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TODIM based method to process heterogeneous information

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Abstract

Multicriteria decision making (MCDM) problems are getting increasingly difficult over the years. Nowadays, it is very common to have a problem where heterogeneous types of information must be processed before making a decision. From a wide variety of MCDM methods, just a few of them are able to process mixed types of information at the same time and most of these methods relies on transformations that may cause problems and/or be unjustified. In this paper we bring a modular interpretation of the TODIM (an acronym in Portuguese for Interative Multi-criteria Decision Making) method to handle heterogeneous data types simultaneously, in a systematic way. We argue that the method have a modular capacity which is the novelty of our approach. Using this interpretation, the whole problem with heterogeneous information is broken into modules and processed in a straightforward way. Two examples are used to illustrate the approach showing the effectiveness and practicability.

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1. Introduction

Multi-criteria decision making (MCDM) methods have been widely applied to support decision makers to select the best alternative regarding to multiple criteria among a finite number of alternatives. Despite their usefulness, many of the existing methods are often applied to data of the same type because of their inability to deal adequately with heterogeneous data type, e.g., crisp, stochastic, linguistic, etc... Indeed, for real world-problems the decision matrix may be filled out with different kind of data. For example, in the decision of choosing which car model to buy, one would analyze how the car models satisfy the attributes that he/she considers important, as the price of the car (crisp), the engine power (crisp), the comfort, which is an imprecise concept that could be evaluated by linguistic variables as *very comfortable*, *comfortable* and *not comfortable*, the fuel consumption that depends on the driver of the car, the conditions of the roads, the average speed, etc... that could be evaluated as a random variable.

Another problem that arises in MCDM is that, as shown in [1], the human thinking presents a strong bias in situation involving risks. In this same work, the Prospect Theory was proposed. The TODIM method [2] is one of the first MCDM methods based on the Prospect Theory. The idea of the TODIM is to compare the alternatives with respect to each criterion in a pairwise fashion in terms of gains and losses. The gain and losses are then passed to the prospect function to get the partial dominances and, then, the partial dominances are aggregated to form the final dominance. The rank order of the alternatives is basically based on this final dominance. In the standard formulation, the TODIM method only deals with crisp numbers. However, it was

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extended to deal with fuzzy numbers [3], intuitionistic fuzzy information [4], intuitionistic fuzzy information in a random environment [5], interval-valued intuitionistic fuzzy information [6], probability distributions [7] and hesitant fuzzy [8].

Although there are many adaptation of TODIM, to deal with several types of information, little efforts were made to try to adapt them to be able to compute with heterogeneous data. The TODIM method was extended to deal with crisp numbers, interval-valued numbers and fuzzy numbers at the same time in [9]. In this paper, we argue that the TODIM method use a modular strategy to process the information. The method break the whole problem into small modules, process these modules separately and, only then, aggregate the results of each module to obtain a general quantity. Using this interpretation, the TODIM is able to deal with heterogeneous data types directly, independently of the type of the information, in a systematic way without the need of data transformation. All these advantages are obtained in a very intuitive and simple manner where the TODIM have a huge resemblance with the standard formulation.

The remainder of this paper is organized as follows. In Section 2, some preliminary background on interval data, fuzzy sets, intuitionistic fuzzy sets are provided. In Section 3, we shortly revise the TODIM. In Section 4, the Modular-TODIM method is presented. A discussion of some aspects that may affect the Modular approach is presented in Section 5. In Section 6, two examples are presented to illustrate the method and the results show the feasibility of the approach. In Section 7, some conclusions and directions for future work are given.

2. Basic concepts and definitions

In this section we present the definitions of the data types that will be used in this paper, which are, crisp numbers, interval numbers, fuzzy sets and intuitionistic fuzzy sets. Also, some necessary concepts are introduced. The crisp numbers are denoted by lower case letters (a), interval numbers are denoted by bold lower case letters ($\mathbf{a}$), fuzzy sets are denoted by lower case letters with tilde ($\tilde{a}$) and intuitionistic fuzzy sets are denoted by capital letters with tilde ($\tilde{A}$).

Consider the problem of selecting one between m alternatives. Each alternative is evaluated with respect to n criteria. Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set with the m alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set with the n criteria. In general, the criteria can be classified into two types: *benefit* and *cost*. For *benefit* criterion, higher value is better while for the *cost* criterion is valid the opposite. We can summarize the Multicriteria Decision Making (MCDM) problem into the following matrix:

$$DM = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{pmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{21} & s_{22} & \ldots & s_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{m1} & s_{m2} & \ldots & s_{mn} \end{pmatrix} \end{array}$$

where s_{ij} represents the rating of the ith alternative evaluated with respect to the jth criterion. In this work the ratings s_{ij} can assume any of the already mentioned forms.

2.1. Interval-numbers

Next, we provide some basic definitions to work with interval numbers. We start with two definitions: interval number and Euclidean distance.

Definition 2.1. *[10]. The object* $\mathbf{a} = \left[a^L, a^U\right]$*, where* $a^L \leq a^U$*, defined on the real line, is called interval number. The values* a^L *and* a^U *stand for the lower and upper bounds of* $\mathbf{a}$*, respectively. The center and the width of an interval number* $\mathbf{a} = \left[a^L, a^U\right]$ *are given by* $m(\mathbf{a}) = (a^L + a^U)/2$ *and* $w(\mathbf{a}) = (a^U - a^L)$*, respectively.*

Definition 2.2. *[11]. Let* $\mathbf{a} = \left[a^L, a^U\right]$ *and* $\mathbf{b} = \left[b^L, b^U\right]$ *be two interval numbers. The Hamming distance between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$d(\mathbf{a}, \mathbf{b}) = \frac{1}{2}\left(\left|a^L - b^L\right| + \left|a^U - b^U\right|\right) \tag{1}$$

Now, it is necessary to define a way to rank the interval numbers. Since there is uncertainty with interval numbers, these rankings are not likely to be complete certain. Facing this problem, [12] has proposed a quantity, called degree of preference, to measure the degree of preference of an interval number over another one.

Definition 2.3. *[12]. Let* $\mathbf{a} = \left[a^L, a^U\right]$ *and* $\mathbf{b} = \left[b^L, b^U\right]$ *be two interval numbers. The degree of preference of* $\mathbf{a}$ *over* $\mathbf{b}$ *is given by*

$$P(\mathbf{a} > \mathbf{b}) = \frac{\max\left\{0, a^U - b^L\right\} - \max\left\{0, a^L - b^U\right\}}{a^U - a^L + b^U - b^L} \tag{2}$$

Definition 2.4. *[12] Let* $\mathbf{a} = \left[a^L, a^U\right]$ *and* $\mathbf{b} = \left[b^L, b^U\right]$ *be two interval numbers. We say that* $\mathbf{a}$ *is superior to* $\mathbf{b}$*, denoted by* $\mathbf{a} > \mathbf{b}$*, if* $P(\mathbf{a} > \mathbf{b}) > P(\mathbf{b} > \mathbf{a})$*. If* $P(\mathbf{a} > \mathbf{b}) = P(\mathbf{b} > \mathbf{a})$*, then we say that* $\mathbf{a}$ *is indifferent to* $\mathbf{b}$*, denoted by* $\mathbf{a} = \mathbf{b}$.

For the generalization of TODIM, we use the Definition 2.3 only through the Definition 2.4. Due this fact we present the following corollary to simplify the method.

Corollary 2.1. *Let* $\mathbf{a} = \left[a^L, a^U\right]$ *and* $\mathbf{b} = \left[b^L, b^U\right]$ *be two interval numbers, then* $\mathbf{a} > \mathbf{b}$*, in the sense of Definition 2.4, if and only if* $m(\mathbf{a}) > m(\mathbf{b})$.

By Corollary 2.1 we can determine which interval is preferable in the sense of Definition 2.4 simply comparing the center of the intervals. The last definition about interval numbers provides a way to normalize interval data.

Definition 2.5. *Let* $\mathbf{s}_{ij} = \left[s_{ij}^L, s_{ij}^U\right]$ *be an interval numbers used to evaluate the ith alternative with respect to jth criterion. The normalization of the interval number is given according to the following expressions*

$$r_{ij}^L = \frac{s_{ij}^L}{\max\limits_i s_{ij}^U} \qquad \text{and} \qquad r_{ij}^U = \frac{s_{ij}^U}{\max\limits_i s_{ij}^U}, \qquad i = 1, \ldots, m \tag{3}$$

2.2. Fuzzy sets

In this section we provide some basic definitions of fuzzy sets.

Definition 2.6. *[13]. A fuzzy set* $\tilde{a}$ *in a universe of discourse* X *is characterized by a membership function* $\mu_{\tilde{a}} : X \to [0, 1]$*. In other words, a fuzzy set* $\tilde{a}$ *is a mathematical object of the form*

$$\tilde{a} = \{\langle x, \mu_{\tilde{a}}(x)\rangle : x \in X\}$$

While there is no restrictions on the form of the membership function, besides $\mu_{\tilde{a}}(x) \leq 1\ \forall x$, a special case commonly used is the trapezoidal membership functions.

Definition 2.7. *A fuzzy set* $\tilde{a}$ *is a trapezoidal fuzzy number (TFN), denoted by* $\mathbf{a} = (a_1, a_2, a_3, a_4)$*, if it is defined on the real line with membership function given by:*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & a_1 < x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & a_3 < x < a_4 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$*. A triangular fuzzy number is a special case of a TFN when* $a_2 = a_3$.

Definition 2.8. *Let* $\tilde{a} = (a_1, a_2, a_3, a_4)$ *and* $\tilde{b} = (b_1, b_2, b_3, b_4)$ *be two TFN. A distance measure between* $\tilde{a}$ *and* $\tilde{b}$ *is given by*

$$d(\tilde{a}, \tilde{b}) = \frac{1}{4}\sum_{i=1}^{4} |a_i - b_i| \tag{5}$$

Now, it is necessary a way to compare two TFN. The next two definitions provide a way to do so.

Definition 2.9. *[14]. Let* $\tilde{b} = (b_1, b_2, b_3, b_4)$ *be a TFN, then defuzzified value of* $\tilde{b}$ *is given by*

$$m(\tilde{a}) = \frac{b_1 + b_2 + b_3 + b_4}{4} \tag{6}$$

Observe that for TrFN, $b_2 = b_3$.

Definition 2.10. *Let* $\tilde{a}$ *and* $\tilde{b}$ *be two TFN. We say that* $\tilde{a}$ *is superior to* $\tilde{b}$ *if* $m(\tilde{a}) > m(\tilde{b})$*. If* $m(\tilde{a}) = m(\tilde{b})$ *then one say that* $\tilde{a}$ *is indifferent to* $\tilde{b}$.

The next definition provides a way to normalize a TFN.

Definition 2.11. *Let* $\tilde{a}_{ij} = \left(a^1_{ij}, a^2_{ij}, a^3_{ij}, a^4_{ij}\right)$ *be a TFN used to evaluate the ith alternative with respect to jth criterion. To normalize the TFN of criterion j we use the following formula:*

$$\tilde{r}_{ij} = \left(\frac{a^1_{ij}}{\max\limits_i a^4_{ij}}, \frac{a^2_{ij}}{\max\limits_i a^4_{ij}}, \frac{a^3_{ij}}{\max\limits_i a^4_{ij}}, \frac{a^4_{ij}}{\max\limits_i a^4_{ij}}\right),\ i = 1, \ldots, m \tag{7}$$

2.3. Intuitionistic fuzzy sets

In this section we provide some basic definitions of intuitionistic fuzzy sets.

Definition 2.12. *[15]. An intuitionistic fuzzy set* $\tilde{A}$ *in* X *is a mathematical object of the form*

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle : x \in X\}$$

where $\mu_{\tilde{A}}, \nu_{\tilde{A}} : X \to [0, 1]$*, such that* $\mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1,\ \forall x \in X$*. Here,* $\mu_{\tilde{A}}$ *stands for the membership function and* $\nu_{\tilde{A}}$ *for the non-membership function. The hesitancy degree of* $x \in X$ *is given by* $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$.

Again, there is no restrictions on the form of the membership and non-membership, as long as they satisfy the restrictions on Definition 2.12. However, the intuitionistic trapezoidal fuzzy number is commonly applied [16, 17, 18, 19, 5].

Definition 2.13. *Let* $\tilde{A}$ *be an IFS in X. We say that* $\tilde{A} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), \tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}}\rangle$ *is an intuitionistic trapezoidal fuzzy set (ITFS) if its membership function and non-membership function are, respectively, given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}\tilde{\mu}_{\tilde{A}} & a_1 < x < a_2 \\ \tilde{\mu}_{\tilde{A}} & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}\tilde{\mu}_{\tilde{A}} & a_3 < x < a_4 \\ 0 & otherwise \end{cases} \tag{8}$$

$$\nu_{\tilde{A}}(x) = \begin{cases} \frac{1-\tilde{\nu}_{\tilde{A}}}{b_1-b_2}(x - b_1) + 1 & b_1 < x < b_2 \\ \tilde{\nu}_{\tilde{A}} & b_2 \leq x \leq b_3 \\ \frac{1-\tilde{\nu}_{A}}{b_4-b_3}(x - b_4) + 1 & b_3 < x < b_4 \\ 1 & otherwise \end{cases} \tag{9}$$

where $\tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}} \in [0, 1]$ *represent the maximum of the membership degree and the minimum of non-membership degree, respectively. Also,* $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$*. If* $a_1 = b_1$*,* $a_2 = b_2$*,* $a_3 = b_3$ *and* $a_4 = b_4$*, we represent the IFS just as* $\tilde{A} = \langle (a_1, a_2, a_3, a_4), \tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}}\rangle$

Definition 2.14. *Let* $\tilde{A} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), \tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}}\rangle$ *and* $\tilde{B} = \langle (c_1, c_2, c_3, c_4), (d_1, d_2, d_3, d_4), \tilde{\mu}_{\tilde{B}}, \tilde{\nu}_{\tilde{B}}\rangle$*, then the distance between then is given by*

$$d(\tilde{A}, \tilde{B}) = \frac{1}{2}\left[d_\mu(\tilde{A}, \tilde{B}) + d_\nu(\tilde{A}, \tilde{B})\right] \tag{10}$$

where

$$d_\mu(\tilde{A}, \tilde{B}) = \frac{1}{4}\Big[\, |a_1 - c_1| + (1 + |\tilde{\mu}_A - \tilde{\mu}_B|)\,(1 + |a_2 - c_2| + |a_3 - c_3|) - 1 + |a_4 - c_4|\, \Big] \tag{11}$$

and

$$d_\nu(\tilde{A}, \tilde{B}) = \frac{1}{4}\Big[\, |b_1 - d_1| + (1 + |\tilde{\nu}_A - \tilde{\nu}_B|)\,(1 + |b_2 - d_2| + |b_3 - d_3|) - 1 + |b_4 - d_4|\, \Big] \tag{12}$$

Definition 2.15. *[5]. Let* $\tilde{A}$ *be an IFS in a bounded X. A score function for IFS is*

$$S(\tilde{A}) = \tilde{\mu}_{\tilde{A}} E_{\mu_{\tilde{A}}} - E_{\nu_{\tilde{A}}} \tag{13}$$

where $E_{\mu_{\tilde{A}}} = \int_X x\mu'_{\tilde{A}}(x)dx$*,* $\mu'_{\tilde{A}} = \frac{\mu_{\tilde{A}}}{\int_X \mu_{\tilde{A}}(x)dx}$*, and* $E_{\nu_{\tilde{A}}} = \int_{\inf\{x:\mu(x)>0\}}^{\sup X} x\nu_{\tilde{A}}(x)dx$*. If* $\mu_{\tilde{A}}$ *is a degenerated function, with* $a_1 = a_2 = a_3 = a_4$ *then* $E_{\mu_{\tilde{A}}} = a_1$.

The Definition 2.15 presents a way to rank IFN. The bigger the score function S is, the bigger is the IFN. For intuitionistic trapezoidal fuzzy number in the interval [0,1], we have the following:

$$E_{\mu_{\tilde{A}}} = \frac{(a_4 + a_3)^2 - (a_2 - a_1)^2 - a_4 a_3 + a_1 a_2}{3(a_2 - a_1) + 6(a_3 - a_2) + 3(a_4 - a_3)} \tag{14}$$

$$\begin{aligned} E_{\nu_{\tilde{A}}} = & \frac{3a_1^2\,(b_2 - b_1\nu) + 2a_1^3(\nu - 1) - b_2^3(2\nu + 1) + 3b_1 b_2^2 \nu}{6(b_1 - b_2)} - \frac{\nu\left(b_2^2 + b_3^2\right)}{2} \\ & - \frac{(b_3 - b_4)\,[b_3(2\nu + 1) + b_4(\nu + 2)]}{6} + \frac{1 - b_4^2}{2} \end{aligned} \tag{15}$$

Definition 2.16. *Let $\tilde{A}$ and $\tilde{B}$ be two TFN. We say that $\tilde{A}$ is superior to $\tilde{B}$ is $S(\tilde{A}) > S(\tilde{B})$. If $S(\tilde{A}) = S(\tilde{B})$ then we say that $\tilde{A}$ is indifferent to $\tilde{B}$.*

Definition 2.17. *Let $\tilde{A}_{ij} = \langle \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right), \left(b_{ij}^1, b_{ij}^2, b_{ij}^3, b_{ij}^4\right), \tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}} \rangle$ be an ITFN used to evaluate the ith alternative with respect to jth criterion. To normalize the ITFNs of criterion j we use the following formula:*

$$\tilde{R}_{ij} = \left\langle \left(\frac{a_{ij}^1}{c}, \frac{a_{ij}^2}{c}, \frac{a_{ij}^3}{c}, \frac{a_{ij}^4}{c} \right), \left(\frac{b_{ij}^1}{c}, \frac{b_{ij}^2}{c}, \frac{b_{ij}^3}{c}, \frac{b_{ij}^4}{c} \right), \tilde{\mu}_{\tilde{A}}, \tilde{\nu}_{\tilde{A}} \right\rangle, \; i = 1, \ldots, m \tag{16}$$

where $c = \max_i b_{ij}^4$.

3. The standard TODIM

In this section, we describe the TODIM method [2]. In the original proposal all the s_{ij} are real numbers. Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be the weight vector of the criteria $C_1, C_2 \ldots, C_n$, where $0 \leq w_i \leq 1$ and $\sum_{i=1}^n w_i = 1$. It is necessary that the decision maker defines a reference criterion, usually the criterion with the highest weight. Let the C_r, $1 \leq r \leq n$ be such criterion. Define $w_{rj} = w_j/w_r$. The TODIM(θ), $\theta > 0$, method consists in:

1. Define and normalize the decision matrix.
2. Calculate the final dominance of A_i over each alternative A_j by $\delta\left(A_i, A_j\right) = \sum_{c=1}^n \phi_c\left(A_i, A_j\right), \forall (i, j)$ where,

$$\phi_c\left(A_i, A_j\right) = \begin{cases} \sqrt{\frac{w_{rc}}{\sum_c w_{rc}}\left(s_{ic} - s_{jc}\right)} & \text{if } s_{ic} \geq s_{jc} \\ -\frac{1}{\theta}\sqrt{\frac{\sum_c w_{rc}}{w_{rc}}\left(s_{jc} - s_{ic}\right)} & \text{otherwise} \end{cases} \tag{17}$$

3. The global valor of alternative i is obtained by

$$\varepsilon_i = \frac{\sum_j \delta(A_i, A_j) - \min_i \sum_j \delta(A_i, A_j)}{\max_i \sum_j \delta(A_i, A_j) - \min_i \sum_j \delta(A_i, A_j)} \tag{18}$$

4. Sort the alternatives by their value ε_i.

The parameter θ in TODIM controls the impact caused in case of losses. We have that, if $\theta < 1$ the losses are amplified and if $\theta > 1$ the losses are attenuated. The prospect theory states that the individuals are more sensitive to losses than to gains, suggesting $\theta < 1$. This parameter can considerably affect the ranking order of the alternatives. If we choose a small θ we are looking for an alternative that provides small losses in all criteria, on the other hand, if we choose big values for θ we are looking for an alternative that provides more gains, even if we have losses in some criteria.

Recently, [5] pointed out an unexpected behavior of the TODIM method and suggested the following modification in the ϕ_c function,

$$\phi_c\left(A_i, A_j\right) = \begin{cases} \sqrt{w_c\left(s_{ic} - s_{jc}\right)} & \text{if } s_{ic} \geq s_{jc} \\ -\frac{1}{\theta}\sqrt{w_c\left(s_{jc} - s_{ic}\right)} & \text{otherwise} \end{cases} \tag{19}$$

4. A modular formulation of the TODIM method

An important characteristic of TODIM is that the comparisons among the alternatives are made isolatedly on each criterion and only then, they are aggregated. In TODIM, the partial dominances are evaluated separately for each criterion and then they are aggregated into the final dominance. By doing so, the partial analysis of the alternatives on each criterion are independent of their ratings on a different criterion. Therefore, if we have a decision matrix where each one of the criterion has a different type of information, since they are analyzed by the TODIM separately, it is not necessary to transform the different types of information in a common type to compare the alternatives. However, we must guarantee that the partial quantities, i.e., the quantities obtained in each module, are compatible, otherwise a module may be overweighted in the aggregation step. We discuss more about this in Section 5. The Mo-TODIM is presented in the following steps:

1. Define and normalize the decision matrix. Each criterion of the decision matrix will be considered as a module of the decision making problem.

2. For each module $c = 1, \ldots, n$, calculate the partial dominance of A_i over A_j, $\phi_c\left(A_i, A_j\right)$, $i, j = 1, \ldots, m$, where

$$\phi_c\left(A_i, A_j\right) = \begin{cases} \sqrt{w_c d\left(s_{ic}, s_{jc}\right)} & \text{if } s_{ic} \geq s_{jc} \\ -\frac{1}{\theta}\sqrt{w_c d\left(s_{ic}, s_{jc}\right)} & \text{otherwise} \end{cases} \tag{20}$$

   for benefit criterion and

$$\phi_c\left(A_i, A_j\right) = \begin{cases} \sqrt{w_c d\left(s_{ic}, s_{jc}\right)} & \text{if } s_{ic} \leq s_{jc} \\ -\frac{1}{\theta}\sqrt{w_c d\left(s_{ic}, s_{jc}\right)} & \text{otherwise} \end{cases} \tag{21}$$

   for cost criterion. We recall that the comparisons $s_{ic} \geq s_{jc}$ depends on the type of information being used. The Definition 2.4 is used for interval number, the Definition 2.10 for TFN and Definition 2.16 for IFS.
3. For all $i, j \in (1, \ldots, m)$, calculate the final dominance of A_i over each alternative A_j, by

$$\delta\left(A_i, A_j\right) = \sum_{c=1}^{n} \phi_c\left(A_i, A_j\right) \tag{22}$$

4. Calculate the global value of alternative i by

$$\varepsilon_i = \frac{\sum_j \delta(A_i, A_j) - \min_i \sum_j \delta(A_i, A_j)}{\max_i \sum_j \delta(A_i, A_j) - \min_i \sum_j \delta(A_i, A_j)} \tag{23}$$

5. Sort the alternatives by their value ε_i.

This is a very simple and intuitive extension of TODIM that makes it capable of processing many types of information. As one can see, the resemblance with the standard version is huge. The key change of this method is that in each module the distance formula may be different, in such a way that allows the method to process different type of information in a very natural way.

5. A note about the role of distances measures in the modular approach

In this section we discuss some issues that can affect the results of Mo-TODIM and we try to clarify what we meant by compatible quantities in Section 4.

Despite the easiness that the modular interpretation of Mo-TODIM provides, we must consider that this modular approach connects different versions of the TODIM (which are in essence different methods) altogether. This can bring problems that we would not have by analyzing every module with the same version. To illustrate that, let us introduce a new distance formula for ITFN.

Definition 5.1. *[5]. Let $\tilde{A}$ and $\tilde{B}$ be two ITFS. The distance between $\tilde{A}$ and $\tilde{B}$ is defined by*

$$\begin{aligned} d(\tilde{A}, \tilde{B}) \quad = \quad & \sqrt{\int_0^{\min\{\tilde{\mu}_A, \tilde{\mu}_B\}} \left[a_\mu^L(\lambda) - b_\mu^L(\lambda)\right]^2 + \left[a_\mu^R(\lambda) - b_\mu^R(\lambda)\right]^2 d\lambda} + \\ & + \sqrt{\int_{\max\{\tilde{\nu}_A, \tilde{\nu}_B\}}^1 [a_\nu^L(\lambda) - b_\nu^L(\lambda)]^2 + [a_\nu^R(\lambda) - b_\nu^R(\lambda)]^2 \, d\lambda} + \\ & + \sqrt{\frac{1}{2}\left[(\tilde{\mu}_A - \tilde{\mu}_B)^2 + (\tilde{\nu}_A - \tilde{\nu}_B)^2 + (\tilde{\mu}_A + \tilde{\nu}_A - \tilde{\mu}_B - \tilde{\nu}_B)^2\right]} \end{aligned}$$

Consider two TFN, $\tilde{a} = (0.1, 0.2, 0.4, 0.5)$ and $\tilde{b} = (0.2, 0.4, 0.6, 0.8)$, and two ITFN, $\tilde{A} = \langle(0.1, 0.2, 0.4, 0.5), (0.1, 0.2, 0.4, 0.5), 1, 0\rangle$ and $\tilde{B} = \langle(0.2, 0.4, 0.6, 0.8), (0.2, 0.4, 0.6, 0.8), 1, 0\rangle$. In this case, since the intuitionistic fuzzy sets are a natural generalization of the fuzzy sets and you can precisely represent the fuzzy sets as intutionistic fuzzy sets, we can state that the information of $\tilde{a}$ and $\tilde{A}$ are the same. The same is valid for $\tilde{b}$ and $\tilde{B}$. Now, suppose that you are applying the Mo-TODIM and in criterion C_1 you have information as TFN and in criterion C_2 you have information as ITFN. Then, you apply Definition 2.8 to calculate the distance between $\tilde{a}$ and $\tilde{b}$ and apply the Definition 5.1 to calculate the distance between $\tilde{A}$ and $\tilde{B}$. For the first, you will get a distance of 0.2 and, for the second, you will get a distance of 0.416. This is not an isolated example. In fact, the distance in Definition 5.1 tends to be higher than the distance in Equation 2.8 for values in [0, 1]. Due this fact, despite being considered the same information, the distance in Definition 5.1 will dominate in value the distance in Definition 2.8. So, trying to apply Mo-TODIM that uses both distances can be misleading, since the module that applies the Definition 5.1 will overestimate the gains/losses in comparison to Definition 2.8. Then, the partial dominance of such criterion can potentially have more impact on the final dominance matrix than the other partial dominances. This would not be a problem if the method applied the same distance in all modules.

Although, the distances must be approximately the same, there is no need, in general, that they must be the same. After all, in the general cases, the distances will be dealing with different information types. For example, consider the case where one of the module is interval number and one of the modules is TFN. Consider two interval values $\mathbf{a} = [0.1, 0.2]$ and $\mathbf{b} = [0.1, 0.9]$ and two TFN $\tilde{a} = (0.1, 0.1, 0.2, 0.2)$ and $\tilde{b} = (0.1, 0.1, 0.9, 0.9)$. Note that for $\tilde{a}$ the values in interval $\mathbf{a} = [0.1, 0.2]$ have membership degree equal to one and the values outside this interval has membership degree equal to zero. The same occurs for $\tilde{b}$ and interval $\mathbf{b}$. Therefore, despite the different types of information, each pair $(\mathbf{a}, \tilde{a})$ and $(\mathbf{b}, \tilde{b})$ have similar content of information, with the similar being the key word here, they are similar but by all means not equal. For example, interval value does not have the concept of membership and non-membership degree associated with it. So, when we assume that the membership degree for the values in the interval is equal to one or any other value is an ad-hoc approach. But still, they are similar information. So, the distances between $\tilde{a}$ and $\tilde{b}$ and $\mathbf{a}$ and $\mathbf{b}$ should be approximately the same. Let us consider another distance measure, presented in Definition 5.2.

Definition 5.2. *[20]. Let $\tilde{c} = (c_1, c_2, c_3, c_4)$ and $\tilde{d} = (d_1, d_2, d_3, d_4)$ be two TFN. A distance measure between $\tilde{c}$ and $\tilde{d}$ is given by*

$$d(\tilde{c}, \tilde{d}) = \sqrt{\frac{1}{6}\left[\sum_{i=1}^{4}(c_i - d_i)^2 + \sum_{i\in\{1,3\}}(c_i - d_i)(c_{i+1} - d_{i+1})\right]} \tag{24}$$

By using the distance in Definition 2.2 we have that $d(\mathbf{a}, \mathbf{b}) = 0.35$ and by using Definition 5.2 we have $d(\tilde{a}, \tilde{b}) = 0.495$. In this situation, one could argue that this is a significant difference and this distances are not compatible. However, these distances tend to be very close to each other, with occasional higher differences. So, it should be fine to use both these distances. As a matter of fact, if we apply Definition 5.2 instead of Definition 2.8, every result presented in the Section 6 holds with only minor differences in few cells of the tables presented. The large majority of the values will be just the same. In general, minor incompatibilities between the distances may not cause serious impact but, in situations where the alternatives have fairly close performance these incompatibilities may influence the results of Mo-TODIM.

In this work, we use the distances in Definitions 2.2, 2.8 and 2.14. For crisp numbers, it is applied the absolute difference. These distances matches perfectly in the sense discussed in this section.

6. Simulation Results

In this section we discuss two case studies of Mo-TODIM. The purpose is to illustrate the method as well as to validate the method by comparing the results with those already reported in literature. The first case study is discussed in [9], where a version of the TODIM, which we will call Extended-TODIM, that process hybrid data types was applied. So, we can analyze the results obtained by the Mo-TODIM and the Extended-TODIM. For the second case study, we will apply the Mo-TODIM in the instance discussed in the [21].

6.1. Case Study 1

This problem is discussed in [9]. A company wants to determine which one of three product projects is better. Each one of these projects are evaluated according to three criteria: cost of each product (C_1), payback period (C_2) and chance of success (C_3). The criteria C_1, C_2 and C_3 are evaluated as crisp number, interval number and triangular fuzzy number, respectively. It is clear that the criteria C_1 and C_2 are cost criteria and the criterion C_3 is a benefit criterion. The weights of the criteria are given by $\mathbf{w} = (0.4, 0.3, 0.3)$. The decision matrix is presented in Table 1. The first step is to normalize the decision matrix.

Table 1: Data of the product project problem: raw (left) and normalized (right).

Alternatives	C_1	C_2	C_3	C_1	C_2	C_3
A_1	660	[1,3]	(4,5,6)	1	[0.33, 1]	(0.50, 0.63, 0.75)
A_2	630	[2,3]	(4,6,8)	0.95	[0.67, 1]	(0.50, 0.75, 1.00)
A_3	650	[2,3]	(6,7,8)	0.98	[0.67, 1]	(0.75, 0.88, 1.00)

Table 2: The partial dominance of the alternatives for criteria C_1, C_2 and C_3 using $\theta = 0.4$

$\boldsymbol{\phi}_1$	A_1	A_2	A_3	$\boldsymbol{\phi}_2$	A_1	A_2	A_3	$\boldsymbol{\phi}_3$	A_1	A_2	A_3
A_1	0.00	-0.34	-0.19	A_1	0.00	0.22	0.22	A_1	0.00	-0.48	-0.68
A_2	0.13	0.00	0.11	A_2	-0.56	0.00	0.00	A_2	0.19	0.00	-0.48
A_3	0.08	-0.28	0.00	A_3	-0.56	0.00	0.00	A_3	0.27	0.19	0.00

Table 3: The partial dominance of the alternatives for criteria C_1, C_2 and C_3 using $\theta = 2.5$

$\boldsymbol{\phi}_1$	A_1	A_2	A_3	$\boldsymbol{\phi}_2$	A_1	A_2	A_3	$\boldsymbol{\phi}_3$	A_1	A_2	A_3
A_1	0.00	-0.05	-0.03	A_1	0.00	0.22	0.22	A_1	0.00	-0.08	-0.11
A_2	0.13	0.00	0.11	A_2	-0.09	0.00	0.00	A_2	0.19	0.00	-0.08
A_3	0.08	-0.04	0.00	A_3	-0.09	0.00	0.00	A_3	0.27	0.19	0.00

Once the decision matrix is normalized, we calculate the partial dominance of the alternatives in each criterion. The partial dominances for $\theta = 0.4$ and $\theta = 2.5$ are presented in Tables 2 and 3.

From Tables 2 and 3 it is clear the effects of θ on the impact of the losses. For $\theta < 1$ the losses are amplified whereas for $\theta > 1$ the losses are attenuated. Next, we calculate the δ, which is just the sum of the matrices of partial dominance, and then we obtain ε. The final rank orders obtained for Mo-TODIM(θ), considering $\theta \in \{0.2, 0.4, 0.6, 0.8, 1, 1.5, 2, 2.5, 5\}$, are presented in Table 4.

Table 4: The ranking of the alternatives provided by Extended-TODIM [9] and obtained by Mo-TODIM, using different values for θ in the case of the best project problem.

Rank	1	2	3
Extended-TODIM(1)	A_3	A_2	A_1
Extended-TODIM(2)	A_3	A_2	A_1
Extended-TODIM(3)	A_2	A_3	A_1
Extended-TODIM(4)	A_2	A_3	A_1
Extended-TODIM(5)	A_2	A_3	A_1
Mo-TODIM(0.2)	A_3	A_2	A_1
Mo-TODIM(0.4)	A_3	A_2	A_1
Mo-TODIM(0.6)	A_3	A_2	A_1
Mo-TODIM(0.8)	A_3	A_2	A_1
Mo-TODIM(1)	A_3	A_2	A_1
Mo-TODIM(1.5)	A_3	A_2	A_1
Mo-TODIM(2)	A_3	A_2	A_1
Mo-TODIM(2.5)	A_3	A_2	A_1
Mo-TODIM(5)	A_3	A_2	A_1

We can note some differences in the results of Mo-TODIM and Extended-TODIM. The Extended-TODIM method ranked the alternatives as $A_3 \succ A_2 \succ A_1$ for $\theta = 1$ and $\theta = 2$. As we can see in Table 4, the Mo-TODIM provides this same rank order for all values of θ. But, for $\theta \in \{3, 4, 5\}$ the Extended-TODIM provided $A_2 \succ A_3 \succ A_1$ disagreeing with the Mo-TODIM in which alternative is the best one. Why does that happen?

Let's analyze the performances of A_3 and A_2. By looking at Table 1, we see that A_2 is superior considering C_1 and A_3 is superior considering C_2. In both criteria A_1 has the worst rating. In criterion C_2, A_2 and A_3 have the same losses. So, each alternative is better according to one criterion and both have the same losses in criterion C_2. Looking the criterion C_1, we have that the rating of A_2 is 630 and the rating of A_3 is 650, i.e., A_3 is approximately 3.2% worse than A_2. Under the criterion C_3, we have that A_3 has a rating of $(6, 7, 8)$ which leads to a deffuzified value of 7 and A_2 has a rating of $(4, 6, 8)$ which leads to a deffuzified value of 6. Therefore, we have that A_2 is approximately 14% inferior to A_3 under criterion C_3. Then the gain of A_3 in C_3 is much bigger than the gain of A_2 in C_1. Even giving more weight for the criterion C_1 we would have $0.4 \times 0.032 \approx 0.01$ and $0.3 \times 0.14 \approx 0.04$. Since in C_2 both have the same rating, we can conclude that A_3 provides more gains than A_2 and, at the same time, A_3 suffer less losses. Therefore, independently if we are amplifying or attenuating the losses, i.e., independently of the value of θ, we can intuitively see that A_3 is superior. So, it makes sense that Mo-TODIM does not change the rank order of the alternative when we change the value of θ.

6.2. Case Study 2

Now we discuss the application provided in [21] where we have four alternatives and five criteria. In this instance we have the following data types: C_1 crisp numbers, C_2 and C_3 triangular fuzzy numbers and C_4 and C_5 interval numbers. All criteria are considered as benefit criteria. The data are shown in Table 5. In their work, [21] obtain from incomplete information the weight vector $\mathbf{w} = (0.103, 0.45, 0.067, 0.3, 0.08)$, which will be used here.

Table 5: Decision matrix of case study 2.

Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	2.0	$(0.4, 0.5, 0.6)$	$(0.8, 0.9, 1.0)$	$[55, 56]$	$[345.91, 404.09]$
A_2	2.5	$(0.2, 0.3, 0.4)$	$(0.4, 0.5, 0.6)$	$[30, 40]$	$[359.66, 428.34]$
A_3	1.8	$(0.6, 0.7, 0.8)$	$(0.6, 0.7, 0.8)$	$[50, 60]$	$[319.26, 392.74]$
A_4	2.2	$(0.4, 0.5, 0.6)$	$(0.4, 0.5, 0.6)$	$[35, 45]$	$[432.26, 505.743]$

First we must normalize the matrix. The normalized matrix is presented in Table 6.

Table 6: Normalized decision matrix of case study 2.

Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	0.80	$(0.50, 0.63, 0.75)$	$(0.80, 0.90, 1.00)$	$[0.917, 0.933]$	$[0.684, 0.799]$
A_2	1.00	$(0.25, 0.37, 0.50)$	$(0.40, 0.50, 0.60)$	$[0.500, 0.667]$	$[0.711, 0.847]$
A_3	0.72	$(0.75, 0.87, 1.00)$	$(0.60, 0.70, 0.80)$	$[0.833, 1.000]$	$[0.631, 0.777]$
A_4	0.88	$(0.50, 0.63, 0.75)$	$(0.40, 0.50, 0.60)$	$[0.583, 0.750]$	$[0.855, 1.000]$

We again apply the Mo-TODIM method using $\theta \in \{0.2, 0.4, 0.6, 0.8, 1, 1.5, 2, 2.5, 5\}$, the results are presented in Table 7. The Mo-TODIM agreed with the method proposed in [21] in the vast majority of θ values. However, we can see that for small values of θ, i.e., when the losses are strongly penalized, the alternative A_1 is preferred than the alternative A_3. By Table 5 we can see that alternative A_1 is better than A_3 in C_1, C_3, C_4 and C_5, although in C_4 the advantage for A_1 is very small (degree of preference of 0.545). So, for small values of θ, the losses of A_3 turns to be too expensive, even considering the small weights in such criteria. When θ gets bigger and bigger the impact of the losses in such criteria goes down and the gains that A_3 has over A_1 in criterion C_2, which has the highest weight, starts to prevail.

Table 7: The ranking of the alternatives provided by Mo-TODIM using different values for θ for case study 2.

Rank	1	2	3	4
Method [21]	A_3	A_1	A_4	A_2
Mo-TODIM(0.2)	A_1	A_3	A_4	A_2
Mo-TODIM(0.4)	A_1	A_3	A_4	A_2
Mo-TODIM(0.6)	A_3	A_1	A_4	A_2
Mo-TODIM(0.8)	A_3	A_1	A_4	A_2
Mo-TODIM(1)	A_3	A_1	A_4	A_2
Mo-TODIM(1.5)	A_3	A_1	A_4	A_2
Mo-TODIM(2)	A_3	A_1	A_4	A_2
Mo-TODIM(2.5)	A_3	A_1	A_4	A_2
Mo-TODIM(5)	A_3	A_1	A_4	A_2

7. Concluding remarks

In this paper we argued that, with some adaptations, the TODIM method is able to process different types of information without any transformation, as long as the data types are homogeneous in each criterion. Avoiding to transform the data types to a common data type has two main advantages: first, the method is much simpler to understand and to apply, and second we prevent some potential drawbacks that the transformation step may cause.

There are some extensions of TODIM method to deal with a variety of information types. Once there is a version of TODIM for one type of information, the modular strategy of evaluation of TODIM, where each criterion can be considered as a separated module, allows the method to process those information and then aggregating the modules results in a natural way. We also discussed the importance that the distances measures used in the modules be compatible, in a broad sense. The Mo-TODIM method is simple and intuitive, very similar to the standard formulation.

In order to analyze the behavior of the method, it was investigated in two case studies. In both examples, the method behaved as expected, with just minor and justified differences of the results obtained by previous works. Also, we analyzed the sensitivity of the Mo-TODIM in relation to the parameter θ and provided the rank order for several different values of θ.

In this work, we illustrate the method with four different information types but, one could easily add more modules to deal with other types of information like interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets, probability distributions and so on. Since that are some proposed generalizations of TODIM to process specific data types, one could use these different versions in each module to achieve a very broad method to model the problem in hands. Eventually, a modification/adaptation of the distance measures may be necessary which might be considered a limitation. Other methods can be easily extended based on the modular approach, which is an interesting future research topic.

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