Recursive Quantum Functions, Avoidable Points, & Shadow Points in Recursive Analysis

Iraj Kalantari 1,2
Department of Mathematics
Western Illinois University
Macomb, IL, USA

Larry Welch 3
Department of Mathematics
Western Illinois University
Macomb, IL, USA

Abstract
In this paper we summarize our approach and findings in a point-free setting for recursive topology and recursive analysis. Recursive analysis has received intensive attention recently and our approach, while differing from those of other schools, does have connections to them. Here, we also present some of our findings on quantum and total recursive functions on the reals, introduce our classification of nonrecursive points, and remark on the connections between our work and the works of others.

1 Introduction

In real analysis, a common way to ‘construct’ the real numbers from the rationals is through Cauchy sequences. A sequence \( \{a_n\} \) is Cauchy if for every \( n \) there is a number \( f(n) \) such that the elements of the sequence beyond \( \{a_{f(n)}\} \) are closer to each other than \( 2^{-n} \). If both the sequence and \( f \) are computable, then the sequence converges to a recursive real. The recursive reals and recursive analysis have been extensively researched; instead of enumerating the long list of the early authors in that field, we refer the reader to the comprehensive (in 1998 & albeit not complete now) bibliography by Brattka & Kalantari [3].

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2 Email: i-kalantari@wiu.edu
3 Email: l-welch@wiu.edu

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In general topology there are three classical approaches: Fréchet’s abstract spaces, Hausdorff’s neighborhood classes and Kuratowski’s closure classes. In all these approaches, points are 
*a priori* objects with which one works. But an approach to a point through a sequence of other points or a nested sequence of neighborhoods is often fruitful.

In recognition of this, we have chosen to use a point-free approach to build a link between classical topology and recursive topology (where a point-free approach is particularly germane since recursive points are defined as limits of sequences of approximants). The objects we handle are therefore not points, but neighborhoods in the sense of Hausdorff. These neighborhoods form a subbasis of the space. The point-free theory exists at the level of our object language. At the level of the metalanguage we do not hesitate to refer to points, and many of our results are theorems about the properties of points.

Our machinery is basis-dependent, and therefore for each space we must choose a subbasis with certain specific properties in order for our approach to yield what we want. As it happens, many commonly studied spaces, such as $\mathbb{R}^n$, have such subbases, which are among the most commonly used.

In analogy with $\mathbb{R}^n$ we have chosen to consider connected, second countable, regular spaces with at least two points. The criterion of second countability is necessary if we are to study a space from a point-free, recursive perspective because each member of our subbasis must be named with a nonnegative integer. The regularity and second countability in effect make each of our spaces homeomorphic to a nontrivial connected subspace of the Hilbert cube $[0,1]^\omega$. As a result, our spaces are metrizable, at least from the perspective of the classical mathematician. Some, such as $\mathbb{R}$ topologized by open intervals with rational endpoints, are clearly also recursively metrizable. Whether all of our spaces are recursively metrizable is an open question; we conjecture that they are not. Up to the present we have neither studied metrizability on our spaces nor developed a formal definition of a recursive metric that would be of use in our setting.

Besides these criteria on each space we study, we also impose a few constraints on the subbasis we choose for it. Of course we take that subbasis to be countable and indexed by the natural numbers. We also require that each member have compact closure; this guarantees that a sequence of basic open sets $\{\alpha_i : i \in \omega\}$ with each $\alpha_{i+1} \subseteq \alpha_i$ will have nonempty intersection. For the sake of ease of topological reasoning we require that each member of our subbasis be connected. Finally, for the recursiveness properties of points to work out correctly we also require that our space be 
*semi-recursively presented*, by which we mean that for our subbasis the relations $\alpha \cap \beta = \emptyset$, $\alpha \subseteq \beta$, and $\overline{\alpha} \subseteq \beta$ should be recursively decidable.

Our approach to addressing points bears a close similarity to that of Weihrauch in that we ‘name’ each point with a sequence of open sets containing it. In clear distinction to Weihrauch, we use a recursive one-to-one numerical indexing of the subbasis. This does not in any way constrain our
development, because the semi-recursive presentation of our space allows for this to be done. Though we could have chosen to use as a name for a point a sequence of all members of the subbasis containing it, we choose instead to name it with a closure-nested sequence converging to it, which we call a *sharp filter*. In a semi-recursive space, the one approach to naming points may be readily substituted for the other. By applying our naming method to any space, we can obtain, by a natural procedure, a new space whose points can be studied by the point-free method. When the original space is as described above, the new space is homeomorphic to it.

Our study is meant to shed light on some of the constructions of researchers into recursive analysis in the 1960’s, and in particular those of the Russians such as Cėitin, Shanin, Zaslavski̇i, and Orevkov. These researchers asked the natural question, ‘In the context of an effective platform for the reals, which theorems of classical analysis carry over and which don’t, and how badly do some fail?’ For example, effective versions of the Intermediate Value Theorem and the Max-Min Theorem hold, and while the effective version of the Max-Min Theorem for a sequence of functions holds, the same is not true for the Intermediate Value Theorem. Many of their constructions are ‘recursive functions’ that, when studied only with recursive points in mind, seem to violate the classical behavior of continuous functions. For instance, Specker [31] proved the existence of a recursive function on $I = [0, 1]$ such that $f(I) \subseteq I$, and although $f$ achieves a maximum recursive value, that value is not achieved at a recursive real. In a similar vein, Orevkov [25] constructs a ‘recursive function’ from the unit square to itself that violates Brouwer’s Fixed Point Theorem in that it is continuous but has no recursive fixed point. Cėitin [4] used ideas similar to Specker’s to show that there exists a recursive function $f$ on $[-1, 1]$ which is defined at 0 but not defined on all points of any neighborhood of 0. Zaslavski̇i [35] showed, among other things, the existence of an unbounded recursive function on $I$.

Certain of the Russian researchers, Cėitin, Shanin, and Zaslavski̇i, in particular, frequently considered only the behavior of their functions at recursive points. In contrast, Goodstein [8], Pour-El & Richards [26], and others have extensively investigated recursive functions that are entire on a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$. The findings of Pour-El & Richards are significant contributions since the works of the Russian school. More recent approaches, such as those of Weihrauch and Brattka, allow for investigation of functions that need not be total. We add our voice to these, with the proviso that since we are specifically interested in the work of the Russians, we choose always to have our functions defined on all the recursive points.

With this in mind, we consider two distinct effectivizations for functions. First are those that are defined at all points of the space, which we simply call recursive functions. When the space is a closed interval in $\mathbb{R}$, our recursive functions are precisely the classically continuous recursive functions studied by Aberth [1], Goodstein [8], Lacombe [20], Mazur [22], Myhill [24], Pour-El
& Richards [26], Rice [27], Shanin [29], and many others. Next are those that fail to be defined at some nonrecursive points, which we call recursive quantum functions. Those points that can be excluded from domains of recursive quantum functions we call avoidable points, and those that cannot we call shadow points. Because we are interested in analyzing the Russian work, we require that these functions be defined on all recursive points, at a minimum. Though a recursive quantum function on a closed interval in $\mathbb{R}$ may be undefined at certain points of that interval, and may behave pathologically in some regards, it also displays some of the properties of classically continuous functions on the interval; for instance, it satisfies an effective version of the Intermediate Value Theorem. Such functions have been studied by Cenit [4], Cenit & Zaslavski [5], Shanin [29], Zaslavski [35], et al.

A question that can be raised in this context, and that is just as natural as the original question raised by the researchers of the 1960’s, is which of the results of classical recursive analysis about total recursive functions carry over to all quantum recursive functions, and which can in some way be refuted when totality is not considered.

In this paper we summarize our approach and several of our findings. The proofs of some of our theorems can be found in our papers [13], [14], [15], and [16]; others are forthcoming.

2 Points & functions

The basic way in which we deal with points is through sharp filters. It is important to have a criterion that states in a point-free way that two sharp filters converge to the same point, and to be able to say what it means for a sharp filter to converge to a point $x$.

Definition 2.1 Let $X$ be a topological space with basis $\Delta = \{\delta_i : i \in \omega\}$. A sequence $\{\alpha_i : i \in \omega\} \subseteq \Delta$ is a sharp filter if the following conditions hold:

\[(\forall i)(\alpha_{i+1} \subseteq \alpha_i), \text{ and } (\forall \beta, \gamma)[(\beta \subseteq \gamma) \Rightarrow \exists i[(\alpha_i \cap \beta = \emptyset) \lor (\alpha_i \subseteq \gamma)]]\].

We say $A = \{\alpha_i : i \in \omega\}$ converges to $x$, or $x$ is the limit of $A$, and write $A \xrightarrow{\omega} x$, if $\bigcap \alpha_i = \{x\}$. For $A = \{\alpha_i : i \in \omega\}$ and $B = \{\beta_i : i \in \omega\}$ sharp filters, we say $A$ is equivalent to $B$, and write $A \equiv B$, if $(\forall i)[(\alpha_i \cap \beta_i \neq \emptyset)].$ Let $A = \{\alpha_i : i \in \omega\}$ be a sharp filter in $\Delta$. Then $A$ is recursive if there is a recursive function $f : \omega \to \omega$ such that for every $i$, $\alpha_i = \delta_{f(i)}$. Let $\text{Rec}(X) = \{x : (A \xrightarrow{\omega} x) \land (A \text{ is a recursive sharp filter)}\}$.

It turns out that working with the classes of sharp filters each converging to the same point of $X$ is equivalent to working with the points of $X$. Namely, in [13] we show that for a space $X$ of our type, the space of sharp filters, $\mathcal{X}^* = \{[A] : A \text{ is a sharp filter in } \Delta\}$, where $[A] = \{B : (B \text{ is a sharp filter}) \land (A \equiv B)\}$, is homeomorphic to $X$, where the topology on $\mathcal{X}^*$ is induced by the topology on $X$ in a natural way.

A full recursive function on one of our spaces is generated by a set function,
called a \textit{recursive correspondence}, on a subbasis for the space. A recursive quantum function is similarly generated by a \textit{recursive quantum correspondence}.

\textbf{Definition 2.2} Let $(X, \Delta_X)$ and $(Y, \Delta_Y)$ be topological spaces. A partial recursive function $F : \Delta_X \rightarrow \Delta_Y$ is a \textit{recursive correspondence} if

(i) $(\forall \alpha, \beta)[F(\alpha) \downarrow \land F(\beta) \downarrow \land \alpha \subseteq \beta] \Rightarrow [F(\alpha) \subseteq F(\beta)]$,

(ii) $(\forall \alpha, \beta)[F(\alpha) \downarrow \land F(\beta) \downarrow \land \alpha \subseteq \beta] \Rightarrow [F(\alpha) \subseteq F(\beta)]$

(if $F$ has properties (1) and (2), we say $F$ is \textit{monotone}); and

(iii) $(\forall B$ a sharp filter in $\Delta_X)(\exists A$ a sharp filter in $\Delta_X)$

$[(A \equiv B) \land (F(A) \downarrow) \land (F(A) is a sharp filter in \Delta_Y)]$.

A partial recursive function $F : \Delta_X \rightarrow \Delta_Y$ is a \textit{recursive quantum correspondence} if

(i) $F$ is monotone, and

(ii) $(\forall B$ a recursive sharp filter in $\Delta_X)(\exists A$ a recursive sharp filter in $\Delta_X)$

$[(A \equiv B) \land (F(A) \downarrow) \land (F(A) is a sharp filter in \Delta_Y)]$.

\textbf{Definition 2.3} Let $F : \Delta_X \rightarrow \Delta_Y$ be a correspondence. Define $f_F : X \rightarrow Y$ by $f_F(x) =$ the unique point in $\bigcap F(A)$, where $x \in X$, and $A$ is a sharp filter in $\Delta_X$ such that $A \nsubseteq x$, $F(A) \downarrow$ and $F(A)$ is a sharp filter in $\Delta_Y$.

For a recursive correspondence $F$ we refer to the function it generates on the space, $f_F$, as a \textit{recursive function}. If $F$ is a recursive quantum correspondence, we refer to the function $f_F$ as a \textit{recursive quantum function}.

When we wish to forcefully distinguish recursive correspondences from recursive quantum correspondences, we refer to the former as \textit{recursive (full) correspondences}.

We have to be careful that these set functions truly generate mappings from points to points.

\textbf{Definition 2.4} A recursive quantum correspondence $G : \Delta_X \rightarrow \Delta_Y$ is called \textit{honest} if for every sharp filter $A$ (recursive or not), with $A \subseteq \text{dom}(G)$, there is a sharp filter $B \subseteq A$ where $G(B)$ is a sharp filter in $\Delta_Y$.

This matter of \textit{honesty} is crucial to our work, since there exists a recursive quantum correspondence that is not honest in $(\mathbb{R}, \Delta_{\mathbb{R}})$ (see [15]). As it happens, though, any such function that behaves properly with respect to the recursive points, even if \textit{dishonest} on some nonrecursive input, has an honest equivalent.

\textbf{Theorem 2.5} Every recursive quantum correspondence has an honest equivalent. That is, if $F$ is a recursive quantum correspondence, then there is an honest recursive quantum correspondence $G \subseteq F$ such that $f_G = f_F$. 

83
3 Classical recursive analysis

It is essential to note that our sharp filters and recursive (full) correspondences generate the same recursive points and recursive functions that others have studied. Let $\text{Rec}(\mathbb{R})$ be the set of recursive reals in our setting (that is recursive sharp filters). Let $\mathbb{R}_{\text{recursive}}$ be the set of points in $\mathbb{R}$ which are recursive according to Goodstein/Pour-El-Richards classification.

**Theorem 3.1** $\text{Rec}(\Delta \mathbb{R}) = \mathbb{R}_{\text{recursive}}$.

**Theorem 3.2** Any recursive (full) correspondence from reals to reals generates a classically recursive function from reals to reals. For any classically recursive function $f$ from the reals to the reals, there is a recursive (full) correspondence from the reals to the reals which generates exactly that $f$.

One of the primary facts about recursive functions on the real numbers is the following.

**Theorem 3.3** (Grzegorczyk, Ceitín, Kreisel-Lacombe-Schoenfield) Every total recursive function on $I$ (an interval in $\mathbb{R}$) is continuous.

This theorem also carries over to recursive quantum correspondences too, in that a recursive quantum function, though not necessarily defined on all of $X$, is still continuous on its domain, and its domain contains uncountably many nonrecursive points.

A recursive quantum correspondence on a closed interval $[a, b]$ has a couple of properties that it shares with continuous functions that have that interval as their domain: It satisfies an effective version of the Intermediate Value Theorem just as if it were continuous on the whole interval $[a, b]$; and hence, if its range also is a subset of $[a, b]$, it also satisfies an effective version of the Brouwer Fixed Point Theorem.

Also recursive quantum correspondences have an effective version of a classical intersection property:

**Theorem 3.4** Let $\{F_k : k \in \omega\}$ be a uniformly recursive sequence of recursive quantum correspondences from $\Delta_X$ to $\Delta_Y$ such that for all $x \in \text{Rec}(X)$ and all $j, k \in \omega$, $f_{F_j}(x) = f_{F_k}(x)$. Then there is a recursive quantum correspondence $H \subseteq F_0$ such that $f_H = \bigcap_k f_{F_k}$.

Note, though, that a recursive quantum function can be far from total. For there is a recursive quantum function on $[0, 1]$ that is not just partial, but nonextendible to a continuous function of larger domain. This function is of unbounded variation, and its domain, which is an open set, can be made to be as small in Lebesgue measure as desired.

Furthermore, any recursive quantum function on a (possibly infinite) interval can be restricted to a domain of Lebesgue measure zero in such a way that the restriction is also a recursive quantum function. Because this is true of the nonextendible function as well as of any other, there is a recursive quantum
function of unbounded variation on \([0, 1]\) which cannot be extended to a full recursive function on \([0, 1]\), whose domain is of Lebesgue measure zero.

The proofs of these results can be adapted to reach findings similar to those of Specker, Cėitin and Zaslavskiǐ cited above, while the resulting functions remain nonextendible.

4 Avoidable and shadow points

An avoidable point is one for which there is a recursively enumerable sequence of ‘witnesses’ to the fact that it is not a recursive point. Each of these witnesses is a basic open set excluding the point, and the entire sequence of witnesses forms a cover of the set of the recursive points of the space. In this sense the avoidable points are those that are ‘recursively bounded away from’ the recursive points. In section 5 we relate avoidable points to domains of quantum recursive correspondences.

Definition 4.1 Let \(\langle X, \Delta X \rangle\) be a topological space. Suppose there is a partial recursive function \(\phi : \omega \to \omega\) and \(x \in X\), such that for any \(n\)

(i) if \(\psi_n\) is a recursive sharp filter (in a standard enumeration of all sharp filters), then \(\phi(n) \downarrow\); and

(ii) if \(\phi(n) \downarrow\) and \(\psi_n(\phi(n)) \downarrow\) (which is true if \(\psi_n\) is a sharp filter), then \(x \notin \psi_n(\phi(n))\).

We then say \(\phi\) is an avoidance function for \(x\), or \(x\) is avoidable via \(\phi\). If \(x\) is avoidable via some \(\phi\), we say \(x\) is avoidable. \(\phi\) is an avoidance function if it is an avoidance function for some \(x\). For an avoidance function \(\phi\), let \(S_\phi = \{ x \in X : \phi\) is an avoidance function for \(x\}\), and refer to \(S_\phi\) as the spectrum of \(\phi\).

If a point \(x\) is nonrecursive and not avoidable, we say \(x\) is a shadow point.

Similar to \(\text{Rec}(X)\) denoting the set of recursive points of \(X\), we use \(\text{Av}(X)\) and \(\text{Shad}(X)\) to denote the set of avoidable and shadow points of \(X\) respectively. Thus \(X = \text{Rec}(X) \cup \text{Av}(X) \cup \text{Shad}(X)\).

Proposition 4.2 Let \(\phi\) be an avoidance function. Then \(S_\phi\) is a nowhere dense, perfect, closed set, containing no isolated points, and every point of \(S_\phi\) is a point of condensation of \(S_\phi\).

Theorem 4.3

(i) The set of all avoidable points in \(X\) is of first category.

(ii) The set of shadow points in \(X\) is of second category.

(iii) Furthermore, \(\text{Shad}(X)\) condenses at every point of \(X\).

Specker [31] proved existence of a recursive sequence of recursive reals whose limit is not a recursive real. We have extended that result to a bounded sequence of points that may not necessarily be recursive and may not have a
unique limit point.

**Theorem 4.4** For every $\alpha \in \Delta$, there is a recursive sequence $R = \{\rho_n \subseteq \alpha : n \in \omega\}$ of basic open sets such that:

(i) $(\forall i, j)[i \neq j \Rightarrow \overline{\rho_i} \cap \overline{\rho_j} = \emptyset]$; and

(ii) if $S = \{y_n : n \in \omega\}$ is any set of points where $y_n \in \rho_n$ for each $n$, then all limit points of $S$ are avoidable.

When $S$ is a sequence of recursive points we get this generalization of Specker’s theorem.

**Theorem 4.5** For any $\alpha \in \Delta$, there exists a uniformly recursively enumerable sequence of recursive points in $\alpha$ all of whose limit points are avoidable.

5 Domains & exdomains

Let $f : X \rightarrow Y$ be a recursive quantum function. Because the domain of $f$ may not be all of $X$, and because that fact is of importance to us, we define the **exdomain** of $f$ to be $\text{exdom}(f) = X - \text{dom}(f)$. The essential theorem linking exdomains with the avoidable and shadow points is this.

**Theorem 5.1** Let $F : \Delta_X \rightarrow \Delta_Y$ be a recursive quantum correspondence. Then $\text{exdom}(f_F) \subseteq \text{Av}(X)$, and thus $\text{dom}(f_F) \supseteq \text{Shad}(X)$.

Thus the domain of a recursive quantum function $f_F$ contains all the recursive and shadow points, and at least some of the avoidable points, while $\text{exdom}f_F$ consists solely of avoidable points. In fact, we prove in [13] that the avoidable points are exactly those points that can be excluded from domains of recursive quantum functions.

Because recursive quantum functions are continuous on their domains, and because the $\text{Rec}(X)$ is dense in $X$, it follows that if two recursive quantum functions agree on recursive points, then they also agree on all of the shadow points and avoidable points which are in the domains of both.

Given that an exdomain consists entirely of avoidable points, the question of how spectra interact with exdomains naturally arises.

**Theorem 5.2** If $F : \Delta_X \rightarrow \Delta_Y$ is a recursive quantum correspondence, and $\phi$ is a recursive avoidance function, then there is a recursive quantum correspondence $G \subseteq F$ such that $\text{dom}(f_G) = \text{dom}(f_F) - S_\phi$.

By starting with a function $f_F$ which is total on $X$, we get $\text{exdom}(f_G) = S_\phi$, proving that a spectrum is one type of exdomain.

The avoidable points have certain density properties which are best stated in terms of domains and exdomains of recursive quantum functions. Let $F$ be a recursive quantum correspondence, Then every point of $X$ is a point of condensation of $\text{Av}(X) \cap \text{dom}(f_F)$ and thus is a point of condensation of
Av(\(X\)). Also each \(x \in \text{exdom}(f_F)\) is a point of condensation of \(\text{exdom}(f_F)\). Finally, \(\text{dom}(f_F)\) is \(G_\sigma\) while \(\text{exdom}(f_F)\) is \(F_\sigma\) and contains no isolated points.

6 Trees of sharp filters

Recursive quantum correspondences turn out to be \(\Pi^0_1\) objects, so we apply \(\Pi^0_1\) trees to our study of them. Through the use of recursively bounded \(\Pi^0_1\) trees, we can capture nowhere dense perfect sets of avoidable points, and we can use this ability to determine some of the properties of domains of recursive quantum functions. We do this by putting basic open sets on the nodes of a tree \(T\) in such a way that its infinite branches form sharp filters, and so that the set of points \(T\) to which those branches converge is nowhere dense.

The principal fact for us about \(\Pi^0_1\) trees is this: If \(T\) is a recursively bounded \(\Pi^0_1\) tree of sharp filters with no recursive branch then \(T\) is the spectrum of an avoidance function, so that \(T \subseteq \text{Av}(X)\). Furthermore, there is a recursive quantum correspondence \(G\) such that \(\text{exdom}(f_G) = T\).

Similarly we have the following important result.

**Theorem 6.1** Let \(F : \Delta_X \to \Delta_Y\) be a recursive quantum correspondence and \(\alpha \in \Delta\). Then there is a recursive quantum correspondence \(G \subseteq F\) such that \(\|\text{dom}(f_F) - \text{dom}(f_G)\| = T \subseteq \alpha \cap \text{Av}(X)\), for some recursively bounded \(\Pi^0_1\) tree of sharp filters \(T\) with \(2^{\aleph_0}\) infinite branches. Consequently, \(\|\text{dom}(f_F) - \text{dom}(f_G)\| \cap \alpha\| = 2^{\aleph_0}\).

One interesting fact about the reals is that for any \(x \in \text{Av}(\mathbb{R})\), there is a complete recursive tree \(T\) of sharp filters in \(\mathbb{R}\) such that \(x \in T\). Whether for each such \(x\) there is a tree of avoidable points that contains \(x\), we do not know.

By building a nonrecursively bounded \(\Pi^0_1\) tree through a finite injury priority argument, we can produce a nowhere dense perfect set of shadow points.

**Theorem 6.2** Given any \(\delta \in \Delta\), there is a \(\Pi^0_1\) tree \(T\) of sharp filters having \(2^{\aleph_0}\) infinite branches, such that \(T \subseteq \delta \cap \text{Shad}(X)\).

Trees of our kind are nowhere dense, and there are only countably many recursive trees. But \(\text{Shad}(X)\) is of second category, so not every shadow point can be found on such a tree.

A very comprehensive exposition of the theory of \(\Pi^0_1\) sets and their use in recursive mathematics can be found in Cenzer & Remmel [6].

7 Nondensity & interpolation theorems

We saw in Theorem 6.1 that any recursive quantum function \(f_F\) can be restricted to a recursive quantum subfunction \(f_G\) whose domain differs from the domain of \(f_F\) only on a basic open set \(\alpha\), with \(\|\text{dom} f_F - \text{dom} f_G\| = 2^{\aleph_0}\). In fact, we can extend this result to other cardinalities.
Theorem 7.1 Let $F : \Delta_X \rightarrow \Delta_Y$ be a recursive quantum correspondence, and let $\alpha \in \delta$. Then there are recursive quantum correspondences $G$ and $H$ such that $G \subseteq H \subseteq F$ and $\text{dom}(f_H) - \text{dom}(f_G)$ contains exactly one point, and that point lies in $\alpha$. (And thus no proper recursive quantum correspondence can be interpolated between $G$ and $H$.)

Corollary 7.2 Let $F : \Delta_X \rightarrow \Delta_Y$ be a recursive quantum correspondence and let $\kappa \leq \aleph_0$. Then there are recursive quantum correspondences $G$ and $H$ such that $G \subseteq H \subseteq F$ and $\|\text{dom}(f_H) - \text{dom}(f_G)\| = \kappa$.

A consequence of Theorem 7.1 is that we cannot always interpolate a recursive quantum function between two nested recursive quantum functions. In contrast to this we have the following theorem for the case of two nested recursive quantum functions whose domains differ by at least two points.

Theorem 7.3 Let $F$ and $G$ be recursive quantum correspondences from $\Delta_X$ to $\Delta_Y$ with $G \subseteq F$, and let $\text{dom}(f_F) - \text{dom}(f_G)$ contain at least two points. Then there is a recursive quantum correspondence $H : \Delta_X \rightarrow \Delta_Y$ such that $H \subseteq F$ and $f_G \subsetneq f_H \subsetneq f_F$.

Thus interpolation of a quantum recursive function between two others is possible in exactly the same circumstances in which it is possible for the classical mathematician to interpolate a function between them.

References


