \subset P \text{ and so we have established so far that } P_0 \subset P^n \text{ entails } n = 1, \text{i.e., } RP_0 = P_1 \cdot \cdot \cdot P_m. \text{ Choose } 0 \neq \bar{x} \in (R/RP_0)_n \text{ and let } x \in R_n - RP_0 \text{ represent } \bar{x}. \text{ From } R_n \notin P_1, \text{ hence } R_{-n} \notin P_1, \text{ it follows that we may select } y \in R_n - RP_0 \text{ so that } 0 \neq \bar{x}y \in R_0/P_0. \text{ By Lemma 1.5., } P_0 \text{ is a maximal ideal of } R_0, \text{ therefore, } \bar{x} \text{ is invertible in } R/RP_0 \text{ and hence } RP_0 \text{ is a maximal graded ideal.} \]

**Corollary 2.2.** In a Gr-Dedekind ring R such that RR_1 = R every graded ideal is generated by its part of degree 0.

**Proof.** The proof is straightforward.

The Rees ring associated to an ideal I of R is defined to be $R(I) = R + IX + \cdot \cdot \cdot + I^nX^n + \cdot \cdot \cdot$. This is a positively graded subring of $R[X]$ which is isomorphic to $R \oplus I \oplus \cdot \cdot \cdot \oplus I^n \oplus \cdot \cdot \cdot$.

If I is an invertible ideal of R, we define the generalized Rees ring associated to I as being the ring $R(I) = \sum_{n \in \mathbb{Z}} I^nX^n$ with the obvious gradation. This generalized Rees ring is a graded subring of $Q(R)[X, X^{-1}]$, where $Q(R)$ is the field of fractions of the domain R. If R is Noetherian, then $R(I) = \bar{R}(I)$ is Noetherian, cf. [2, Proposition 3.12]. On the other hand, the proof of Proposition 3.12 may be adapted to yield that $\bar{R}(I)$ is Noetherian too. But then [2, Proposition 3.2] entails that $R(I)$ is a Noetherian ring.

**Theorem 2.3.** Let I be an ideal of a Dedekind ring R, then $\bar{R}_0(I)$ is a Gr-Dedekind ring.

**Proof:** We have to establish that $\bar{R}_0(I)$ is graded integrally closed in its ring of fractions and that graded prime ideals of $\bar{R}_0(I)$ are maximal graded ideals. Write $R = \bar{R}_0(I)$ and let R be the integral closure of R in its graded field of fractions $Q^g(R)$. Suppose $y_n \in \bar{R} - R$ for some $n \in \mathbb{Z}$. Clearly, $R_{-n}y_n \subset \bar{R}$, but $R_{-n}y_n \notin R$ (since $R_{-n}y_n \subset R$ yields $RR_{-n}y_n \subset R$ hence $y_n \in R$). So for some $z_{-n} \in R_{-n}, c = z_{-n}y_n \in (\bar{R} - R)_0$. If c satisfies $T^n + a_{n-1}T^{n-1} + \cdot \cdot \cdot + a_0$ with $a_i \in R$, then also $T^n + (a_{n-1})_0T^{n-1} + \cdot \cdot \cdot + (a_0)_0$ with $(a_i)_0 \in R_0$. Because $R_0$ is a Dedekind ring, $c \in R_0$, contradiction. Therefore $\bar{R} = R$. To prove that graded prime ideals of R are 0 or maximal graded ideals it will suffice to show that for any prime ideal $P_0$ of $R_0$, $RP_0$ is a maximal graded ideal of R. By definition of $\bar{R}_0(I)$ it follows that $(R/RP_0)_n = I^nX^n/P_0I^nX^n$ and hence $(R/RP_0)_n \cong R_0/P_0$ because $I^n/P_0I^n$ is a simple $R_0$-module. Hence $R/RP_0$ and $R_0/P_0[X, X^{-1}]$ are isomorphic as graded rings (note: deg $X = 1$ since RR_1 = R) and therefore $RP_0$ is a maximal graded ideal.

**Remark 2.4.** $\bar{R}_n(I) = \bar{R}_n(J)$ if and only if I and J belong to the same element of C($R_0$).
Proof. Assume that $I$ and $J$ are integral ideals, then $\tilde{R}_0(I)I = \tilde{R}_0(I)X^{-1}$, whence: $I = \tilde{R}(I)_1X^{-1}$. Since $\tilde{R}_0(J)_1 = JY$, $\tilde{R}_0(J) = \sum_{n \in \mathbb{Z}} J^nY^n$ with $Y^{-1} \in \tilde{R}_0(J)$, it follows (up to isomorphism) that $IJ^{-1} = R_0XY^{-1}$. Conversely, if $I = Jx$ for some $x \in Q(R_0)$, then

$$\tilde{R}_0(I) = \sum_{n \in \mathbb{Z}} (Jx)^nX^n = \sum_{n \in \mathbb{Z}} J^n(xX)^n \cong \sum_{n \in \mathbb{Z}} J^nY^n = \tilde{R}_0(J).$$

Theorem 2.5. Every Gr-Dedekind ring $R$ with $RR_1 = R$ is of the form $\tilde{R}_0(I)$ for some ideal $I$ of $R_0$. There is an exact sequence of multiplicative groups: $I \rightarrow \langle \tilde{I} \rangle \rightarrow C(R_0)x^\times C(R) \rightarrow 1$, where $\langle \tilde{I} \rangle$ is the subgroup of $C(R_0)$ generated by the class $\tilde{I}$ of $I$. The epimorphism $\pi$ is thus an isomorphism if and only if $I$ is a principal ideal and in this case $R \cong R_0[X, X^{-1}]$.

Proof. Put $S = R_0 - \{0\}$, $Q_0 = S^{-1}R_0$. If $0 \neq x_n \in R_n$, then there exists $0 \neq y_{-n} \in R_{-n}$ and thus it is sufficient to make $x_ny_{-n} \in S$ invertible in order to obtain the graded field of fractions of $R$, $S^{-1}R$, say. Take $x_{-1} \in R_{-1} - \{0\}$. Then $x_{-1}^{-1} \in S^{-1}R$ and hence $S^{-1}R = Q_0[x_{-1}, x_{-1}^{-1}]$. Let $J$ be the maximal fractional ideal of $R_0$ such that $Jx_1 \subset R$ and $I$ the maximal ideal of $R_0$ such that $Ix_{-1} \subset R$. We have: $Jx_{-1} = R_{-1}$, $Ix_{-1} = R_1$ and since $R_1R_{-1} = R_0$ we obtain that $IJ = R_0$, i.e., $J = I^{-1}$. Similarly if $I_2$ is the ideal of $R_0$ for which $I_2x_1^2 = R_2$, then $I_2I^{-1} \subset I$ follows from $R_{-1}R_2 \subset R_1$, hence $I_2 \subset I^2$. On the other hand $I^2x_1^2 = (Ix_1)^2 \subset R_2$ yields $I^2 \subset I_2$. Repeating this argument we find that $R = \tilde{R}_0(I)$. It is obvious that we may define a mapping $\pi: C(R_0) \rightarrow C(R)$ which is induced by extension of ideals of $R_0$ to ideals of $R$ and it is easily seen that $\pi$ is a group epimorphism containing $\tilde{I}$ in its kernel. If $H$ is an ideal of $R_0$ such that $RH$ is principal in $R$, then we may write $RH = Rh_j^I$ for some $h_j \in RH$. Taking parts of degree $n$ in $RH$ yields: $(RH)_n = H^nX^n = R_{n-j}h_j^I$ with $h_j^I = h_jX^I$ for some $h_j \in I^j$; whence: $H^n = I^{n-j}h_j$ and thus $\tilde{H} \in \langle \tilde{I} \rangle$ in $C(R_0)$. Finally, if $I = R_0i$ is principal in $R_0$, then $R = R_0[iX, (IX)^{-1}]$ and $C(R) \cong C(R_0)$. Conversely, if $\pi$ is injective, then $RI = RX^{-1}$ entails that $I$ is principal.

Proposition 2.6. If $R$ is a graded Noetherian domain such that $RR_1 = R$ and $R_0$ is a Dedekind ring, then $R = \tilde{R}_0(I)$ for some ideal $I$ of $R$.

Proof. Attentive reading and economizing of the foregoing proof of Theorem 2.5.

Although the condition $RR_1 = R$ is a rather natural one (it should be compared to the assumption that $R$ is generated as an $R_0$ ring by $R_1$ in common theory of positively graded rings!) we can deal with the general case to some extent. First note
Remark 2.7. If $R$ is a positively graded Gr-Dedekind ring, then $R \cong k[X]$ with $k$ a field, $X$ a variable.

Proof. $M = \sum_{i>0} R_i$ is a graded prime ideal hence maximal as a graded ideal, so $R_0$ is a field. Moreover, since $M$ is the unique maximal graded ideal it follows that $R$ is a Gr-principal ideal ring. Write $\sum_{i>0} R_i = Ra$ for some homogeneous $a \in R$ and then it is easy to show $R = R_0[a] \cong k[X]$.

If $R$ is a graded ring, then for any $0 \neq e \in \mathbb{N}$ we put $R^{(e)} \cong \bigoplus_{n \in \mathbb{Z}} R_{en}$ with gradation defined by $R_k^{(e)} = R_{ek}$.

Lemma 2.8. If $R$ is a Gr-Dedekind ring, then $R^{(e)}$ is a Gr-Dedekind ring for any $e \neq 0$ in $\mathbb{N}$.

Proof. Reference [2, Corollary 3.11] yields that $R^{(e)}$ is Noetherian. Neglecting the gradations we may consider $R^{(e)}$ as a subring of $R$ and the graded field of quotients $Q\hat{e}(R^{(e)})$ as a subring of $Q\hat{e}(R)$. Therefore any $x \in Q\hat{e}(R^{(e)})$ graded integral over $R^{(e)}$ is integral over $R$ (in $Q\hat{e}(R)$) hence in $R$. Obviously $Q\hat{e}(R^{(e)}) = S^{-1}R^{(e)}$ with $S = R_0\setminus\{0\}$, i.e., $Q\hat{e}(R^{(e)}) = (Q\hat{e}(R))^{(e)}$ and $R \cap (Q\hat{e}(R))^{(e)} = R^{(e)}$ as ungraded rings. Consequently $x \in R^{(e)}$. The correspondence $P \to P^{(e)}$ defines a one-to-one correspondence between spec$_g R$ and spec$_g R^{(e)}$, the inverse correspondence being given by $Q \to \text{rad } R(\sum_{m \in \mathbb{Z}} Q_m)$. Hence graded prime ideals of $R^{(e)}$ are maximal graded ideals.

Lemma 2.9. If $R$ is a Gr-Dedekind ring and $P$ is a graded prime ideal of $R$, then the graded ring of fractions at $P$ is obtained by localizing at the multiplicatively closed set $R_0 - P_0$ and the localized ring is a graded discrete valuation ring.

Proof. Since $R/P$ is a graded field $x_n \in R_n - P$ yields that there is an $y_n \in R_n - P$ and it suffices to invert $x_n y_n \in R_0$ in order to invert $x_n$. The localized ring is a local graded principal ideal domain and therefore it is a graded discrete valuation ring (see [3] for details on graded valuation rings). Note that here a direct construction of a $\mathbb{Z}$-valued valuation function $v_p$ is possible by setting $v_p(x)$ equal to the exponent of $P$ in the decomposition of $Rx$ if $x$ is homogeneous and $v_p(x_1 + \cdots + x_k) = \min\{v_p(x_1), \ldots, v_p(x_k)\}(1)$

Theorem 2.10. If $R$ is a Gr-Dedekind ring, then there exists $e \in \mathbb{N}$ and an ideal $I$ of $R_0$ such that $R^{(e)} = R_0(I)$.

Proof. Write $RR_1 = P^m_1 \cdots P^m_n$. If all $v_i$ are zero, then $RR_1 = R$ and we take $e = 1$ and use Theorem 2.5, otherwise we deduce from the structure of the graded field $R/P$ that there exists $e \in \mathbb{N}$ such that $P \supset R_{me+r}$ for all $m$ and all $0 < r < e$. If $e = 0$, then $P \supset R_t$ for all $t \neq 0$. Consider the graded
localized ring $Q^g_p(R)$ of $R$ at $P$ (for generalities on graded rings of quotients, cf. [4]). Then $Q^g_t(P) \supset Q^g_p(R)$, for all $t \neq 0$ since $Q^g_p(R)$ is obtained by inverting elements of degree 0. The Lemma 2.9 yields that $Q^g_p(R)$ is a graded discrete valuation ring so we may choose $a \in Q^g_p(R)_0$ such that $v_p(a) = n \neq 0$ and such that $v_p(\alpha)$ is minimal amongst nonzero $v_p(\beta)$ with $\beta \in Q^g_p(R)_0$. Choose any $x \in Q^g_p(p) - Q^g_p(p)^*$, i.e. $v_p(x) = 1$. Then $v_p(\alpha^{-1}x^n) = 0$ or $\alpha^{-1}x^n$ is invertible in $Q^g_p(R)$, contradicting $Q^g_p(R) \subset Q^g_p(P)$ unless $P = R$, $P_0 = R_0$.

Actually, for each $P_i \supset R_1$ we find a number $e_i$ such that $e_i > 0$ and $P_i^{(e_i)} \subset R^{(e_i)}$ has the property: $P_i^{(e_i)} \not\supset (R^{(e_i)})_1$. Put $e = \text{the least common multiple of the } e_i$ then, for each $i$ $P_i^{(e)} \supset R_i^{(e)}$. The structure theorem for Gr Dedekind rings with $RR_1 = R$ may therefore be applied to $R^{(e)}$, because $R_i^{(e)}$ is not contained in any graded prime ideal of $R^{(e)}$ (indeed, for $P_i \supset R_1$ this has been shown before and if $Q \not\subset R$, then $Q \not\subset R_n$ yields $Q^{(n)} \not\subset (R^{(n)})_1$). Therefore $R^{(e)} = \overline{R}_0(I)$ for some ideal $I$ of $R_0$.

**Corollary 2.11.** If $R$ is a Gr-Dedekind ring, then there exists $n \in \mathbb{N}$ such that the diagram of group homomorphism is commutative,

$$
1 \rightarrow \langle I \rangle \rightarrow C(R_0) \rightarrow C(R^{(n)}) \rightarrow 1.
$$

**Proof.** Find $n \in \mathbb{N}$ as in Theorem 2.10. Extension of ideals from $R_0$ to $R^{(n)}$, "from" $R^{(n)}$ to $R$ and from $R_0$ to $R$ defines the arrows in the diagram. Use Theorem 2.5 and check commutativity.

We call the ideal $R_1R_{-1}$ of $R_0$ the discriminator of the Gr-Dedekind ring $R$, and denote it by $\delta(R)$.

**Lemma 2.12.** If $R$ is a Gr-Dedekind ring, then $S = Q^g_{\delta(R)}(R)$ is a generalized Rees ring with $C(S) = C(R)/(P_1, \ldots, P_s)$, where $P_1, \ldots, P_s$ are the graded prime ideals containing $R_1$.

**Proof.** Since $R$ is a Gr-Dedekind ring every graded localization is perfect, i.e., has graded property $T$ in the sense of [4]. For every ideal $L$ of $S$ we have $L = S$. ($L \cap R$) and therefore we obtain an epimorphism,

$$
\gamma: C(R) \rightarrow C(S).
$$

If $J$ is an ideal of $R$ such that $SJ = Sa$ for some homogeneous $a \in S$, then $(RR_1)^N a \subset J$ for some $N \in \mathbb{N}$. The Gr-Dedekind property yields $(RR_1)^N a = JH$, where $H$ is such that $SH = S$. From this it is easily seen that
Note 2.13. \( Q_{R_{-1}}^g(R) \cong Q_{RR_1}^g(R) \) as graded rings.

**Corollary 2.14.** We have a commutative diagram

\[
\begin{array}{ccc}
C(R^{(n)}) & \rightarrow & C(R) \\
\downarrow & & \downarrow \\
C(R_0) & \rightarrow & C(Q_{S(R_0)}) = C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s \rangle \\
\downarrow & & \downarrow \\
C(Q_{S(R_0)}) = C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s \rangle & \rightarrow & C(Q_{RR_1}^g(R)) = (C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s \rangle)/\langle \bar{I}_{S(R_0)} \rangle
\end{array}
\]

where \( p_1, \ldots, p_s \) are the prime ideals of \( R_0 \) containing \( \delta(R) \), i.e., \( p_i = P_i \cap R_0 \), and \( \bar{I}_{S(R)} \) is the ideal of \( Q_{S(R_0)}^g(R_0) \) defining the generalized Rees ring structure of \( Q_{RR_1}^g(R) \).

**Proposition 2.15.** Let \( R \) be a Gr-Dedekind ring such that there is an \( N \in \mathbb{N} \) such that the generalized Rees ring \( R^{(N)} \) has the property that \( C(R^{(N)}) \rightarrow C(R) \) is injective. Then \( C(Q_{RR_1}^g(R)) \cong C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s, \bar{I} \rangle \), where \( I \) is the ideal of \( R_0 \) determining the structure of the generalized Rees ring \( R^{(N)} \) and \( p_i, i = 1, \ldots, s \) are the prime factors of the discriminator in \( R_0 \).

**Proof.** In Corollary 2.14 we write \( I_{S(R)} = Q_{S(R_0)}^g(J') \) for some ideal \( J' \) of \( R_0 \). Commutativity of the diagram yields that \( J' \) is in the kernel of the composition \( C(R_0) \rightarrow C(R^{(N)}) \rightarrow C(R) \rightarrow C(S) \). Hence if we write \( J' = p_0^{e_0} q_1^{w_1} \cdots q_s^{w_s} \), then \( R\hat{J}' = \hat{p}_1^{e_1} \cdots \hat{p}_s^{e_s} \hat{Q}_1^{w_1} \cdots \hat{Q}_s^{w_s} \) (where \( e_j, j = 1, \ldots, s \), are the ramification indices of \( p_1, \ldots, p_s \), respectively) and \( S\hat{J}' = (S\hat{Q}_1)^{w_1} \cdots (S\hat{Q}_s)^{w_s} \). Consequently \( (S\hat{Q}_1)^{w_1} \cdots (S\hat{Q}_s)^{w_s} \) is a product of the \( \hat{p}_1 \cdots \hat{p}_s \) up to principal \( \hat{R} \)-ideals. Since \( C(R^{(N)}) \rightarrow C(R) \) is injective it follows that the class of \( q_1^{w_1} \cdots q_s^{w_s} \) is in the group generated by \( \bar{p}_1, \ldots, \bar{p}_s, \bar{I} \), whence \( J' \in \langle \bar{p}_1, \ldots, \bar{p}_s, \bar{I} \rangle \) follows. Therefore \( C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s, \bar{I} \rangle = C(R_0)/\langle \bar{p}_1, \ldots, \bar{p}_s, \bar{I} \rangle \).

Next we turn to Gr-principal ideal rings.

**Theorem 2.16.** If \( R \) is a Gr-principal ideal domain such that \( RR_1 = R \), then \( R_0 \) is a principal ideal domain and \( R \cong R_0[X, X^{-1}] \).
Proof. It is straightforward to show the fact that \( R \) is a graded unique factorization domain implies that \( R_0 \) is a unique factorization domain, hence as \( R_0 \) is also a Dedekind domain, \( R_0 \) has to be a principal ideal domain. Theorem 2.5 then yields that \( R \cong R_0[X, X^{-1}] \).

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