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Computational method of rapidly propagating cracks using continuous dislocations model

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Abstract

The method of continuously distributed dislocations model is applied to the problem of rapidly propagating cracks. Weertman's solution of a running edge dislocation considering the effect of inertia (1961) has been utilized. A semi-infinite length crack running in a strip with clamped sides was analyzed. The singular integral equations representing the boundary condition of this problem were solved numerically by the newly proposed method based on the boundary collocation. The stress intensity factor was calculated and it has been made clear that precise solutions can be obtained even when the number of the collocation points is small. The crack opening displacement was also calculated and a startling result has been obtained. In the neighborhood of the crack tip, the crack opening displacement exceeds the applied displacement between both sides of the strip.

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Keywords: Dynamic crack propagation, Continuous dislocations model, Moving dislocation, Stress intensity factor, Crack opening displacement.

1. Introduction

The method of continuously distributed dislocations model is one of the classical methods for the analysis of crack problems. For the last few decades, classical methods have been giving way to numerical methods such as FEM (Finite Element Method), especially in the analysis of complicated or practical crack problems. However, the method of continuously distributed dislocations model is a powerful technique for expressing the singularities at the

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crack tips and in some special two dimensional cases, this method can give extremely precise solutions. Regarding this method, we can appreciate an excellent textbook by Hills et al. (1996).

The method of continuously distributed dislocations model can be applied to dynamic crack problems as well as static ones considering the effect of inertia due to the rapid crack propagation using Weertman's solution of a moving dislocation (1961). By this method, a running crack in a strip with clamped sides under anti-plane shear loading (mode III) was analyzed by Shioya et al. (1983). Related in-plane problem (mixed mode of I and II) was also analyzed by Fujimoto et al. (1985). From these problems, this paper focuses on a semi-infinite length mode I crack running rapidly in a strip with clamped sides and the numerical method of the singular integral equations for the problem is newly proposed. In the singular integral equations of this problem, the unknown functions are infinite or semi-infinite in their domains. Therefore, the current numerical method (e.g. Fujimoto (1991)) is not applicable because this method is for the problems in which the unknown functions are finite in their domains. Not only stress intensity factors but also crack opening displacements were calculated. The convergence with the increase of the number of the collocation points was investigated.

Nomenclature

G	shear modulus
ν	Poisson's ratio
κ	$=3-4\nu$ for the plane strain condition, $(3-\nu)/(1+\nu)$ for the plane stress condition
ρ	mass density
h	width of the strip
V	velocity of crack
x, y	moving coordinates with the velocity V in the x-direction
s_1	$=\sqrt{1-(V/c_1)^2}$
s_2	$=\sqrt{1-(V/c_2)^2}$
c_1	$=\sqrt{(\kappa+1)/(\kappa-1)}\sqrt{G/\rho}$: longitudinal wave velocity
c_2	$=\sqrt{G/\rho}$: shear wave velocity
u, υ	displacement components in the x- and y-directions, respectively

2. Model and formulation using continuous dislocations model

As shown in Fig. 1, let's consider a semi-infinite length crack propagating with a constant velocity V in a strip with an infinite length and clamped sides. This strip is assumed to be linearly elastic, homogeneous and isotropic. The boundary condition at both sides is as follows.

$$u=0,\ v=+v_0/2 \text{ on } y=+h/2; \quad u=0,\ v=-v_0/2 \text{ on } y=-h/2 \tag{1}$$

The running crack in Fig. 1 can be replaced with the continuous array of running dislocations. Let's denote the density of the dislocation (Burgers vector per unit crack length) by $f(x)\,(-\infty<x<0)$ and introduce the function $p(x)\,(-\infty<x<\infty)$ for satisfying the clamp condition at both sides (2). According to the reference by Fujimoto et al. (1985), we can obtain the following singular integral equations as the stress free condition on the crack surface and the clamp condition at both sides, respectively.

$$\int_{-\infty}^{0} M_1(x,\xi)f(\xi)d\xi+\int_{-\infty}^{\infty} N_1(x,\xi)p(\xi)d\xi+\frac{\kappa+1}{\kappa-1}\frac{Gv_0}{2h}=0 \quad (-\infty<x<0) \tag{2}$$

$$\int_{-\infty}^{0} M_2(x,\xi)f(\xi)d\xi+\int_{-\infty}^{\infty} N_2(x,\xi)p(\xi)d\xi=0 \quad (-\infty<x<\infty) \tag{3}$$

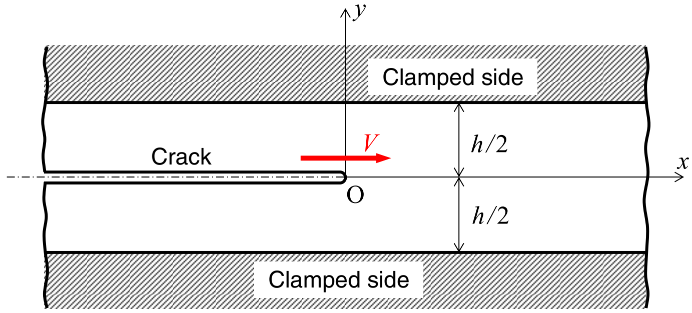

Fig. 1. Model of a semi-infinite length crack running in a strip with clamped sides.

The kernel functions in the equations (5) and (6) are as follows.

$$M_1(x,\xi)=\frac{G}{h(1-s_2{}^2)}\left\{-\frac{(1+s_2{}^2)^2}{2s_1{}^2}\coth\frac{\pi(x-\xi)}{s_1h}+2\coth\frac{\pi(x-\xi)}{s_2h}\right\}$$

$$N_1(x,\xi)=\frac{1}{h(1-s_2{}^2)}\left\{\frac{1+s_2{}^2}{2s_1{}^2}\tanh\frac{\pi(x-\xi)}{s_1h}-\tanh\frac{\pi(x-\xi)}{s_2h}\right\}$$

$$M_2(x,\xi)=\frac{1}{h(1-s_2{}^2)}\left\{\frac{1+s_2{}^2}{2s_1{}^2}\tanh\frac{\pi(x-\xi)}{s_1h}-\tanh\frac{\pi(x-\xi)}{s_2h}\right\}$$

$$N_2(x,\xi)=\frac{1}{Gh(1-s_2{}^2)}\left\{-\frac{1}{2s_1{}^2}\coth\frac{\pi(x-\xi)}{s_1h}+\frac{1}{2}\coth\frac{\pi(x-\xi)}{s_2h}\right\} \tag{4}$$

In the first term of the lhs of the equation (2) and the second term of the lhs of (3), there are singularities in the integrands. In this case, the principal value of the Cauchy type is taken. It should be noted that in deriving these equations, the stress fields of a moving dislocation (Weertman (1961)) and a moving point force (Fujimoto (1985)) were utilized. Further, we must consider the following conditions additionally.

$$\int_{-\infty}^{0} f(\xi)d\xi = v_0\,,\qquad \int_{-\infty}^{\infty} p(\xi)d\xi = -\frac{3-\kappa}{\kappa-1}Gv_0 \tag{5}$$

The stress intensity factor K_{I} and the crack opening displacement $v_{\mathrm{COD}}(x)$ can be expressed using the dislocation density function $f(x)$ as

$$K_{\mathrm{I}}=\lim_{x\to+0}\sqrt{2\pi x}\,\sigma_y(y=0)=\sqrt{\frac{\pi}{2}}\frac{4s_1s_2-(1+s_2{}^2)^2}{s_1(1-s_2{}^2)}G\lim_{x\to-0}\sqrt{-x}\,f(x)\,,\qquad v_{\mathrm{COD}}(x)=\int_x^0 f(\xi)d\xi\,. \tag{6}$$

3. Numerical method

In solving the singular integral equations (2) and (3) numerically with the additional conditions (5), normalization and transformation of variables are conducted in these equations. The variables with the domain [–∞, 0] are changed to those with the domain [–1, 1] by the following transformation.

$$x=h\log\frac{1+t}{2}\,,\qquad \xi=h\log\frac{1+\tau}{2} \tag{7}$$

Similarly, regarding the variables with the domain [–∞, ∞], the following transformation is conducted to have the new domain [–1, 1].

$$x = \frac{h}{2}\log\frac{1+t}{1-t}, \qquad \xi = \frac{h}{2}\log\frac{1+\tau}{1-\tau} \tag{8}$$

Then, we obtain from the equations (2), (3) and (5),

$$\int_{-1}^{1} M_1^*(t,\tau)F^*(\tau)d\tau + \int_{-1}^{1} N_1^*(t,\tau)P^*(\tau)d\tau + \frac{1}{2}\frac{\kappa+1}{\kappa-1} = 0 \quad (-1<t<1), \tag{9}$$

$$\int_{-1}^{1} M_2^*(t,\tau)F^*(\tau)d\tau + \int_{-1}^{1} N_2^*(t,\tau)P^*(\tau)d\tau = 0 \quad (-1<t<1), \tag{10}$$

$$\int_{-1}^{1} F^*(\tau)d\tau = 1, \tag{11}$$

$$\int_{-1}^{1} P^*(\tau)d\tau = -\frac{3-\kappa}{\kappa-1}, \tag{12}$$

where

$$M_1^*(t,\tau) = \frac{h}{G}M_1\left(h\log\frac{1+t}{2},\ h\log\frac{1+\tau}{2}\right), \qquad N_1^*(t,\tau) = hM_1\left(\frac{h}{2}\log\frac{1+t}{2},\ \frac{h}{2}\log\frac{1+\tau}{1-\tau}\right),$$

$$M_2^*(t,\tau) = hM_2\left(h\log\frac{1+t}{1-t},\ h\log\frac{1+\tau}{2}\right), \qquad N_2^*(t,\tau) = GhM_1\left(\frac{h}{2}\log\frac{1+t}{1-t},\ \frac{h}{2}\log\frac{1+\tau}{1-\tau}\right),$$

$$F^*(\tau) = \frac{h}{v_0}\frac{f\left(h\log\frac{1+\tau}{2}\right)}{1+\tau}, \qquad P^*(\tau) = \frac{h}{Gv_0}\frac{p\left(\frac{h}{2}\log\frac{1+\tau}{1-\tau}\right)}{1-\tau^2}. \tag{13}$$

Next, let's consider the equations (9) and (10) to be satisfied at the following collocation points, respectively.

$$t = \cos\frac{2i-1}{2m-1}\pi \quad (i = 1, 2, 3, \cdots, m-1), \qquad t = \cos\frac{i}{n}\pi \quad (i = 1, 2, 3, \cdots, n-1) \tag{14}$$

Based on the numerical method by Fujimoto (1991), equations (9)-(12) are discretized and approximated as follows.

$$\frac{h_f}{2}M_1^{(i,1)}F^{(1)} + h_f\sum_{j=2}^{m} M_1^{(i,j)}F^{(j)} + h_p\sum_{j=1}^{n} N_1^{(i,j)}P^{(j)} = -\frac{1}{2}\frac{\kappa+1}{\kappa-1} \quad (i = 1, 2, 3, \cdots, m-1), \tag{15}$$

$$\frac{h_f}{2}M_2^{(i,1)}F^{(1)} + h_f\sum_{j=2}^{m} M_2^{(i,j)}F^{(j)} + h_p\sum_{j=1}^{n} N_2^{(i,j)}P^{(j)} = 0 \quad (j = 1, 2, 3, \cdots, n-1), \tag{16}$$

$$\frac{h_f}{2}F^{(1)} + h_f\sum_{j=2}^{m} F^{(j)} = 1, \tag{17}$$

$$h_p\sum_{j=1}^{n} P^{(j)} = -\frac{3-\kappa}{\kappa-1}, \tag{18}$$

where

$$M_1^{(i,j)} = M_1^*\left(\cos\frac{2i-1}{2m-1}\pi,\ \cos\frac{2(j-1)}{2m-1}\pi\right) \quad (i=1,2,3,\cdots,m-1;\ j=1,2,3,\cdots,m),$$

$$N_1^{(i,j)} = N_1^*\left(\cos\frac{2i-1}{2m-1}\pi,\ \cos\frac{2j-1}{2n}\pi\right) \quad (i=1,2,3,\cdots,m-1;\ j=1,2,3,\cdots,n),$$

$$M_2^{(i,j)} = M_2^*\left(\cos\frac{i}{n}\pi,\ \cos\frac{2(j-1)}{2m-1}\pi\right) \quad (i=1,2,3,\cdots,n-1;\ j=1,2,3,\cdots,m),$$

$$N_2^{(i,j)} = N_2^*\left(\cos\frac{i}{n}\pi,\ \cos\frac{2j-1}{2n}\pi\right) \quad (i=1,2,3,\cdots,n-1;\ j=1,2,3,\cdots,n),$$

$$F^{(j)} = F^*\left(\cos\frac{2(j-1)}{2m-1}\pi\right)\cdot\sin\frac{2(j-1)}{2m-1}\pi \quad (j=1,2,3,\cdots,m)$$

$$P^{(j)} = P^*\left(\cos\frac{2j-1}{2n}\pi\right)\cdot\sin\frac{2j-1}{2n}\pi \quad (j=1,2,3,\cdots,n),$$

$$h_f = \frac{2\pi}{2m-1}, \quad h_p = \frac{\pi}{n}. \tag{19}$$

Solving the linear algebraic equations (15)-(18), we can obtain $F^{(j)}$ $(j=1,2,3,\cdots,m)$ and $P^{(j)}$ $(j=1,2,3,\cdots,n)$. The stress intensity factor and the crack opening displacement can be calculated by

$$K_{\mathrm{I}} = \sqrt{\frac{\pi}{2}}\frac{4s_1s_2-(1+s_2{}^2)^2}{s_1(1-s_2{}^2)}\frac{Gv_0}{\sqrt{h}}F^{(1)}, \tag{20}$$

$$v_{\mathrm{COD}}\left(x = h\log\frac{1+\cos\theta_k}{2}\right) = v_0\left\{\frac{h_f}{2}F^{(1)} + h_f\sum_{j=2}^{k-1}F^{(j)} + \frac{h_f}{2}F^{(k)}\right\}, \quad \theta_k = \frac{2(k-1)}{2m-1}\pi \quad (k=3,4,5,\ \cdots,m). \tag{21}$$

4. Numerical results

In this chapter, some numerical results are shown. Table 1 shows the convergence of the calculated non-dimensional stress intensity factors with the increase of the number of the collocation points, where non-dimensional stress intensity factor is defined as

$$\overline{K}_{\mathrm{I}} = \frac{\sqrt{h}}{Gv_0}K_{\mathrm{I}}. \tag{22}$$

Further, exact solution by Nilsson (1972) is as follows.

$$K_{\mathrm{I}} = \sqrt{\frac{\kappa+1}{\kappa-1}\frac{4s_1s_2-(1+s_2{}^2)^2}{s_1(1-s_2{}^2)}}\frac{Gv_0}{\sqrt{h}}. \tag{23}$$

This value is also indicated in this table. It can be seen from this table that the calculated stress intensity factor converges rapidly to the exact solution with the increase of m and n when the crack velocity V is not large. In the case when V is close to the Rayleigh wave velocity c_R ($c_R \cong 0.927c_2$ when $\kappa = 1.8$), many collocation points are needed for the convergence. However, precise numerical solutions can be obtained without fail by increasing m and n. Rapid convergence of the crack opening displacement was also verified. Figure 2 is the calculated crack opening displacements for various crack velocities with $m = n = 200$. As described in the reference by Fujimoto (1985), in the neighborhood of the crack tip, there is a region where the opening displacement exceeds the value v_0, the applied displacement between both sides and the larger the crack velocity is, the larger this excess becomes.

Table 1. Chart of the calculated non-dimensional stress intensity factors with the increase of *m* and *n* (in the case κ = 1.8).

n	m	$V/c_2=0.1$	$V/c_2=0.5$	$V/c_2=0.8$	$V/c_2=0.9$	$V/c_2=0.92$	$V/c_2=0.925$
10	10	2.229573	2.004926	1.408738	0.7532961	0.3745994	0.1928547
	20	2.229578	2.004928	1.408671	0.7532891	0.3735255	0.1928240
	30	2.229578	2.004929	1.408678	0.7532922	0.3735057	0.1927809
	40	2.229578	2.004929	1.408677	0.7532910	0.3735064	0.1927844
	50	2.229578	2.004929	1.408677	0.7532908	0.3735060	0.1927852
	100	2.229578	2.004929	1.408677	0.7532908	0.3735061	0.1927851
	200	2.229578	2.004929	1.408677	0.7532908	0.3735061	0.1927851
20	10	2.227804	2.000985	1.391263	0.7453340	0.4047537	0.2287867
	20	2.227803	2.000984	1.391259	0.7455369	0.4037181	0.2295843
	30	2.227804	2.000984	1.391259	0.7455378	0.4037189	0.2295799
	40	2.227804	2.000985	1.391259	0.7455369	0.4037180	0.2295795
	50	2.227804	2.000985	1.391259	0.7455369	0.4037179	0.2295795
	100	2.227804	2.000985	1.391259	0.7455369	0.4037180	0.2295797
	200	2.227804	2.000985	1.391259	0.7455369	0.4037180	0.2295797
30	10	2.227804	2.000983	1.391170	0.7442750	0.4043652	0.2317081
	20	2.227804	2.000983	1.391165	0.7444869	0.4032325	0.2326493
	30	2.227804	2.000983	1.391165	0.7444868	0.4032325	0.2326491
	40	2.227804	2.000983	1.391165	0.7444865	0.4032319	0.2326486
	50	2.227804	2.000983	1.391165	0.7444869	0.4032325	0.2326492
	100	2.227804	2.000983	1.391165	0.7444869	0.4032324	0.2326491
	200	2.227804	2.000983	1.391165	0.7444869	0.4032324	0.2326491
40	10	2.227804	2.000983	1.391169	0.7442539	0.4043060	0.2317098
	20	2.227804	2.000983	1.391165	0.7444668	0.4031778	0.2326301
	30	2.227804	2.000983	1.391165	0.7444668	0.4031778	0.2326299
	40	2.227804	2.000983	1.391165	0.7444668	0.4031778	0.2326299
	50	2.227804	2.000983	1.391165	0.7444668	0.4031779	0.2326300
	100	2.227804	2.000983	1.391165	0.7444668	0.4031778	0.2326299
	200	2.227804	2.000983	1.391165	0.7444668	0.4031778	0.2326299
50	10	2.227804	2.000983	1.391169	0.7442537	0.4043046	0.2317074
	20	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326280
	30	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326278
	40	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326278
	50	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326279
	100	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326278
	200	2.227804	2.000983	1.391165	0.7444665	0.4031762	0.2326278
100	10	2.227804	2.000983	1.391169	0.7442537	0.4043046	0.2317072
	20	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326279
	30	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	40	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	50	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	100	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	200	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
200	10	2.227804	2.000983	1.391169	0.7442537	0.4043046	0.2317072
	20	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326279
	30	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	40	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	50	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	100	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
	200	2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278
Exact (Nilsson (1972))		2.227804	2.000983	1.391165	0.7444665	0.4031761	0.2326278

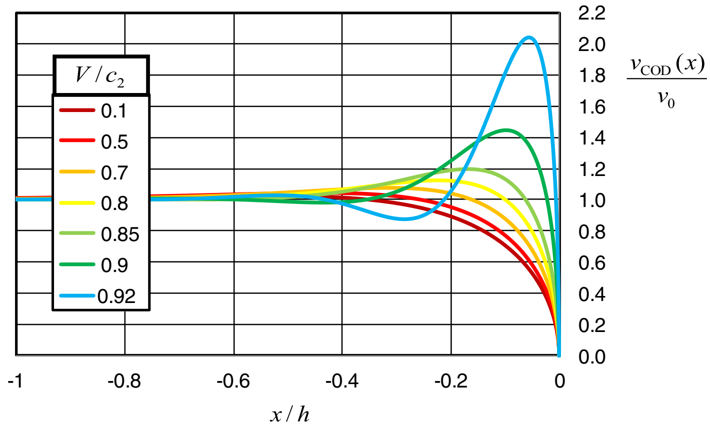

Fig.2. Calculated crack opening displacement for various crack velocity V (in the case $\kappa = 1.8$).

5. Conclusions

In this paper, a crack running in a strip with fixed sides was analyzed by the method of continuously distributed dislocations model with considering the effect of the inertia due to rapid crack propagation. The problem was reduced into a set of singular integral equations. A simple numerical method was proposed and applied to the problem. As the results, the followings have been obtained.

(1) Rapid convergence with the increase of the number of the collocation points was verified both for the calculated stress intensity factors and the crack opening displacements.

(2) The converged stress intensity factors agreed perfectly with the exact solutions.

(3) The crack is opening beyond the applied displacement between both sides of the strip. This phenomenon is remarkable when the crack velocity is large.

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