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# Is tame open? ☆

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**Abstract**

Is tame open? No answer so far. One may pose the Tame-Open Conjecture: Tame is open. But how to support it? No effective way to date. In this note, the rank of a wild algebra is introduced. The Wild-Rank Conjecture, which implies the Tame-Open Conjecture, is formulated. The Wild-Rank Conjecture is improved to the Basic-Wild-Rank Conjecture. A covering criterion on the rank of a basic wild algebra is given, which can be effectively applied to verify the Basic-Wild-Rank Conjecture for concrete algebras. It makes all conjectures much reliable.


Throughout $k$ denotes a fixed algebraically closed field. By an algebra we mean a finite-dimensional associative $k$-algebra with identity. By a module we mean a left module of finite $k$-dimension except in the context of covering theory. We denote by mod $A$ the category of finite-dimensional left $A$-modules. For terminology in the representation theory of algebras, we refer to [ARS,R2].

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## 1. Tame-Open Conjecture

For $d \in \mathbb{N}_1 := \{1, 2, 3, \ldots\}$, $\mathcal{A}_d$ denotes the affine variety of associative algebra structures with identity on $k^d$ (cf. [Ga1, Section 2.1]). The linear group $GL_d(k)$ operates on $\mathcal{A}_d$ by transport of structure (cf. [Ga1, Section 2.2]). One remarkable result in the geometry of representations is: *the finite representation type is open*, i.e., all $d$-dimensional $k$-algebras of finite representation type form an open subset of $\mathcal{A}_d$ (cf. [Ga1,Kr,Ge1]). Inspired by this, Geiss asked whether tame is open (cf. [Ge1,Ge2])? Of course one may pose a conjecture as follows:

**Tame-Open Conjecture.** *For any $d \in \mathbb{N}_1$, all tame algebras in $\mathcal{A}_d$ form an open subset of $\mathcal{A}_d$.*

How to support the Tame-Open Conjecture? An obvious way is to verify it for each dimension $d$. In case $1 \leqslant d \leqslant 3$, $\mathcal{A}_d = \{\text{all } d\text{-dimensional tame algebras}\}$. Thus Tame-Open Conjecture holds for $1 \leqslant d \leqslant 3$. In case $d = 4$, one can easily determine the representation type of all 4-dimensional algebras listed in [Ga1, Section 5]. Apply the upper semicontinuity of the function $A \mapsto \dim_k \mathrm{Aut}(A) = \dim_k \mathrm{End}(A)$ (cf. [Kr, Proposition 6.3]), one can show that Tame-Open Conjecture holds for $d = 4$ as well. However, for $d \geqslant 5$, even for $d = 5$ only, the problem becomes too complicated to be dealt with (cf. [Hap,Ma]). Thus it seems that it is difficult to go further along this way.

Note that the Tame-Open Conjecture was also studied by Kasjan from the viewpoint of model theory. He proved that the class of tame algebras is axiomatizable, and finite axiomatizability of this class is equivalent to the Tame-Open Conjecture (cf. [Kas]). Nevertheless, it seems that this cannot support Tame-Open Conjecture.

## 2. Wild-Rank Conjecture

A finite-dimensional $k$-algebra $A$ is called *wild* if there is a finitely generated $A$-$k\langle x, y\rangle$-bimodule $M$ which is free as a right $k\langle x, y\rangle$-module and such that functor $M \otimes_{k\langle x,y\rangle} -$ from $\mathrm{mod}\, k\langle x, y\rangle$ to $\mathrm{mod}\, A$ preserves indecomposability and isomorphism classes (cf. [CB1]). We say that $A$ is *strictly wild* if in addition the functor $M \otimes_{k\langle x,y\rangle} -$ is full. In a natural way, we can define notions of wildness or strictly wildness for a full subcategory of the module category over an algebra. If the algebra $A$ is wild then we denote by $r_A$ the number $\min\{\mathrm{rank}_{k\langle x,y\rangle} M \mid M$ is a finitely generated $A$-$k\langle x, y\rangle$-bimodule which is free as a right $k\langle x, y\rangle$-module and such that the functor $M \otimes_{k\langle x,y\rangle} -$ from $\mathrm{mod}\, k\langle x, y\rangle$ to $\mathrm{mod}\, A$ preserves indecomposability and isomorphism classes$\}$. By [C, Corollary 2.4.3], $k\langle x, y\rangle$ is a free ideal ring. By [C, Corollary 1.1.2], $k\langle x, y\rangle$ is an IBN ring. Thus the rank of a free $k\langle x, y\rangle$-module is unique. Hence $r_A$ is well defined and called the *rank* of the wild algebra $A$. Similarly, we may define the rank $r_{\mathcal{C}}$ of a wild subcategory $\mathcal{C}$ of $\mathrm{mod}\, A$. Obviously, $r_A \leqslant r_{\mathcal{C}}$.

In this paper, we do not distinguish $d$-dimensional algebras from points in $\mathcal{A}_d$. Put $\mathcal{T}_d := \{A \in \mathcal{A}_d \mid A \text{ tame}\}$ and $\mathcal{W}_d := \{A \in \mathcal{A}_d \mid A \text{ wild}\}$.

**Wild-Rank Conjecture.** *There is a function* $f:\mathbb{N}\to\mathbb{N}$ *such that* $r_A \leqslant f(d)$ *for all* $A \in \mathcal{W}_d$.

**Remark 1.** In some sense, the Wild-Rank Conjecture is an analogue of the numerical criterion of the finite representation type (cf. [B, Theorem]).

If an algebraic group $G$ acts on a variety $X$ then the *number of parameters* of $G$ on $X$ is $\dim_G X := \max\{\dim X_{(s)} - s \mid s \geqslant 0\}$ where $X_{(s)}$ is the union of orbits of dimension $s$ (cf. [Kac, p. 71] or [KR, p. 125] or [CB2, p. 399]). If $A$ is a finite-dimensional $k$-algebra then the set $\operatorname{mod}(A,n)$ of the $n$-dimensional representations of $A$ is the closed subset of $\operatorname{Hom}_k(A, M(n,k))$ consisting of all $k$-algebra homomorphisms from $A$ to the algebra $M(n,k)$ of $n\times n$ matrices. There is a natural conjugation action of $GL_n(k)$ on $\operatorname{mod}(A,n)$. Put $\mathcal{A}_{d,\leqslant n} := \{A\in\mathcal{A}_d \mid \dim_{GL_n(k)} \operatorname{mod}(A,n) \leqslant n\}$ and $\mathcal{A}_{d,>n} := \{A\in\mathcal{A}_d \mid \dim_{GL_n(k)} \operatorname{mod}(A,n) > n\}$.

**Lemma 1** ([Ge1, Proposition 1], [CB2, Proof of Theorem B]). $\mathcal{A}_{d,\leqslant n}$ *is an open subset of* $\mathcal{A}_d$ *and* $\mathcal{A}_{d,>n}$ *is a closed subset of* $\mathcal{A}_d$ *for all* $d$ *and* $n$.

Put $\mathcal{A}_d^{\leqslant n} := \bigcap_{i=1}^n \mathcal{A}_{d,\leqslant i}$ and $\mathcal{A}_d^{>n} := \bigcup_{i=1}^n \mathcal{A}_{d,>i}$. Then $\mathcal{A}_d^{\leqslant 1} \supseteq \mathcal{A}_d^{\leqslant 2} \supseteq \cdots$ and $\mathcal{A}_d^{>1} \subseteq \mathcal{A}_d^{>2} \subseteq \cdots$. By Lemma 1, $\mathcal{A}_d^{\leqslant n}$ is an open subset of $\mathcal{A}_d$ and $\mathcal{A}_d^{>n}$ is a closed subset of $\mathcal{A}_d$ for all $d$ and $n$.

**Lemma 2** ([D, Proposition 2], [Ge1, Proposition 2], [CB2, Lemma 3]).

$$\mathcal{T}_d = \bigcap_{i\in\mathbb{N}_1} \mathcal{A}_{d,\leqslant i} = \bigcap_{i\in\mathbb{N}_1} \mathcal{A}_d^{\leqslant i} \quad \textit{and} \quad \mathcal{W}_d = \bigcup_{i\in\mathbb{N}_1} \mathcal{A}_{d,>i} = \bigcup_{i\in\mathbb{N}_1} \mathcal{A}_d^{>i}.$$

**Theorem 1.** *The Wild-Rank Conjecture implies the Tame-Open Conjecture.*

**Proof.** If the Wild-Rank Conjecture holds then there is a function $f:\mathbb{N}\to\mathbb{N}$ such that $r_A \leqslant f(d)$ for all $A\in\mathcal{W}_d$ and $d\in\mathbb{N}_1$. Let $A\in\mathcal{W}_d$. Then there is a finitely generated $A$-$k\langle x,y\rangle$-bimodule $M$ which is free of rank $r_A$ over $k\langle x,y\rangle$ such that the functor $M\otimes_{k\langle x,y\rangle} -$ from $\operatorname{mod} k\langle x,y\rangle$ to $\operatorname{mod} A$ preserves indecomposability and isomorphism classes. Note that $\phi := M\otimes_{k\langle x,y\rangle} - : \operatorname{mod}(k\langle x,y\rangle, t) \to \operatorname{mod}(A, r_A t)$ is a regular map (cf. [DS, p. 67]). Consider stratifications

$$\operatorname{mod}\bigl(k\langle x,y\rangle,t\bigr) = \bigcup_i \operatorname{mod}\bigl(k\langle x,y\rangle,t\bigr)_{(i)} \quad \text{and} \quad \operatorname{mod}(A, r_A t) = \bigcup_j \operatorname{mod}(A, r_A t)_{(j)}.$$

Since $\operatorname{mod}(k\langle x,y\rangle,t)$ is irreducible and

$$\operatorname{mod}\bigl(k\langle x,y\rangle,t\bigr) = \bigcup_{i,j}\Bigl(\operatorname{mod}\bigl(k\langle x,y\rangle,t\bigr)_{(i)} \cap \phi^{-1}\bigl(\operatorname{mod}(A, r_A t)_{(j)}\bigr)\Bigr),$$

there are $i$ and $j$ such that the constructible subset

$$X := \mathrm{mod}\big(k\langle x, y\rangle, t\big)_{(i)} \cap \phi^{-1}\big(\mathrm{mod}(A, r_A t)_{(j)}\big)$$

is irreducible and dense in $\mathrm{mod}(k\langle x, y\rangle, t)$. Thus $\phi(X)$ is an irreducible and constructible subset of $\mathrm{mod}(A, r_A t)_{(j)}$. Consider the restriction of $\phi$ on $X$ and $\phi(X)$. By [Mu, Section I.8, Theorem 3], $\dim\phi(X) - \dim X = \dim\phi^{-1}(y)$ for some $y \in \phi(X)$. Take any $x \in \phi^{-1}(y)$. Since the inverse image of an orbit under $\phi$ is an orbit, $\phi$ induces a regular map $\psi$ from the orbit $GL_t(k)\cdot x$ to the orbit $GL_{r_A t}(k)\cdot y$. Applying [Mu, Section I.8, Theorem 3] again, we have $\dim\phi^{-1}(y) = \dim\psi^{-1}(y) = \dim GL_{r_A t}(k)\cdot y - \dim GL_t(k)\cdot x = j - i$. Therefore

$$\begin{aligned}\dim_{GL_{r_A t}(k)} \mathrm{mod}(A, r_A t) &\geqslant \dim \mathrm{mod}(A, r_A t)_{(j)} - j \geqslant \dim\phi(X) - j \\ &= \dim X + (j - i) - j = \dim \mathrm{mod}\big(k\langle x, y\rangle, t\big) - i \\ &> \dim \mathrm{mod}\big(k\langle x, y\rangle, t\big) - \dim GL_t(k) = 2t^2 - t^2 = t^2\end{aligned}$$

for all $t$. In particular, take $t = r_A$ then $\dim_{GL_{r_A^2}(k)} \mathrm{mod}(A, r_A^2) > r_A^2$. This implies that for any $A \in \mathcal{W}_d$,

$$A \in \mathcal{A}_{d,\,>r_A^2} \subseteq \mathcal{A}_d^{>r_A^2} \subseteq \mathcal{A}_d^{>f^2(d)}.$$

By Lemma 2, $\mathcal{W}_d = \mathcal{A}_d^{>f^2(d)}$ is a closed subset of $\mathcal{A}_d$. □

## 3. Morita equivalence

Now we study changes of the rank of a wild algebra under Morita equivalence and factor algebra. The following result implies that to prove the Wild-Rank Conjecture suffices to show it for all basic algebras.

**Theorem 2.** *If a $d$-dimensional wild algebra $A$ is Morita equivalent to a basic algebra $B$ then $r_A \leqslant d \cdot r_B$.*

**Proof.** Suppose $A = \bigoplus_{i=1}^m n_i P_i$ with $n_i \geqslant 1$ and $P_i$, $1 \leqslant i \leqslant m$, being the nonisomorphic indecomposable projective $A$-modules. Let $P = \bigoplus_{i=1}^m P_i$. Then $B \cong \mathrm{End}_A(P)^{op}$. Consider the evaluation functor $e_P = \mathrm{Hom}_A(P, -): \mathrm{mod}\, A \to \mathrm{mod}\, B$. Note that $e_P$ is an equivalence of categories with quasi-inverse $P \otimes_B -$ (cf. [ARS, Corollary II.2.6.] and [AF, Theorem 22.2]). Since $B$ is wild, there is a $B$-$k\langle x, y\rangle$-bimodule $M$ which is free of rank $r_B$ over $k\langle x, y\rangle$ such that the functor $M \otimes_{k\langle x,y\rangle} -$ from $\mathrm{mod}\, k\langle x, y\rangle$ to $\mathrm{mod}\, B$ preserves indecomposability and isomorphism classes. Note that $P$ is also projective over $B$. Decompose $P$ as the direct sum of the indecomposable projective right $B$-modules, set $P = \bigoplus_{i=1}^t Q_i$. For $Q_i$ there is a projective right $B$-module $Q_i'$ such that $Q_i \oplus Q_i' = B$. Thus there is a projective right $B$-module $P'$ such that $P \oplus P' = B^t$. Further $(P \otimes_B M) \oplus (P' \otimes_B M) = B^t \otimes_B M$ which is free of rank $t \cdot r_B \leqslant \dim_k P \cdot r_B \leqslant \dim_k A \cdot r_B = d \cdot r_B$. Since $P \otimes_B M$ is finitely generated projective over $k\langle x, y\rangle$, by [C, Theorem 1.4.1], it is free over $k\langle x, y\rangle$.

Moreover, its rank is at most $d \cdot r_B$. Consider the composition $P \otimes_B M \otimes_{k\langle x,y\rangle} -$, we have $r_A \leqslant d \cdot r_B$. □

From now on, unless stated otherwise, *we assume that all algebras are basic*. Thus any algebra $A$ can be written as $kQ/I$ where $Q$ is the Gabriel quiver of $A$ and $I$ is an admissible ideal of the path algebra $kQ$. For a quiver $Q$ we denote by $Q_0$ (respectively $Q_1$) the set of vertices (respectively arrows) of $Q$. The next result implies that to prove the Wild-Rank Conjecture suffices to show it for all minimal wild algebras. Here *minimal wild* means no proper factor algebra is wild.

**Lemma 3.** *If $I$ is an ideal of an algebra $A$ and $A/I$ is wild then $r_A \leqslant r_{A/I}$.*

**Proof.** If $M$ is a finitely generated $A/I$-$k\langle x,y\rangle$-bimodule which is free of rank $r_{A/I}$ over $k\langle x,y\rangle$ such that the functor $M \otimes_{k\langle x,y\rangle} -$ from $\operatorname{mod} k\langle x,y\rangle$ to $\operatorname{mod} A/I$ preserves indecomposability and isomorphism classes, then $M$ is also a finitely generated $A$-$k\langle x,y\rangle$-bimodule which is free of rank $r_{A/I}$ over $k\langle x,y\rangle$ such that the functor $M \otimes_{k\langle x,y\rangle} -$ from $\operatorname{mod} k\langle x,y\rangle$ to $\operatorname{mod} A$ preserves indecomposability and isomorphism classes. □

## 4. Covering criterion

In this section, we shall provide a covering criterion which can be effectively applied to provide an anticipated upper bound for the rank of a concrete wild algebra. For the knowledge of Galois covering theory, we refer to [BG,Ga2,MP].

A *minimal wild concealed algebra* means a concealed algebra of a minimal wild hereditary algebra. Unless stated otherwise, the word *minimal* in *minimal wild hereditary algebra* or in *minimal wild concealed algebra* is always used in the sense of [Ke1]. First of all, we provide upper bounds for ranks of some strictly wild subcategories in the module categories over minimal wild concealed algebras.

**Lemma 4.** *Ranks of all minimal wild hereditary algebras are bounded by a fixed number.*

**Proof.** Note that the underlying diagrams of the quivers of all minimal wild hereditary algebras are listed in [Ke1, p. 443]. Denote by $|Q|$ the underlying diagram of the quiver $Q$. Then there are at most $2^{|Q_1|}$ quivers with underlying diagram $|Q|$. Thus (up to isomorphism) there are finitely many minimal wild hereditary algebras. □

Let $A = kQ/I$. For an $A$-module $M$ we define its *support* $\operatorname{Supp}(M)$ to be the subset of $Q_0$ consisting of all $x \in Q_0$ satisfying $M(x) \neq 0$. An $A$-module $M$ is called *sincere* if $\operatorname{Supp}(M) = Q_0$.

**Lemma 5.** *Ranks of all minimal wild concealed algebras are bounded by a fixed number.*

**Proof.** It is enough to show that (up to isomorphism) there are only finitely many minimal wild concealed algebras. This is clear by [U1,U2]. Here we give some details. Let $A$ be a minimal wild concealed algebra of type $H$. Let $T = \bigoplus_{i=1}^{n} T_i$ be a preprojective tilting $H$-module such that $A = \mathrm{End}_H(T)$. Then $T_i = \tau^{-m_i} P_i$ for some indecomposable projective $H$-module $P_i$ and some nonnegative integer $m_i$. Here $\tau$ denotes Auslander–Reiten translation. Thus $T = \tau^{-\min\{m_i \mid 1 \leqslant i \leqslant n\}} T_1$ with $T_1 = P \oplus \tau^{-1} T_2$, where $P$ is a projective $H$-module and $\tau^{-1} T_2$ has no projective direct summand. By [R2, p. 76, (6)] we have $\mathrm{Ext}^1_H(T_1, T_1) = 0$. Thus $T_1$ is still a preprojective tilting $H$-module. By [ARS, Proposition 1.9(b)] we have $\mathrm{End}_H(T_1) = \mathrm{End}_H(T) = A$. Let $P = He$ and $H' = H/\langle e\rangle$ where $\langle e\rangle$ is the two-sided ideal of $H$ generated by $e$. Then $\mathrm{Hom}_H(P, T_2) = \mathrm{Hom}_H(P, \tau\tau^{-1} T_2) = D\,\mathrm{Ext}^1_H(\tau^{-1} T_2, P) = 0$. Thus $T_2$ is an $H'$-module. In particular $T_2$ is a non-sincere preprojective $H$-module. Since there are only finitely many non-sincere indecomposable preprojective $H$-modules (cf. [Ke3, Corollary 3.9]), there are only finitely many square-free preprojective tilting $H$-modules with projective summands. Therefore there are only finitely many minimal wild concealed algebras of type $H$. By the proof of Lemma 4, the number of minimal wild hereditary algebras is finite, so is the number of minimal wild concealed algebras. □

Denote by $(\mathrm{mod}\,A)_s$ the full subcategory of $\mathrm{mod}\,A$ consisting of all $A$-modules whose indecomposable direct summands are all sincere. Note that this notation is different from that in [E,Han2].

**Lemma 6.** *If $A = kQ/I$ is a strictly wild algebra and $A/\langle e_i\rangle$ is not strictly wild for any primitive idempotent corresponding to a vertex $i$ in $Q_0$, then $(\mathrm{mod}\,A)_s$ is strictly wild.*

**Proof.** The proof is almost the same as that of [Han2, Lemma (3.1)]. Denote by $\mathbb{K}_3$ the quiver with two vertices $1, 2$ and three arrows $\alpha, \beta, \gamma$. First of all, there is a fully faithful exact functor $\mathcal{F}: \mathrm{mod}\,k\mathbb{K}_3 \to \mathrm{mod}\,k\langle x, y\rangle$ sending $(V_1, V_2; \alpha, \beta, \gamma)$ to

$$\left( (V_1 \oplus V_2)^7; \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha' & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta' & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma' & 1 & 0 \end{bmatrix} \right),$$

where all entries of these two matrices are $2 \times 2$ matrices and

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \delta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \alpha' = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix},$$

$$\beta' = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} \quad \text{and} \quad \gamma' = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}.$$

Moreover, there is also a fully faithful exact functor $\mathcal{G}: \mathrm{mod}\,k\langle x, y\rangle \to \mathrm{mod}\,k\mathbb{K}_3$ which is defined by sending $(V; x, y)$ to $(V, V; 1, x, y)$. Since $A$ is strictly wild, there exists a fully faithful exact functor $\mathcal{H}: \mathrm{mod}\,k\mathbb{K}_3 \to \mathrm{mod}\,A$. By assumption, we know that

$\mathrm{Supp}(\mathcal{H}(S_1)) \cup \mathrm{Supp}(\mathcal{H}(S_2)) = Q_0$, where $S_i$ is the simple $k\mathbb{K}_3$-module corresponding to vertex $i$. It is easy to see that both $\mathcal{GF}(S_1)$ and $\mathcal{GF}(S_2)$ are sincere $k\mathbb{K}_3$-modules, i.e. for each $i$, $\mathcal{GF}(S_i)$ is an extension of $S_1^{m_i}$ by $S_2^{n_i}$ for some positive integers $m_i$ and $n_i$. Hence $\mathcal{HGF}(S_1)$ and $\mathcal{HGF}(S_2)$ are sincere $A$-modules. Since the functor $\mathcal{HGF}$ is fully faithful and exact, it preserves indecomposability. Hence each indecomposable direct summand of each $A$-module in $\mathrm{Im}\,\mathcal{HGF}$ is an image of a module in $\mathrm{mod}\,k\mathbb{K}_3$. Thus all $A$-modules in $\mathrm{Im}\,\mathcal{HGF}$ are contained in $(\mathrm{mod}\,A)_s$. Finally $\mathcal{HGFG}$ defines a strictly wild functor from $\mathrm{mod}\,k\langle x, y\rangle$ to $(\mathrm{mod}\,A)_s$. □

The constant $b$ in the next lemma is very important and will appear frequently.

**Lemma 7.** *Ranks of* $(\mathrm{mod}\,A)_s$ *where $A$ runs through all minimal wild concealed algebras are bounded by a fixed number. Suppose $b$ is the smallest bound.*

**Remark 2.** It should be interesting to evaluate the number $b$.

**Proof.** It follows from [Ke2, Corollary 2.2] that $\mathrm{mod}\,A$ is strictly wild. It is well known that minimal wild concealed algebras are minimal wild in the sense of [Ke1] (cf. [U2, p. 146]). By Lemma 6, $(\mathrm{mod}\,A)_s$ is strictly wild as well. By the proof of Lemma 5, there are only finitely many minimal wild concealed algebras. □

A quiver with relations $(Q, I)$ is called a *factor quiver* of a quiver with relations $(Q', I')$ if $Q_0$ is a subset of $Q'_0$, $Q_1$ is a subset of the subset of $Q'_1$ obtained from $Q'_1$ by excluding all the arrows starting or ending at some vertex in $Q'_0\backslash Q_0$, and $I$ is the admissible ideal of $kQ$ obtained from $I'$ by replacing each arrow in $Q'_1\backslash Q_1$ in each element of $I'$ by zero (cf. [Han2]). Note that in this case, $kQ/I$ is a factor algebra of $kQ'/I'$. A Galois covering of quiver with relation $\pi : (Q', I') \to (Q, I)$ is said to be *wild concealed* if there is a finite factor quiver $(\widetilde{Q}, \tilde{I})$ of $(Q', I')$ such that $k\widetilde{Q}/\tilde{I}$ is a minimal wild concealed algebra. The following result including its proof is a modification of [E, Proposition I.10.6].

**Lemma 8.** *Let $\pi : (Q', I') \to (Q, I)$ be a Galois covering of a quiver with relations with a torsion-free Galois group $G$ and $(\widetilde{Q}, \tilde{I})$ a finite factor quiver of $(Q', I')$. Then*

(1) *The restriction $F_\lambda : (\mathrm{mod}\,k\widetilde{Q}/\tilde{I})_s \to \mathrm{mod}\,kQ/I$ preserves indecomposability and isomorphism classes.*
(2) *There is a finitely generated $kQ/I$-$k\widetilde{Q}/\tilde{I}$-bimodule $M$ which is free of rank $|\widetilde{Q}_0|$ over $k\widetilde{Q}/\tilde{I}$ and such that $F_\lambda \cong M \otimes_{k\widetilde{Q}/\tilde{I}} -$ on $(\mathrm{mod}\,k\widetilde{Q}/\tilde{I})_s$.*

**Proof.** (1) $F_\lambda$ preserves indecomposability: Take $N$ indecomposable in $(\mathrm{mod}\,k\widetilde{Q}/\tilde{I})_s$ and consider $N$ as a $kQ'/I'$-module. By [Ga2, Lemma 3.5], it suffices to show that ${}^gN \ncong N$ for $1 \neq g \in G$. If $1 \neq g$ then, since $G$ is torsion-free, $({}^g\widetilde{Q})_0 \neq \widetilde{Q}_0$. Hence $\mathrm{Supp}({}^gN) \neq \mathrm{Supp}(N)$. Thus ${}^gN \ncong N$.

$F_\lambda$ preserves isomorphism classes: Let $F_\lambda(N_1) \cong F_\lambda(N_2)$. Let $N_j = \bigoplus_{i=1}^{n_j} N_{ji}$ be the direct sum decomposition of $N_j \in (\mathrm{mod}\,k\widetilde{Q}/\tilde{I})_s$, $j = 1, 2$, into indecomposables. Then, by the paragraph above and Krull–Schmidt theorem, we have $n_1 = n_2$ and $F_\lambda(N_{1i}) \cong F_\lambda(N_{2t_i})$, $1 \leqslant t_i \leqslant n_1$, $i = 1, \ldots, n_1$. Considering $N_{ji}$, $j = 1, 2$, $i = 1, \ldots, n_1$ as $kQ'/I'$-module. By [Ga2, Lemma 3.5], we have $N_{1i} \cong {}^{g_i}N_{2t_i}$ for some $g_i \in G$ and $i = 1, \ldots, n_1$. Thus $\widetilde{Q}_0 = \mathrm{Supp}(N_{1i}) = \mathrm{Supp}({}^{g_i}N_{2t_i}) = {}^{g_i}\widetilde{Q}_0$. Since $G$ is torsion-free, we have $g_i = 1$ and $N_{1i} \cong N_{1t_i}$, $i = 1, \ldots, n_1$. Hence $N_1 \cong N_2$.

(2) The $kQ/I$-$k\widetilde{Q}/\tilde{I}$-bimodule $M$: For a free basis $\{b_i \mid i \in \widetilde{Q}_0\}$, define $M$ to be the free $k\widetilde{Q}/\tilde{I}$-module $\bigoplus_{i \in \widetilde{Q}_0} b_i(k\widetilde{Q}/\tilde{I})$. We define a left $kQ/I$-module structure on $M$ as follows: Let $i \in Q_0$, $s \in \widetilde{Q}_0$, and $\sigma \in k\widetilde{Q}/\tilde{I}$. We denote by $e_s$ the idempotent of $k\widetilde{Q}$ corresponding to $s$, and we set

$$e_i(b_s\sigma) = \begin{cases} b_s(e_s\sigma) & \text{if } \pi(s) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\alpha : i \to j$ is an arrow in $Q$. If $s \in \widetilde{Q}_0$ with $\pi(s) = i$ and $\tilde{\alpha} : s \to t$ is an arrow in $\widetilde{Q}$ with $\pi(s) = i$ and $\pi(\tilde{\alpha}) = \alpha$ then we define $\alpha(b_s\sigma) = b_t(\tilde{\alpha}\sigma)$, and set $\alpha(b_s\sigma) = 0$ otherwise. We claim that this is a $kQ/I$-module action: Suppose $\rho \in I$. Note that every relation is a sum of minimal and zero relations (cf. [MP]). To prove $\rho(b_s\sigma) = 0$ for $\sigma \in k\widetilde{Q}/\tilde{I}$ suffices to show it for a minimal or zero relation $\rho \in I$. We assume $\rho \in e_j(kQ)e_i$ for $i, j \in Q_0$. If there is no $s \in \widetilde{Q}_0$ such that $\pi(s) = i$ then we have $\rho(b_s\sigma) = 0$. If there is $s \in \widetilde{Q}_0$ such that $\pi(s) = i$ then there is $\rho' \in I' \cap e_t(kQ')e_s$ such that $\pi(\rho') = \rho$. By replacing each arrow in $Q'_1 \backslash \widetilde{Q}_1$ by zero, we obtain $\tilde{\rho} \in \tilde{I} \cap e_t(k\widetilde{Q})e_s$ from $\rho'$. Clearly, $\rho(b_s\sigma) = b_t(\tilde{\rho}\sigma) = 0$.

Now let $N \in \mathrm{mod}\, k\widetilde{Q}/\tilde{I}$; we will show that $F_\lambda(N) = M \otimes_{k\widetilde{Q}/\tilde{I}} N$ canonically. Since for any arrow $\tilde{\alpha} \in \widetilde{Q}$ we have that $(b_s\tilde{\alpha}) \otimes N = b_s \otimes (\tilde{\alpha}N) \subseteq b_s \otimes N$, the module $M \otimes_{k\widetilde{Q}/\tilde{I}} N$ has underlying space $\bigoplus_{s \in \widetilde{Q}_0}(b_s \otimes N)$. Let $i \in Q_0$. If $\pi(s) \neq i$ then $e_i(b_s \otimes N) = 0$. If $\pi(s) = i$ then $e_i(b_s \otimes N) = (b_s e_s) \otimes N = b_s \otimes e_s N = b_s \otimes N(s)$. So we may identify $e_i(M \otimes N)$ with $(F_\lambda(N))(i) = \bigoplus_{\pi(s)=i} N(s)$. Now consider the action of an arrow $\alpha : i \to j$ in $Q$. Let $\tilde{\alpha} : s \to t$ be an arrow in $\widetilde{Q}$ with $\pi(s) = i$, $\pi(\tilde{\alpha}) = \alpha$ and hence $\pi(t) = j$. Then $\alpha(b_s \otimes N) = (b_t\tilde{\alpha}) \otimes N = b_t \otimes (\tilde{\alpha}N) = b_t \otimes (\tilde{\alpha}e_s N) = b_t \otimes (\tilde{\alpha}N(s)) = b_t \otimes N(\tilde{\alpha})(N(s))$, and this is just the action of $\alpha$ on the space $(F_\lambda(N))(i)$. □

**Theorem 3** (covering criterion). *Let $A = kQ/I$ be a wild algebra and $\pi : (Q', I') \to (Q, I)$ a wild concealed Galois covering of quivers with relations with torsion-free Galois group. Then $r_A \leqslant 10b$.*

**Proof.** Let $(\widetilde{Q}, \tilde{I})$ be a finite factor quiver of $(Q', I')$ such that $k\widetilde{Q}/\tilde{I}$ is a minimal wild concealed algebra. By Lemma 7, there is a finitely generated $k\widetilde{Q}/\tilde{I}$-$k\langle x, y\rangle$-bimodule $M_1$ which is free of rank at most $b$ over $k\langle x, y\rangle$ such that the functor $M_1 \otimes_{k\langle x,y\rangle} -$ from $\mathrm{mod}\, k\langle x, y\rangle$ to $(\mathrm{mod}\, k\widetilde{Q}/\tilde{I})_s$ preserves indecomposability and isomorphism classes. By Lemma 8, there is a finitely generated $kQ/I$-$k\widetilde{Q}/\tilde{I}$-bimodule $M_2$ which is free of rank $|\widetilde{Q}_0|$ over $kQ/I$ such that on $\mathrm{mod}(\widetilde{Q}, \tilde{I})_s$ the pushdown functor $F_\lambda \cong M_2 \otimes_{k\widetilde{Q}/\tilde{I}} -$ preserves indecomposability and isomorphism classes. Consider the composition $M_2 \otimes_{k\widetilde{Q}/\tilde{I}} M_1 \otimes_{k\langle x,y\rangle} -$; we have $r_A \leqslant \mathrm{rank}(M_2 \otimes M_1) \neq |\widetilde{Q}_0| \cdot b \leqslant 10b$. □

According to Theorems 2 and 3, we reformulate the Wild-Rank Conjecture as follows:

**Wild-Rank Conjecture.** *Let $A$ be a $d$-dimensional* (*unnecessarily basic*) *wild algebra. Then $r_A \leqslant 10bd$.*

**Basic-Wild-Rank Conjecture.** *Let $A$ be a $d$-dimensional basic wild algebra. Then $r_A \leqslant 10b$.*

Clearly, Basic-Wild-Rank Conjecture $\Rightarrow$ Wild-Rank Conjecture $\Rightarrow$ Tame-Open Conjecture.

## 5. Applications of the covering criterion

How to support the Basic-Wild-Rank Conjecture? For concrete algebras, our covering criterion is very effective. Indeed, for a concrete basic wild algebra $A$ given by quiver with relations $(Q, I)$, we can find a minimal wild factor algebra $B$ of $A$. Usually, either $B$ is itself a minimal wild concealed algebra or there is an algebra $C \cong B$ such that $C$ admits a wild concealed Galois covering with torsion-free Galois group. Thus we can apply the covering criterion to the algebra $C$.

By the covering criterion, we know the Basic-Wild-Rank Conjecture holds for all well-known wild algebras such as wild local algebras, wild two-point algebras, wild radical square zero algebras, wild finite $p$-group algebras, wild three-point algebras whose quiver is system quiver (cf. [R1,Han3,Han1,Han2,LZ]). This implies that all three conjectures are much reliable.

Certainly one can list many propositions analogous to the following one.

**Proposition.** *Let $A$ be a $d$-dimensional wild local algebra* (*respectively wild two-point algebra, wild radical square zero algebra*)*. Then $r_A \leqslant 10b$.*

**Proof.** Up to duality and isomorphism, $A$ has a minimal wild factor algebra $B$ appearing in the list of [R1, p. 283] (respectively [Han3, Table W], [Han1, p. 98] or [Han2, p. 290]). By check case-by-case, we know that either $B$ is itself a minimal wild concealed algebra or there is an algebra $C \cong B$ such that $C$ admits a wild concealed Galois covering with a torsion-free Galois group. □

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