Two integral operators on the class $\mathcal{N}(\beta)$

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**ABSTRACT**

Let $\mathcal{N}(\beta)$ be the subclass of analytic functions $f(z)$, which satisfies the inequality

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \beta.$$ (for some $\beta > 1$)

In this paper, we determine conditions on $\beta$ such that the integral operators

$$\int_0^z \prod_{i=1}^n \left(\frac{f^{(i)}(t)}{t} \right)^{\alpha_i} dt \quad \text{and} \quad \int_0^z (te^{\xi t})^{\gamma_i} dt$$

will be in the class $\mathcal{N}(\beta)$. © 2011 Elsevier Ltd. All rights reserved.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Further, by $\mathcal{S}$, we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $U$. A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order $\beta$ if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} \right\} > \beta \quad (z \in U)$$

for some $\beta (0 \leq \beta < 1)$. We denote by $\mathcal{S}^*\beta$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order $\beta$ in $U$. Also a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_\beta$ iff

$$\text{Re}\left\{f'(z) \right\} > \beta, \quad (z \in U)$$

for some $\beta (0 \leq \beta < 1)$.

Let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$, consisting of functions $f(z)$, which satisfies the inequality

$$\text{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \beta \quad (\beta > 1, z \in U).$$

The class $\mathcal{N}(\beta)$ was introduced and studied by Uralegaddi et al. in [1] and Owa and Srivastava in [2].

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Let the functions \( f(z) \) be regular in the unit disk \( \mathbb{U} \), with \( f(0) = 0 \). If \( |f(z)| \leq 1 \), for all \( z \in \mathbb{U} \), then
\[
|f(z)| \leq |z|, \quad z \in \mathbb{U}
\]
and equality holds only if \( f(z) = \varepsilon z \), where \( |\varepsilon| = 1 \).

2. Main results

**Theorem 2.1.** Let the functions \( f_i(z) \in \mathcal{A} \) for all \( i = 1, n \) be in the class \( \mathcal{B}(\mu, \beta) \), \( \mu \geq 1, 0 \leq \beta < 1 \). If \( |f_i(z)| < M_i \) \((M_i \geq 1)\), \( z \in \mathbb{U} \) then the integral operator
\[
F_{n, \ldots, n}(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt
\]
is in \( \mathcal{N}(\gamma) \), where
\[
\gamma = \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( (2 - \beta)M_i^{\mu_i-1} + 1 \right) + 1
\]
and \( \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( (2 - \beta)M_i^{\mu_i-1} + 1 \right) > 0 \), \( \alpha_i \in \mathbb{C} - \{0\} \), for all \( i = 1, n \).

**Proof.** Define the function \( F_n(z) \) by
\[
F_n(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt.
\]
Then a computation shows that
\[
\frac{zF_n'(z)}{F_n(z)} = \sum_{i=1}^n \frac{1}{\alpha_i} \left( \frac{zf_i(z)}{f(z)} - 1 \right).
\]
Thus, we have
\[
\text{Re} \left( \frac{zF_n'(z)}{F_n(z)} + 1 \right) = \text{Re} \left( \sum_{i=1}^n \frac{1}{\alpha_i} \left( \frac{zf_i(z)}{f(z)} - 1 \right) + 1 \right)
\]
\[
< \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( \left| \frac{zf_i(z)}{f(z)} \right|^\mu + 1 \right),
\]
\[
= \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( \left| \frac{zf_i(z)}{f(z)} \right|^\mu - \left| \frac{f_i(z)}{f(z)} \right|^\mu \right) + 1.
\]
Since \( f_i(z) \in \mathcal{B}(\mu, \beta) \) and \( |f_i(z)| \leq M_i \), applying Schwarz Lemma, we obtain
\[
\text{Re} \left( \frac{zF_n'(z)}{F_n(z)} + 1 \right) < \sum_{i=1}^n \frac{1}{|\alpha_i|} \left( \left| \frac{zf_i(z)}{f(z)} \right|^\mu \right) M_i^{\mu_i-1} + 1
\]
Therefore $F_{a_1,\ldots,a_n}(z) \in \mathcal{N}(\gamma)$. □

Letting $\mu = 1$ in Theorem 2.1, we have

**Corollary 2.2.** Let the functions $f_i(z) \in \mathcal{A}$ for all $i = 1, n$ be in the class $S_\beta^\gamma$, $0 \leq \beta < 1$. If $|f_i(z)| < M_i (M_i \geq 1)$, $z \in \mathcal{U}$ then the integral operator defined by (2.1) is in $\mathcal{N}(\gamma)$, where

$$
\gamma = \sum_{i=1}^{n} \frac{1}{|\alpha_i|} ((2 - \beta) + 1) + 1
$$

and $\sum_{i=1}^{n} \frac{1}{|\alpha_i|} ((2 - \beta) + 1) > 0, \alpha_i \in \mathbb{C} - \{0\}$, for all $i = 1, n$.

**Theorem 2.3.** Let $f_i(z) \in \mathcal{A}$ for all $i = 1, n$ be in the class $\mathcal{B}(\mu, \beta)$, $\mu \geq 0, 0 \leq \beta < 1$. If $|f_i(z)| \leq M_i (M_i \geq 1)$, $z \in \mathcal{U}$ then the integral operator

$$
G(z) = \int_{0}^{z} (re^{i(\gamma t)})^n dt
$$

is in $\mathcal{N}(\delta)$, where

$$
\delta = \sum_{i=1}^{n} |\gamma_i| ((2 - \beta)M_i^{\mu} + 1) + 1
$$

and $\sum_{i=1}^{n} |\gamma_i| ((2 - \beta)M_i^{\mu} + 1) > 0, \gamma_i \in \mathbb{C}$, for all $i = 1, n$.

**Proof.** From (2.4), we obtain

$$
\frac{zG''(z)}{G'(z)} = \sum_{i=1}^{n} \gamma_i (1 + zf'_i(z)).
$$

Hence

$$
\text{Re} \left( \frac{zG''(z)}{G'(z)} + 1 \right) = \text{Re} \left( \sum_{i=1}^{n} \gamma_i (1 + zf'_i(z)) + 1 \right)
$$

$$
< \sum_{i=1}^{n} |\gamma_i| (1 + zf'_i(z)) + 1
$$

$$
< \sum_{i=1}^{n} |\gamma_i| \left( 1 + |f'_i(z)| \left( \frac{z}{f_i(z)} \right)^\mu \left| \left( \frac{f_i(z)}{z} \right)^\mu \right| |z| \right) + 1. \tag{2.5}
$$

Since $f_i(z) \in \mathcal{B}(\mu, \beta)$ and $|f_i(z)| \leq M_i$, we obtain

$$
\text{Re} \left( \frac{zG''(z)}{G'(z)} + 1 \right) < \sum_{i=1}^{n} |\gamma_i| \left( 1 + \left| f'_i(z) \left( \frac{z}{f_i(z)} \right)^\mu \right| M_i^{\mu-1} \right) + 1
$$

$$
< \sum_{i=1}^{n} |\gamma_i| ((2 - \beta)M_i^{\mu} + 1) + 1 = \delta.
$$

So the integral operator $G(z)$ is in $\mathcal{N}(\delta)$. □

Letting $\mu = 0$ in Theorem 2.3, we have

**Corollary 2.4.** Let $f_i(z) \in \mathcal{A}$ for all $i = 1, n$ be in the class $\mathcal{R}_\beta$, $0 \leq \beta < 1$. If $|f_i(z)| \leq M_i (M_i \geq 1)$, $z \in \mathcal{U}$ then the integral operator defined by (2.4) is in $\mathcal{N}(\delta)$, where

$$
\delta = \sum_{i=1}^{n} |\gamma_i| (3 - \beta) + 1
$$

and $\sum_{i=1}^{n} |\gamma_i| (3 - \beta) > 0, \gamma_i \in \mathbb{C}$, for all $i = 1, n$. 
Letting $\mu = 1$ in Theorem 2.3, we have

**Corollary 2.5.** Let $f_i(z) \in A$ for all $i = 1, n$ be in the class $S^\beta_\gamma$, $0 \leq \beta < 1$. If $|f_i(z)| \leq M_i (M_i \geq 1), z \in U$ then the integral operator defined by (2.4) is in $\mathcal{N}(\delta)$, where

$$\delta = \sum_{i=1}^{n} |\gamma_i| ((2 - \beta)M_i + 1) + 1$$

and $\sum_{i=1}^{n} |\gamma_i| ((2 - \beta)M_i + 1) > 0$, $\gamma_i \in \mathbb{C}$, for all $i = 1, n$.

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