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Noncommutative plurisubharmonic polynomials part II: Local assumptions

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ABSTRACT

We say that a symmetric noncommutative (nc) polynomial is nc plurisubharmonic (nc plush) on an nc open set if it has an nc complex hessian that is positive semidefinite when evaluated on open sets of matrix tuples of sufficiently large size. In this paper, we show that if an nc polynomial is nc plurisubharmonic on an nc open set then the polynomial is actually nc plurisubharmonic everywhere and has the form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{0.1}$$

where the sums are finite and f_j, k_j, F are all nc analytic.

Greene et al. (2011) [1] has shown that if p is nc plurisubharmonic everywhere then p has the form in Eq. (0.1). In other words, [1] makes a global assumption while the current paper makes a local assumption, but both reach the same conclusion. We show that if p is nc plurisubharmonic on an nc "open set" (local) then p is, in fact, nc plurisubharmonic everywhere (global) and has the form expressed in Eq. (0.1).

This paper requires a technique that is not used in [1]. We use a Gram-like vector and matrix representation (called the border vector and middle matrix) for homogeneous degree 2 nc polynomials. We then analyze this representation for the nc complex hessian on an nc open set and positive semidefiniteness forces a very rigid structure on the border vector and middle matrix. This rigid structure plus the theorems in [1] ultimately force the form in Eq. (0.1).

1. Introduction

In [1], it is shown that if p is nc plush everywhere then p has the form in Eq. (0.1). In this paper, we prove a stronger result on a "local implies global" level. We show that if p is nc plush on an nc "open set" (local) then p is, in fact, nc plush everywhere (global) and has the form expressed in Eq. (0.1). Since this paper is a close companion of [1], we refer the reader there, see Section 1.4, for background and motivation.

This paper requires a technique that is not used in [1]. We use a Gram-like vector and matrix representation (called the border vector and middle matrix) for homogeneous degree 2 nc polynomials and apply it to the nc complex hessian of p. When p is convex, its hessian is positive semidefinite and this forces a simple rigid structure on the middle matrix. Analysis of this structure leads to proving, in [2], that p must have degree not greater than 2. Here, we are concerned with a considerably less stringent requirement than convexity, namely that p has a positive semidefinite nc complex hessian. This forces a considerably more complicated structure on our Gram representation. The meat of our proofs is analyzing this structure. Combined with the main theorems in [1], this forces the form in Eq. (0.1).

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We mention that one (open ended) direction for free nc algebras heads toward a geometry for an nc variety V. It concerns the zero set of an nc polynomial, p, and a notion of curvature defined in terms of the hessian, p'', of p restricted to a "tangent plane" to V. If it is positive and p is irreducible in a certain sense, then p has degree 2. This type of geometry is introduced and analyzed in [3,4] using the middle matrix representation heavily. Suppose now that one wants to understand nc varieties on which an nc Levi form is positive semidefinite, an nc analog of what one sees with pseudoconvexity in the subject of several complex variables. This would entail restriction of the nc complex hessian to a type of complex tangent plane and assuming it positive semidefinite. Then, one would use the middle matrix analysis done in this paper. Work has not seriously begun along these lines, but it is a natural extension of the techniques developed in this paper.

Section 1 of [1] provides the necessary background for this paper. [1] introduces the basics such as noncommutative variables, monomials, and polynomials; as well as matrix positivity. In addition, [1] provides details about nc differentiation, the nc complex hessian, and nc plurisubharmonicity. Hence, we begin this paper with direct sums and nc open sets.

1.1. Direct sums

Our definition of the direct sum is the usual one, which for two matrices X_1 and X_2 is given by

$$X_1 \oplus X_2 := \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}.$$

Given a finite set of matrix tuples $\{X^1, \dots, X^t\}$ with

$$X^j = \{X_{j1}, X_{j2}, \dots, X_{jg}\} \in (\mathbb{R}^{n_j \times n_j})^g$$

for $j = 1, \dots, t$, we define

$$\bigoplus_{j=1}^{t} X^j := \left\{ \bigoplus_{j=1}^{t} X_{j1}, \bigoplus_{j=1}^{t} X_{j2}, \dots, \bigoplus_{j=1}^{t} X_{jg} \right\}.$$

For example, if $X^1 = \{X_{11}, \dots, X_{1g}\}$, $X^2 = \{X_{21}, \dots, X_{2g}\}$, and $X^3 = \{X_{31}, \dots, X_{3g}\}$, we get

$$X^1 \oplus X^2 \oplus X^3 = \{X_{11} \oplus X_{21} \oplus X_{31}, \dots, X_{1g} \oplus X_{2g} \oplus X_{3g}\}.$$

Now let

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$$

where $\mathcal{B}_n \subseteq (\mathbb{R}^{n \times n})^g$ for $n = 1, 2, \dots$ is given. The graded set $\mathcal{B}$ *respects direct sums* if for each finite set

$$\{X^1, \dots, X^t\} \quad \text{with } X^j \in \mathcal{B}_{n_j} \quad \text{and} \quad n = \sum_{j=1}^{t} n_j,$$

with repetitions allowed, $\oplus_{j=1}^{t} X^j \in \mathcal{B}_n$.

1.2. Noncommutative open set

A set $\mathcal{G} \subseteq \cup_{n \geq 1} (\mathbb{R}^{n \times n})^g$ is an *nc open set* if $\mathcal{G}$ satisfies the following two conditions:

(i) $\mathcal{G}$ respects direct sums, and
(ii) there exists a positive integer n_0 such that if $n > n_0$, the set $\mathcal{G}_n := \mathcal{G} \cap (\mathbb{R}^{n \times n})^g$ is an open set of matrix tuples.

We say that an nc polynomial, p, is *nc plush on an nc open set*, $\mathcal{G}$, if the nc complex hessian, q, of p satisfies

$$q(X, X^T)[H, H^T] \succeq 0 \tag{1.1}$$

for all $X \in \mathcal{G}$ and all $H \in (\mathbb{R}^{n \times n})^g$ for all $n \geq 1$.

1.3. Main results

As we will see, in Section 4, the nc complex hessian, q, if matrix positive on an nc open set, can be factored as

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T] \tag{1.2}$$

where $D(x, x^T)$ is a diagonal matrix, $L(x, x^T)$ is a lower triangular matrix with ones on the diagonal (we call this a *unit lower triangular matrix*), and $V(x, x^T)[h, h^T]$ is a vector of monomials in x, x^T, h, h^T.

When we take the transpose of a matrix with monomial or polynomial entries (e.g., $L(x, x^T)^T$ or $V(x, x^T)[h, h^T]^T$), we get the matrix obtained by taking the transpose (as a matrix) *and* applying the transpose (involution) to every entry.

Example 1.1. If

$$v = \begin{pmatrix} hxx \\ hx \\ h \end{pmatrix},$$

then

$$v^T = \begin{pmatrix} x^T x^T h^T & x^T h^T & h^T \end{pmatrix}.$$

The next theorem shows the surprising result that the diagonal matrix, $D(x, x^T)$, in Eq. (1.2) does not depend on x, x^T and that $L(x, x^T)$ has nc polynomial entries.

Theorem 1.2. *If p is an nc symmetric polynomial that is nc plurisubharmonic on an nc open set, then q, the nc complex hessian of p, can be written as*

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $V(x, x^T)[h, h^T]$ *is a vector of monomials in* x, x^T, h, h^T, $D = \operatorname{diag}(d_1, d_2, \ldots, d_{\mathcal{N}})$ *is a positive semidefinite constant real matrix, and* $L(x, x^T)$ *is a unit lower triangular matrix with nc polynomial entries.*

Proof. The proof of this theorem requires the rest of this paper and culminates in Section 4.5. $\square$

This gives rise to an extension of the main theorem from [1]. In [1], it is shown that an nc polynomial which is nc plush *everywhere* has the specific form given in Eq. (1.3). In this paper, Theorem 1.3, is a stronger, "local implies global", result in that an nc polynomial that is nc plush just on an nc open set is actually nc plush everywhere (and has the form in Eq. (1.3)).

Theorem 1.3. *If an nc symmetric polynomial, p, is nc plurisubharmonic on an nc open set, then p is, in fact, nc plurisubharmonic everywhere and has the form expressed in* [1]

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T \tag{1.3}$$

where the sums are finite and each f_j, k_j, *and* F *is nc analytic.*

Proof. That $D = D(x, x^T)$, in Theorem 1.2, is a positive semidefinite constant real matrix immediately implies

$$q(X, X^T)[H, H^T] \succeq 0$$

for all $X, H \in \cup_{n \geq 1} (\mathbb{R}^{n\times n})^g$; that is, p is nc plush at all $X \in (\mathbb{R}^{n\times n})^g$. Consequently, Theorem 1.7 in [1] gives that p is of the desired form

$$p = \sum f_j^T f_j + \sum k_j k_j^T + F + F^T$$

where the sums are finite and f_j, k_j, F are nc analytic. $\square$

Note that with an nc polynomial, p, as in Eq. (1.3), the nc complex hessian, q, of p is

$$q = \sum (f_j^T)_{x^T}[h^T](f_j)_x[h] + \sum (k_j)_x[h](k_j^T)_{x^T}[h^T], \tag{1.4}$$

which is obviously matrix positive as it is a sum of squares. From Eq. (1.4), we see that the nc complex hessian for an nc polynomial that is nc plush on an nc open set has even degree.

1.4. Guide to the paper

In Section 2, we introduce a Gram-like representation of nc quadratics. In Section 3, we study this Gram-like representation for the nc complex hessian and prove some properties for this representation. In Section 4, we introduce the LDL^T decomposition of the nc complex hessian and conclude that D is constant.

2. Middle matrix representation for a general NC quadratic

In this section, we turn to a special representation for nc symmetric quadratic polynomials called the *middle matrix representation* (MMR). We represent nc quadratics in a factored form, $v^T M v$. This representation greatly facilitates the study of the positivity of nc quadratics by letting us study the positivity of M. Now we give details.

Any noncommutative symmetric polynomial, $f(x, x^T, h, h^T)$, in the variables $x = (x_1, \dots, x_g)$, $x^T = (x_1^T, \dots, x_g^T)$, $h = (h_1, \dots, h_g)$, and $h^T = (h_1^T, \dots, h_g^T)$ that is degree s in x, x^T and homogeneous of degree two in h, h^T admits a representation of the form

$$f(x, x^T, h, h^T) = V(x, x^T)[h, h^T]^T M(x, x^T) V(x, x^T)[h, h^T] \tag{2.1}$$

where $M(x, x^T)$, called the *middle matrix*, is a symmetric matrix of nc polynomials in x, x^T and $V(x, x^T)[h, h^T]$, called the *border vector*, is given by

$$V(x, x^T)[h, h^T] = \begin{pmatrix} V_s(x, x^T)[h] \\ \vdots \\ V_0(x, x^T)[h] \\ V_s(x, x^T)[h^T] \\ \vdots \\ V_0(x, x^T)[h^T] \end{pmatrix}. \tag{2.2}$$

The $V_k(x, x^T)[h]$ (resp. $V_k(x, x^T)[h^T]$) are vectors of nc monomials of the form $h_j m(x, x^T)$ (resp. $h_j^T m(x, x^T)$) where $m(x, x^T)$ runs through the set of $(2g)^k$ monomials in x, x^T of length k for $j = 1, \dots, g$. Note that the degree of the monomials in V_k is $k + 1$.

We note that the vector of monomials, $V(x, x^T)[h, h^T]$, might contain monomials that are not required in the representation of the nc quadratic, f. Therefore, we can omit all monomials from the border vector that are not required. This gives us a *minimal length* border vector and prevents extraneous zeros from occurring in the middle matrix. The next lemma, Lemma 2.1, says that a minimal length border vector contains distinct monomials.

Lemma 2.1. *If $f(x, x^T, h, h^T)$ is an nc symmetric polynomial that has a middle matrix representation, then there is a middle matrix representation for f such that the border vector contains distinct monomials. Here, distinct precludes one monomial being a scalar multiple of another.*

Proof. Suppose we have f with the representation

$$f(x, x^T, h, h^T) = \begin{pmatrix} m \\ \alpha m \\ n \end{pmatrix}^T \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} m \\ \alpha m \\ n \end{pmatrix}$$

with α a real number and m and n distinct monomials. Write f as

$$f = m^T(p_{11} + \alpha^2 p_{22} + \alpha p_{21} + \alpha p_{12})m + m^T(p_{13} + \alpha p_{23})n + n^T(p_{31} + \alpha p_{32})m + n^T p_{33} n$$

which leads to the representation

$$f(x, x^T, h, h^T) = \begin{pmatrix} m \\ n \end{pmatrix} \begin{pmatrix} p_{11} + \alpha^2 p_{22} + \alpha p_{21} + \alpha p_{12} & p_{13} + \alpha p_{23} \\ p_{31} + \alpha p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}$$

which has distinct monomials in the border vector. □

To aid us in the following sections, we cite a theorem (Theorem 8.3 in [5] and Theorem 6.1 in [2]). Note that in [5], the following theorem is stated for a positivity domain but the proof only uses the fact that positivity domains are nc open sets (satisfy the two conditions in Section 1.2). Hence, we slightly generalize the statement of the theorem to work on a more general nc open set as defined in Section 1.2.

Theorem 2.2. *Consider a noncommutative polynomial $\mathcal{Q}(x, x^T)[h, h^T]$ which is quadratic in the variables h, h^T that is defined on $\mathcal{G} \subseteq \cup_{n \ge 1}(\mathbb{R}^{n\times n})^g$. Write $\mathcal{Q}(x, x^T)[h, h^T]$ in the form $\mathcal{Q}(x, x^T)[h, h^T] = V(x, x^T)[h, h^T]^T M(x, x^T) V(x, x^T)[h, h^T]$. Suppose that the following two conditions hold:*

(i) *the set $\mathcal{G}$ is an nc open set as defined in Section 1.2;*
(ii) *the border vector $V(x, x^T)[h, h^T]$ of the quadratic function $\mathcal{Q}(x, x^T)[h, h^T]$ has distinct monomials.*

Then, the following statements are equivalent:

(a) *$\mathcal{Q}(X, X^T)[H, H^T]$ is a positive semidefinite matrix for each pair of tuples of matrices X and H for which $X \in \mathcal{G}$;*
(b) *$M(X, X^T) \succeq 0$ for all $X \in \mathcal{G}$.*

We will also need the following well known lemma (c.f. [2]). Just for notational purposes of stating the lemma, let $\mathcal{B}(\mathcal{H})^g$ denote all g-tuples of operators on $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space.

Lemma 2.3. *Given d, there exists a Hilbert space $\mathcal{K}$ of dimension $\sum_0^{2d}(2g)^j$ such that if G is an open subset of $\mathcal{B}(\mathcal{K})^g$, if p has degree at most d, and if $p(X) = 0$ for all $X \in G$, then $p = 0$.*

Next we proceed to study this middle matrix representation for the nc complex hessian.

3. Middle matrix representation for the NC complex hessian

In Section 2, we introduced the middle matrix representation for a general nc quadratic polynomial, and this section specializes it to the nc complex hessian. The requirement that the nc complex hessian be positive on an nc open set forces rigid structure to the border vector and middle matrix.

3.1. Border vector for a complex hessian: choosing an order for monomials

Let p be an nc symmetric polynomial in g free variables such that the degree of its nc complex hessian is d. Then the complex hessian will be homogeneous of degree two in h, h^T.

For a fixed degree k, there are g^k nc analytic monomials and g^k nc antianalytic monomials in x, x^T. That means there are $(2g)^k - g^k - g^k = (2g)^k - 2g^k$ 'mixed' monomials of degree k (i.e., monomials that are not nc analytic nor nc antianalytic).

3.1.1. Analytic border vector

For $0 \le k \le d-2$, let $A_k = A_k(x)[h]$ be the vector of nc analytic monomials with entries $h_j m(x)$ where $m(x)$ runs through the set of g^k nc analytic monomials of length k for $j = 1, \ldots, g$. The order we impose on the monomials in this vector is lexicographic order. Thus, the length of $A_k = A_k(x)[h]$ is g^{k+1} and the vector

$$A(x)[h] = \mathrm{col}(A_{d-2}, \ldots, A_1, A_0) \tag{3.1}$$

has length $g^{d-1} + \cdots + g^2 + g = g\nu$ where $\nu = g^{d-2} + \cdots + g^2 + g + 1$.

3.1.2. Antianalytic border vector

Let $A_k^t = A_k(x^T)[h^T]$ be the same as $A_k = A_k(x)[h]$ except replace each h_j with h_j^T and replace each x_i by x_i^T. So A_k^t is the vector of nc antianalytic monomials with entries $h_j^T m(x^T)$ where $m(x^T)$ runs through the set of g^k nc antianalytic monomials of length k for $j = 1, \ldots, g$ (again, the order is lexicographic). Thus, the length of $A_k^t = A_k(x^T)[h^T]$ is g^{k+1} and the vector

$$A(x^T)[h^T] = \mathrm{col}(A_{d-2}^t, \ldots, A_1^t, A_0^t) \tag{3.2}$$

also has length $g\nu$.

3.1.3. Mixed term border vector

Next, we define notation to handle all nonanalytic and nonantianalytic monomials. Let $B_1 = B_1(x, x^T)[h]$ be the vector of monomials with entries $h_j x_i^T$ for $i = 1, \ldots, g$ and $j = 1, \ldots, g$. The length of B_1 is g^2. For $2 \le k \le d-2$, let $B_k = B_k(x, x^T)[h]$ be the vector of monomials with entries $h_j m(x, x^T)$ where $m(x, x^T)$ runs through the set of $(2g)^k - 2g^k$ monomials of length k that are not nc analytic nor nc antianalytic for $j = 1, \ldots, g$. Again, we put the same lexicographic order on the monomials. Thus, the length of $B_k = B_k(x, x^T)[h]$ is $g((2g)^k - 2g^k)$ and the vector

$$B(x, x^T)[h] = \mathrm{col}(B_{d-2}, \ldots, B_2, B_1)$$

has length $g^2 + \sum_{k=2}^{d-2} g((2g)^k - 2g^k)$. Then we can also define $B_1^t = B_1^t(x, x^T)[h^T]$ to be the vector of monomials with entries $h_j^T x_i$ for $i = 1, \ldots, g$ and $j = 1, \ldots, g$. This also has length g^2. Then we define, for $2 \le k \le d-2$, the vector $B_k^t = B_k(x, x^T)[h^T]$ to be the same as B_k except h_j is replaced by h_j^T. In other words, each entry looks like $h_j^T m(x, x^T)$. Then the vector

$$B(x, x^T)[h^T] = \mathrm{col}(B_{d-2}^t, \ldots, B_2^t, B_1^t)$$

has the same length as $B(x, x^T)[h]$.

Note that the degree of the monomials in A_k, A_k^t, B_k, B_k^t is $k+1$.

3.2. The middle matrix of a complex hessian

Now we can represent the nc complex hessian, q, of a symmetric nc polynomial p as

$$q(x, x^T)[h, h^T] = \begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \\ A(x^T)[h^T] \\ B(x, x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_2 & 0 & 0 \\ Q_2^T & Q_4 & 0 & 0 \\ 0 & 0 & Q_5 & Q_6 \\ 0 & 0 & Q_6^T & Q_8 \end{pmatrix} \begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \\ A(x^T)[h^T] \\ B(x, x^T)[h^T] \end{pmatrix} \tag{3.3}$$

where $Q_i = Q_i(x, x^T)$ are matrices with nc polynomial entries in the variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$.

Again, we wish to stress that the vectors $A(x)[h]$, $A(x^T)[h^T]$, $B(x, x^T)[h]$, and $B(x, x^T)[h^T]$ may contain monomials that are not required in the representation of the complex hessian, q. Therefore, we omit all monomials from the border vector that are not required. This gives us a *minimal length* border vector and prevents extraneous zeros from occurring in the middle matrix. Lemma 2.1 says that a minimal length border vector contains only distinct monomials.

The next subsection provides some necessary background on nc differentiation.

3.3. Levi-differentially wed monomials

An extremely important fact about the nc complex hessian, $q(x, x^T)[h, h^T]$, is that it is quadratic in h, h^T and that *each term contains some h_j and some h_k^T*. If a certain monomial m is in q, then any monomial obtained by exchanging h_j with some x_ℓ in m and/or exchanging some h_k^T with some x_j^T in m is also in q. We say two such monomials are *Levi-differentially wed*. Indeed, being Levi-differentially wed is an equivalence relation on the monomials in q with the coefficients of all Levi-differentially wed monomials in q being the same.

Often we write that *terms* (rather than monomials) in the nc complex hessian are Levi-differentially wed. When we write this, we mean that the nc complex hessian, q, contains the terms αm and $\alpha\tilde{m}$ where the monomials m and $\tilde{m}$ are Levi-differentially wed and they have the same coefficient, $\alpha \in \mathbb{R}$.

Example 3.1. The monomials $h^T hx^T x$, $h^T xx^T h$, $x^T hh^T x$, and $x^T xh^T h$ are all Levi-differentially wed to each other.

Example 3.2. None of the monomials $h^T hx^T x$, $h^T xh^T x$, $x^T hx^T h$ are Levi-differentially wed to each other.

The next theorem gives necessary and sufficient conditions as to when an nc polynomial is an nc complex hessian. This theorem is proved in [1] but gets used extensively in this paper.

Theorem 3.3. *An nc polynomial q in x, x^T, h, h^T is an nc complex hessian if and only if the following two conditions hold:*

(P1) *Each monomial in q contains exactly one h_j and one h_k^T for some j, k.*
(P2) *If a certain monomial m is contained in q, any monomial $\tilde{m}$ that is Levi-differentially wed to m is also contained in q with the same coefficient.*

Proof. The proof is provided in [1]. □

Theorem 3.3(P1) shows that every term in the complex hessian, q, has an h_j and h_k^T for some j and k. This structure forces the zeros in the middle matrix in Eq. (3.3).

3.4. Structure of the middle matrix

In this subsection, we prove some properties about the structure of the middle matrix in the MMR for a matrix positive nc complex hessian.

Lemma 3.4. *Let p be an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Then, the MMR in Eq. (3.3) for its nc complex hessian, q, of p has $Q_2 = Q_4 = Q_6 = Q_8 = 0$. Thus,*

$$q = \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1(x, x^T) & 0 \\ 0 & Q_5(x, x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}. \tag{3.4}$$

Proof. We consider the upper left block of the middle matrix in Eq. (3.3)

$$\begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \end{pmatrix}^T \begin{pmatrix} Q_1(x, x^T) & Q_2(x, x^T) \\ Q_2(x, x^T)^T & Q_4(x, x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ B(x, x^T)[h] \end{pmatrix}$$

with the goal of showing $Q_2 = 0$ and $Q_4 = 0$. Thus, suppose the border vector contains a nonzero monomial which is an entry in the vector of mixed monomials, $B(x, x^T)[h]$; i.e., the border vector contains a term

$$h_k m_1(x, x^T) x_j^T m_2(x, x^T) \tag{3.5}$$

for some monomials m_1 and m_2 in the variables $x_1, \ldots, x_g, x_1^T, \ldots, x_g^T$.

Soon we shall look at the diagonal entry, $\mathcal{P}^{(0)}$, in the middle matrix corresponding to this border vector monomial in (3.5) and show it is 0. By Theorem 2.2, we have the middle matrix positive semidefinite for every X in the nc open set, $\mathcal{G}$. By Lemma 2.3, if an nc polynomial is zero on an open set of matrix tuples with sufficiently large dimension, then the nc polynomial is identically zero. Hence, if there is ever a diagonal entry in the middle matrix that is zero on an open set of matrix tuples of large enough dimension, then that diagonal entry is identically zero. Hence, to force matrix positivity, the corresponding row and column in the middle matrix must be zero. This implies that the particular monomial in the border vector is not needed in the representation, thereby contradicting the border vector being of minimal length. Thus, showing $\mathcal{P}^{(0)}$ is 0, a contradiction.

The term(s) in the nc complex hessian corresponding to the diagonal entry $\mathcal{P}^{(0)}$ of the middle matrix and monomial (3.5) in the border vector are

$$m_2^T x_j m_1^T h_k^T \mathcal{P}^{(0)} h_k m_1 x_j^T m_2$$

where $\mathcal{P}^{(0)}$ is some matrix positive polynomial in $x_1, \dots, x_g, x_1^T, \dots, x_g^T$. By Theorem 3.3(P2), q must also contain the Levi-differentially wed term(s)

$$m_2^T h_j m_1^T h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2.$$

This means the border vector must contain the monomial(s)

$$\{h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2\}_{mon} \tag{3.6}$$

where $\{h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2\}_{mon}$ is the list of the monomials that appear as terms in the nc polynomial $h_k^T \mathcal{P}^{(0)} x_k m_1 x_j^T m_2$.

Again, we shall look at the term(s) in q corresponding to the diagonal in the middle matrix corresponding to any one of the border vector monomial(s) in (3.6). Pick $h_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$ as a specific border vector monomial in the list in (3.6). Then, the term(s) in q look like

$$m_2^T x_j m_1^T x_k^T (\widehat{\mathcal{P}}^{(0)})^T h_k \mathcal{P}^{(1)} h_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$$

where $\mathcal{P}^{(1)}$ is a matrix positive polynomial in $x_1, \dots, x_g, x_1^T, \dots, x_g^T$, which is a diagonal entry of the middle matrix. Theorem 3.3(P2) implies q must also contain the Levi-differentially wed term(s)

$$m_2^T h_j m_1^T h_k^T (\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$$

which means the border vector must contain the monomial(s)

$$\{h_k^T (\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2\}_{mon} \tag{3.7}$$

where $\{h_k^T (\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2\}_{mon}$ is the list of the monomials that appear as terms in the nc polynomial $h_k^T (\widehat{\mathcal{P}}^{(0)})^T x_k \mathcal{P}^{(1)} x_k^T \widehat{\mathcal{P}}^{(0)} x_k m_1 x_j^T m_2$.

Note that the border vector monomial in (3.7) has degree at least 2 more than the degree of the border vector monomial in (3.6) which has degree at least 2 more than the degree of the border vector monomial in (3.5). We can continue this process and the degree of the successive border vector monomials will keep increasing by at least 2 at each step. At some step, the degree of the border vector monomial will exceed $d-1$. This contradicts the fact that the border vector monomials must have degree at most $d-1$. Thus, we have shown that $Q_4 = 0$. A similar argument shows that $Q_8 = 0$. Since the middle matrix is positive semidefinite, we also get $Q_2 = 0$ and $Q_6 = 0$, by the argument in the previous paragraph. Hence, the nc complex hessian has the representation in Eq. (3.4), as claimed by the theorem. □

We call an nc polynomial *hereditary* if all $x_1^T, x_2^T, \dots, x_g^T$ variables appear to the *left* of every $x_1, x_2, \dots, x_g$ variable. Similarly, we call an nc polynomial *antihereditary* if all $x_1^T, x_2^T, \dots, x_g^T$ variables appear to the *right* of every $x_1, x_2, \dots, x_g$ variable.

Theorem 3.5. *The nc complex hessian, q, of an nc symmetric polynomial that is nc plush on an nc open set can be written as in Eq.* (3.4)

$$q(x, x^T)[h, h^T] = \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T \begin{pmatrix} Q_1(x, x^T) & 0 \\ 0 & Q_5(x, x^T) \end{pmatrix} \begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}$$

where every nc polynomial entry in $Q_1(x, x^T)$ is hereditary and every nc polynomial entry in $Q_5(x, x^T)$ is antihereditary.

Proof. Suppose, for the sake of contradiction, Q_1 contains an nc polynomial entry which is not hereditary. Without loss of generality, this nc polynomial contains a term of the form

$$m_1(x^T) x_j x_k^T m_2(x, x^T) \tag{3.8}$$

where m_1 is a monomial in x^T and m_2 is a monomial in x and x^T. Since this is part of an entry in the middle matrix, this means that the nc complex hessian must contain a term of the form

$$m_3(x^T) h_\ell^T m_1(x^T) x_j x_k^T m_2(x, x^T) h_s m_4(x)$$

where $m_3(x^T) h_\ell^T$ is a specific monomial entry from the vector $A(x)[h]^T$ and $h_s m_4(x)$ is a specific monomial entry from the vector $A(x)[h]$. Then, Theorem 3.3(P2) implies that the nc complex hessian must also contain the Levi-differentially wed term

$$m_3(x^T) h_\ell^T m_1(x^T) h_j x_k^T m_2(x, x^T) x_s m_4(x).$$

This implies that the border vector must contain the monomial

$$h_j x_k^T m_2(x, x^T) x_s m_4(x)$$

which contradicts having an nc analytic or nc antianalytic border vector, as required by Lemma 3.4. The proof that Q_5 contains antihereditary nc polynomial entries is similar. □

For a real number, r, we define $\lfloor r \rfloor$ as the largest integer less than or equal to r and we define $\lceil r \rceil$ as the smallest integer greater than or equal to r. The next theorem puts an upper bound on the degree of the monomials in the border vector for q.

Lemma 3.6. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set. If the degree of its nc complex hessian, q, is d, then the degree of the border vector monomials is at most $\left\lfloor \frac{d}{2} \right\rfloor$.*

Proof. Write the MMR for $q(x, x^T)[h, h^T]$ as

$$q = V^T M V = \begin{pmatrix} V_1^T & V_2^T \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

with the following property. If d is odd, V_1 contains monomials of degree $1, \dots, \left\lfloor \frac{d}{2} \right\rfloor$ and V_2 contains monomials of degree $\left\lceil \frac{d}{2} \right\rceil, \dots, d-1$. If d is even, V_1 contains monomials of degree $1, \dots, \frac{d}{2}$ and V_2 contains monomials of degree $\frac{d}{2}+1, \dots, d-1$. In either case, polynomials in M_4 correspond to terms in q having degree strictly greater than d. Hence $M_4 = 0$. By Theorem 2.2, $M(X) \succeq 0$ for all X in an nc open set. This forces $M_2(X) = 0$ for all X in an nc open set. Then, by taking X to have large enough size, Lemma 2.3 implies $M_2 = 0$. □

3.4.1. Consequences of positivity of the complex hessian

Now we turn from a description of the middle matrix to describing the structure of the nc complex hessian of an nc polynomial that is nc plush on an nc open set.

Proposition 3.7. *The nc complex hessian, q, of an nc symmetric polynomial that is nc plush on an nc open set is a sum of hereditary and antihereditary polynomials.*

Proof. This follows immediately from Lemma 3.4 and Theorem 3.5. □

Finally, we show that the degree of q must be even when p is nc plush on an nc open set. This fact is obvious if p is assumed nc plush everywhere because then the nc complex hessian is a sum of squares.

Theorem 3.8. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set. Then, the degree of its complex hessian, q, is even.*

Proof. Suppose the degree of q is $2N+1$. Without loss of generality, Proposition 3.7 and Theorem 3.3, requiring the presence of Levi-differentially wed monomials, imply that q must contain a hereditary term of the form

$$x_{i_1}^T x_{i_2}^T \cdots h_{i_s}^T h_{j_1} x_{j_2} \cdots x_{j_\ell}$$

where $s, \ell > 0$, $s + \ell = 2N + 1$, and $i_1, \dots, i_s, j_1, \dots, j_\ell \in \{1, \dots, g\}$. This means that in the middle matrix representation for q, the border vector must contain $h_{j_1} x_{j_2} \cdots x_{j_\ell}$ and $h_{i_s} x_{i_{s-1}} \cdots x_{i_1}$ which have degree ℓ and s, respectively. But since $s + \ell = 2N + 1$ and $s, \ell > 0$, one of either s or ℓ is at least $\left\lceil \frac{2N+1}{2} \right\rceil$. This contradicts Lemma 3.6. □

4. LDL^T decomposition has constant D

This section concerns the "algebraic Cholesky" factorization, LDL^T, of the middle matrix. We will show that for an nc polynomial that is nc plush on an nc open set, this D is a positive semidefinite matrix whose diagonal entries are all nonnegative real constants, and L is unit lower triangular with entries which are nc polynomials. This is a stronger conclusion than one would expect because, typically, such factorizations have nc rational entries; see [5,6]. In our approach, the LDL^T factorization of a symmetric matrix with noncommutative entries will be the key tool for the determination of the matrix positivity of an nc quadratic function.

4.1. The LDL^T decomposition

Begin by considering the block 2×2 matrix

$$M = \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

where A is a constant real symmetric invertible matrix and B and C are matrices with nc polynomial entries with C symmetric. Then, M has the following decomposition

$$M = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - BA^{-1}B^T \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & I \end{pmatrix}, \tag{4.1}$$

where all matrices in this decomposition contain nc polynomial entries. If $C - BA^{-1}B^T$ contains a constant real symmetric invertible matrix somewhere on the diagonal, then we can apply a permutation, Π, on the left of M and its transpose, Π^T, on the right of M to move this constant real symmetric invertible matrix to the first (block) diagonal position of $C - BA^{-1}B^T$. We then pivot off this constant real symmetric invertible matrix, factor $C - BA^{-1}B^T$ as $\hat{L}\hat{D}\hat{L}^T$, and we get

$$\Pi M \Pi^T = \begin{pmatrix} I & 0 \\ BA^{-1} & \hat{L} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \hat{D} \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & \hat{L}^T \end{pmatrix}.$$

This can be continued, provided at each step, a constant real symmetric invertible matrix appears somewhere on the diagonal to obtain $\Pi M \Pi^T = LDL^T$ where L is a unit lower triangular matrix with nc polynomial entries and D is a (block) diagonal matrix with real constant blocks. This special situation is the one which turns out to hold in the derivation which follows.

Indeed, we shall only care about the case where A is a constant real symmetric invertible matrix. For the case where A contains nc polynomial entries and is considered to be "noncommutative invertible", see [5]. In this case, we also have the notion of "noncommutative rational" functions (see [6]). However, as we soon shall see, while nc rationals are mentioned, they never actually appear in any calculations in this paper.

We recall an immediate consequence of Theorem 3.3 in [5]:

Theorem 4.1. *Suppose $M(x, x^T)$ is a symmetric $r \times r$ matrix with noncommutative rational function entries and that $M(X, X^T) \succeq 0$ for all X in some nc open set. Then, there exist a permutation matrix, Π, a diagonal matrix, $D(x, x^T)$, with nc rational entries, and a unit lower triangular matrix, $L(x, x^T)$, with nc rational entries such that*

$$\Pi M(x, x^T)\Pi^T = L(x, x^T)D(x, x^T)L(x, x^T)^T.$$

Remark 4.2. In this paper, we care about the positivity of the middle matrix, $M(x, x^T)$. If Π is a permutation matrix, it is clear that

$$\Pi M(X, X^T)\Pi^T \succeq 0 \iff M(X, X^T) \succeq 0$$

for any $X \in \mathbb{R}^{n\times n}$ and any $n \geq 1$. As a result, for ease of exposition, we will often, without loss of generality, omit the permutation matrix, Π.

Also, there will be some instances where we will, without loss of generality, assume a specific order in the border vector, $V(x, x^T)[h, h^T]$. For example, we may assume a given monomial, say, $hm(x, x^T)$, is the first monomial in $V(x, x^T)[h, h^T]$. This assumption also amounts to a permutation of $V(x, x^T)[h, h^T]$ which, again, does not affect positivity of $M(x, x^T)$ so we omit it from the discussion.

We now proceed to apply the LDL^T factorization to the middle matrix of the nc complex hessian. Let p be an nc symmetric polynomial and let q denote the nc complex hessian of p. Since q is homogeneous of degree 2 in h, h^T, q admits the MMR

$$q = V(x, x^T)[h, h^T]^T M(x, x^T) V(x, x^T)[h, h^T]. \tag{4.2}$$

If p is nc plush on an nc open set, then $M(x, x^T)$ is symmetric and matrix positive on an nc open set and we can factor $M(x, x^T)$ following the process underlying Eq. (4.1) and Theorem 4.1, thus converting Eq. (4.2) to

$$q = V(x, x^T)[h, h^T]^T L(x, x^T) D(x, x^T) L(x, x^T)^T V(x, x^T)[h, h^T] \tag{4.3}$$

up to a harmless rearrangement of the border vector.

In Section 4.5, we prove one of the main theorems of this paper, Theorem 4.13, which was stated in Section 1.3 as Theorem 1.2. We recall that this theorem says that $D(x, x^T)$ in Eq. (4.3) does not depend on x, x^T and is a positive semidefinite constant real diagonal matrix for an nc polynomial that is nc plush on an nc open set. In addition, we will prove that $L(x, x^T)$ contains nc polynomials instead of nc rationals. Now we start the buildup to Section 4.5.

4.2. Properties of LDL^T for NC polynomials that are NC plush on an NC open set

In this subsection, we present properties of the LDL^T factorization of the nc complex hessian for an nc polynomial that is nc plush on an nc open set.

Recall from Section 1.2 that a set $\mathcal{G} \subseteq \cup_{n\geq 1}(\mathbb{R}^{n\times n})^g$ is an nc open set if:

(i) $\mathcal{G}$ respects direct sums, and
(ii) there exists a positive integer n_0 such that if $n > n_0$, the set $\mathcal{G}_n := \mathcal{G} \cap (\mathbb{R}^{n\times n})^g$ is an open set of matrix tuples

and an nc symmetric polynomial, p, is nc plush on an nc open set, $\mathcal{G}$, if p has an nc complex hessian, q, such that $q(X, X^T)[H, H^T]$ is positive semidefinite for all $X \in \mathcal{G}$ and for all $H \in (\mathbb{R}^{n\times n})^g$ for every $n \geq 1$.

Theorem 3.8 shows that the nc complex hessian has even degree; denote it $2N$. We will use this fact throughout the duration of the paper. The next lemma is a stepping stone for Lemma 4.4.

Lemma 4.3. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex hessian, q. Then, q must contain a term of the form*

$$\alpha m^T h^T hm \quad (\text{or } \alpha mhh^T m^T)$$

where m is an nc analytic monomial of degree $N-1$ and α is a positive real constant.

Proof. Proposition 3.7 implies q is a sum of hereditary and antihereditary polynomials. Let w be a term of degree $2N$ in q. Without loss of generality, suppose w is hereditary; i.e., w has the form

$$w = \alpha m_1^T h^T m_2^T m_3 h m_4$$

where $\alpha \in \mathbb{R}$, m_1, m_2, m_3, m_4 are nc analytic monomials in x, and

$$\deg(m_1) + \deg(m_2) + \deg(m_3) + \deg(m_4) = 2N - 2.$$

By Theorem 3.3(P2), q must contain the Levi-differentially wed term

$$\tilde{w} = \alpha \tilde{m}_1^T h^T h \tilde{m}_2$$

where $\tilde{m}_1, \tilde{m}_2$ are nc analytic monomials in x and $\deg(\tilde{m}_1) = \deg(\tilde{m}_2) = N - 1$.

If $\tilde{m}_1 = \tilde{m}_2$, we are done (except for showing $\alpha > 0$). If the conclusion of the lemma is false, so that q contains no term of the form $\alpha m^T h^T hm$, then this implies $\tilde{m}_1 \neq \tilde{m}_2$. Since q is symmetric, q must also contain the term

$$\tilde{w}^T = \alpha \tilde{m}_2^T h^T h \tilde{m}_1.$$

If we partition the border vector so that $e_1^T V = h\tilde{m}_1$ and $e_2^T V = h\tilde{m}_2$, then we get that

$$q = \begin{pmatrix} h\tilde{m}_1 \\ h\tilde{m}_2 \\ \vdots \end{pmatrix}^T \begin{pmatrix} 0 & \alpha & \cdots \\ \alpha & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} h\tilde{m}_1 \\ h\tilde{m}_2 \\ \vdots \end{pmatrix}.$$

This middle matrix is not positive semidefinite for any $X \in \mathcal{G}$. Hence, Theorem 2.2 implies that q is not positive semidefinite for all $X \in \mathcal{G}$. This contradicts the positivity of q on the nc open set, $\mathcal{G}$. Hence, q must contain some term of the form $\alpha m^T h^T hm$.

We now show $\alpha > 0$. Since we know that q contains a term of the form $\alpha m^T h^T hm$ with m an nc analytic or nc antianalytic monomial of degree $N-1$, the real constant α will appear on the diagonal in the middle matrix. Then, Theorem 2.2 implies that this α must be positive. □

When we write e_i, we mean the vector whose ith entry is 1 and every other entry is 0. From Eq. (4.3), we can write q as a sum of outer products

$$\begin{aligned} q &= V(x, x^T)[h, h^T]^T \left(\sum_{i=1}^{\mathcal{N}} (Le_i) d_i (Le_i)^T \right) V(x, x^T)[h, h^T] \\ &= \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i) d_i (Le_i)^T V(x, x^T)[h, h^T]. \end{aligned} \tag{4.4}$$

We stress that in Eq. (4.4), each Le_i and d_i depends on x and x^T. However, the next lemma shows that one element of D is constant and one column of L contains nc polynomials rather than nc rationals.

Lemma 4.4. *Let p be an nc symmetric polynomial that is nc plush on an nc open set. Let $2N$ denote the degree of its nc complex hessian, q. Then, we can write the nc complex hessian, q, as in Eqs. (4.3) and (4.4) where $L(x, x^T)$ is unit lower triangular and $D(x, x^T) = \operatorname{diag}(d_1, \ldots, d_{\mathcal{N}})$ with d_1 a positive real constant.*

Hence, each entry in Le_1, the first column of $L(x, x^T)$, is an nc polynomial rather than an nc rational.

Proof. Theorem 4.1 implies $D(x, x^T)$ is a diagonal matrix. Without loss of generality, Lemma 4.3 implies that q contains a term of the form

$$\alpha m^T h^T hm$$

where $\alpha > 0$ is a positive real constant and m is an nc analytic monomial of degree $N-1$. The MMR of q can be written as

$$q = \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} \alpha & \ell^T \\ \ell & \widehat{M} \end{pmatrix} \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}.$$

Since $\alpha > 0$, we can first pivot off α in computing the LDL^T factorization of the middle matrix to get

$$q = \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ \frac{1}{\alpha}\ell & I \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \widehat{M} - \frac{1}{\alpha}\ell\ell^T \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{\alpha}\ell^T \\ 0 & I \end{pmatrix} \begin{pmatrix} hm \\ \widehat{V} \end{pmatrix}.$$

Now we see that $d_1 = e_1^T D e_1 = \alpha > 0$ and that

$$Le_1 = \begin{pmatrix} 1 \\ \frac{1}{\alpha}\ell \end{pmatrix} \quad \text{and} \quad \widehat{M} - \frac{1}{\alpha}\ell\ell^T$$

contain only nc polynomials as entries. □

The next lemma provides even more specific structure to Le_1 and maintains the nc polynomial structure.

Lemma 4.5. *Under the same hypotheses of Lemma 4.4, either:*

(i) *every entry of Le_1 (the 1st column of $L(x, x^T)$) is an nc antianalytic polynomial, d_1 (the 1st diagonal entry of $D(x, x^T)$) is a positive real constant, and the corresponding monomials in $V(x, x^T)[h, h^T]$ are nc analytic; or*
(ii) *every entry of Le_1 (the 1st column of $L(x, x^T)$) is an nc analytic polynomial, d_1 (the 1st diagonal entry of $D(x, x^T)$) is a positive real constant, and the corresponding monomials in $V(x, x^T)[h, h^T]$ are nc antianalytic.*

Proof. Lemma 3.4 implies that q can be written as

$$q = A(x)[h]^T Q_1(x, x^T) A(x)[h] + A(x^T)[h^T]^T Q_5(x, x^T) A(x^T)[h^T]$$

where each entry of $A(x)[h]$ is an nc analytic monomial and each entry of $A(x^T)[h^T]$ is an nc antianalytic monomial. Also, Q_1 contains hereditary nc polynomials and Q_5 contains antihereditary nc polynomials. Then, we have that

$$\begin{aligned} q &= A(x)[h]^T L_1 D_1 L_1^T A(x)[h] + A(x^T)[h^T]^T L_2 D_2 L_2^T A(x^T)[h^T] \\ &= \underbrace{\begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}^T}_{V(x,x^T)[h,h^T]^T} \underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}^T}_{L^T} \underbrace{\begin{pmatrix} A(x)[h] \\ A(x^T)[h^T] \end{pmatrix}}_{V(x,x^T)[h,h^T]}. \end{aligned} \tag{4.5}$$

By Lemma 4.3, we know that there are two cases to consider: either q contains a term of the form $d_1 m^T h^T hm$ or q contains a term of the form $d_1 mhh^T m^T$. We proceed with the former and assume that q contains a term of the form

$$d_1 m^T h^T hm$$

where m is an nc analytic monomial in x (so that hm is an entry in $A(x)[h]$) of degree $N - 1$ and d_1 is a positive real constant. Lemma 4.4 implies that $e_1^T D_1 e_1 = d_1$ and that each entry of Le_1 is an nc polynomial. From Eq. (4.5), we have that

$$Le_1 = \begin{pmatrix} L_1 e_1 \\ 0 \end{pmatrix}$$

and $(Le_1)^T V = (L_1 e_1)^T A(x)[h]$.

Next, write q as in Eq. (4.4) and see that the first term in this sum becomes

$$V^T (Le_1) d_1 (Le_1)^T V = d_1 ((L_1 e_1)^T A(x)[h])^T ((L_1 e_1)^T A(x)[h]).$$

Proposition 3.7 implies that q is a sum of hereditary and antihereditary polynomials. Therefore, since $A(x)[h]$ contains only nc analytic monomials, this forces $(L_1 e_1)^T$ to contain only nc analytic polynomials (which means that $L_1 e_1$ contains only nc antianalytic polynomials). This completes the proof of Case (i).

The proof of Case (ii) works the same way when, by Lemma 4.3, we assume that q contains a term of the form

$$d_1 mhh^T m^T$$

where m is an nc analytic monomial in x of degree $N - 1$ and d_1 is a positive real constant. □

The next lemma is a technical lemma that is used as a stepping stone to help prove Proposition 4.11.

Lemma 4.6. *Let p be an nc symmetric polynomial that is nc plush on an nc open set. Let $2N$ denote the degree of its nc complex hessian, q. Then, we can write q as in Eq. (4.4)*

$$q = \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i) d_i (Le_i)^T V(x, x^T)[h, h^T]$$

with

$$V(x,x^T)[h,h^T]^T e_1 = x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \quad (\textit{resp.}\ V(x,x^T)[h,h^T]^T e_1 = x_{i_N} \cdots x_{i_2} h_{i_1})$$

in which case, any term in q that has the form

$$d_1 \gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h) \quad (\textit{resp.}\ d_1 \gamma x_{i_N} \cdots x_{i_2} h_{i_1} m(x^T,h^T))$$

where γ is a real constant and $m(x,h)$ is some nc analytic monomial in x, h of degree 1 in h (resp. $m(x^T,h^T)$ is some nc antianalytic monomial in x^T, h^T of degree 1 in h^T), is a term in the nc polynomial

$$d_1 V(x,x^T)[h,h^T]^T (Le_1)(Le_1)^T V(x,x^T)[h,h^T].$$

Moreover, $\gamma m(x,h)$ (resp. $\gamma m(x^T,h^T)$) is a term in the nc analytic (resp. nc antianalytic) polynomial

$$(Le_1)^T V(x,x^T)[h,h^T].$$

Proof. Proposition 3.7 implies q is a sum of hereditary and antihereditary polynomials. Since the degree of q is $2N$, there exists a term, w, in q of degree $2N$. By Lemma 4.3, there are two cases to consider: either w looks like

$$w = d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N}$$

or w looks like

$$w = d_1 x_{i_N} \cdots x_{i_2} h_{i_1} h_{i_1}^T x_{i_2}^T \cdots x_{i_N}^T$$

with $d_1 \in \mathbb{R}_+$. We proceed and supply the details of the former, as the latter is similar. We partition the border vector $V(x,x^T)[h,h^T]$ as

$$V(x,x^T)[h,h^T] = \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix}$$

where $h_{i_1} x_{i_2} \cdots x_{i_N}$ is not a monomial entry in the vector $\widehat{V}$. Then, q becomes

$$\begin{aligned} q &= \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ \ell & \widehat{L} \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & \widehat{D} \end{pmatrix} \begin{pmatrix} 1 & \ell^T \\ 0 & \widehat{L}^T \end{pmatrix} \begin{pmatrix} h_{i_1} x_{i_2} \cdots x_{i_N} \\ \widehat{V} \end{pmatrix} \\ &= d_1 \overbrace{(x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N} + x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \ell^T \widehat{V} + \widehat{V}^T \ell h_{i_1} x_{i_2} \cdots x_{i_N} + \widehat{V} \ell \ell^T \widehat{V})}^{V(x,x^T)[h,h^T]^T (Le_1)(Le_1)^T V(x,x^T)[h,h^T] = (x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T + \widehat{V}^T \ell)(h_{i_1} x_{i_2} \cdots x_{i_N} + \ell^T \widehat{V})} + \widehat{V}^T \widehat{L} \widehat{D} \widehat{L}^T \widehat{V}. \end{aligned} \tag{4.6}$$

Since $x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T$ is not a monomial entry in the vector $\widehat{V}^T$, this shows that any term in q of the form $d_1 \gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h)$, where γ is a real constant and $m(x,h)$ is an nc analytic monomial of degree 1 in h, is a term in the nc polynomial

$$d_1 V(x,x^T)[h,h^T]^T (Le_1)(Le_1)^T V(x,x^T)[h,h^T].$$

Eq. (4.6) implies that either

$$d_1 \gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h) = d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T h_{i_1} x_{i_2} \cdots x_{i_N}$$

or that $d_1 \gamma x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T m(x,h)$ is a term in the nc polynomial

$$d_1 x_{i_N}^T \cdots x_{i_2}^T h_{i_1}^T \ell^T \widehat{V}.$$

This implies that either $\gamma = 1$ and $m(x,h) = h_{i_1} x_{i_2} \cdots x_{i_N}$ or that $\gamma m(x,h)$ is a term in the nc polynomial $\ell^T \widehat{V}$. Hence, $\gamma m(x,h)$ is a term in the nc polynomial

$$h_{i_1} x_{i_2} \cdots x_{i_N} + \ell^T \widehat{V} = (Le_1)^T V(x,x^T)[h,h^T]$$

and Lemma 4.5 implies that $(Le_1)^T V(x,x^T)[h,h^T]$ is nc analytic. □

4.3. 1-differentially wed monomials and NC directional derivatives

Earlier, in Section 3.3, we introduced Levi-differentially wed monomials and provided necessary and sufficient conditions as to when an nc polynomial is an nc complex hessian. Now, we introduce 1-*differentially wed monomials* and state Proposition 4.10, which gives necessary and sufficient conditions as to when a given nc polynomial is an nc directional derivative. This machinery is needed to prove Proposition 4.11.

4.3.1. Notation

Let m be a monomial that is degree one in h or h^T. This means m contains some h_i or some h_i^T. If m contains some h_i, we denote

$$m|_{h_i \to x_i}$$

as the monomial only in x and x^T where x_i replaces h_i in m. If m contains some h_i^T, we denote

$$m|_{h_i^T \to x_i^T}$$

as the monomial only in x and x^T where x_i^T replaces h_i^T in m.

Sometimes, we may write h_i^γ or x_i^γ where γ is either $\emptyset$ or T. When $\gamma = \emptyset$, we define

$$h_i^\emptyset := h_i$$
$$x_i^\emptyset := x_i$$

and when $\gamma = T$, we mean h_i^T or x_i^T.

Example 4.7. If $m = x_1 h_2^T x_1^T$ then $m|_{h_2^T \to x_2^T} = x_1 x_2^T x_1^T$.

4.3.2. 1-differentially wed monomials

For α, β either $\emptyset$ or T, two monomials m and $\tilde{m}$ are called 1-*differentially wed* if both are degree one in h or h^T and if

$$m|_{h_i^\alpha \to x_i^\alpha} = \tilde{m}|_{h_j^\beta \to x_j^\beta}.$$

Example 4.8. The monomials $m = h_1 x_2^T x_1$ and $\tilde{m} = x_1 h_2^T x_1$ are 1-differentially wed.

Example 4.9. The monomials $m = x_2 h_2 x_2$ and $\tilde{m} = x_1 x_2 h_2$ are not 1-differentially wed.

The following theorem gives necessary and sufficient conditions for an nc polynomial to be an nc directional derivative.

Proposition 4.10. *A polynomial p in $x = (x_1, \dots, x_g)$ and $h = (h_1, \dots, h_g)$ is an nc directional derivative if and only if each monomial in p has degree one in h (i.e., contains some h_j) and whenever a monomial m occurs in p, each monomial which is* 1-*differentially wed to m also occurs in p and has the same coefficient.*

Proof. This is proved in [1]. $\square$

4.4. Part of the NC complex hessian is an NC complex hessian

In this subsection, we focus on writing the nc complex hessian, q, as in Eq. (4.4)

$$q = \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i) d_i (Le_i)^T V(x, x^T)[h, h^T].$$

This subsection culminates with the result that the nc polynomial

$$d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T],$$

is the nc complex hessian for some nc polynomial that is nc plush on an nc open set. In order to do this, we first show that the nc polynomial

$$(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc directional derivative for some nc analytic or nc antianalytic polynomial.

Proposition 4.11. *Let p be an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex hessian, q. If we write q as in Eq. (4.4) and d_1 is constant, then the nc polynomial*

$$(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc directional derivative for an nc analytic polynomial or an nc antianalytic polynomial.

In addition, the nc polynomial

$$d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc complex hessian for some nc polynomial that is nc plush on $\mathcal{G}$.

Proof. By Lemma 4.5, there are two cases to consider. We proceed and supply the details for the first, as the second is similar. We assume that

$$(Le_1)^T V(x, x^T)[h, h^T]$$

is an nc analytic polynomial where $V(x,x^T)[h,h^T]$ and Le_1 are partitioned as

$$V(x,x^T)[h,h^T]=\begin{pmatrix}V_N\\V_{N-1}\\\vdots\\V_1\end{pmatrix},\qquad Le_1=\begin{pmatrix}\ell_0\\\ell_1\\\vdots\\\ell_{N-1}\end{pmatrix},\quad \ell_0=\begin{pmatrix}1\\\star\\\vdots\\\star\end{pmatrix},\tag{4.7}$$

where $\star$ is any nc polynomial and V_j is a vector that contains only nc analytic monomials of the form $h_{i_1}x_{i_2}\cdots x_{i_j}$ having total degree j. Each ℓ_j is a vector with the same length as V_j and, by Lemma 4.5, ℓ_j contains only nc antianalytic polynomials (ℓ_j^T contains only nc analytic polynomials). With this setup, we have that

$$\mathcal{F}(x,h):=(Le_1)^TV(x,x^T)[h,h^T]=\sum_{j=0}^{N-1}\ell_j^TV_{N-j}$$

is an nc analytic polynomial in x and h. We define this as $\mathcal{F}(x,h)$ for convenience.

Lemma 4.4 implies $d_1\in\mathbb{R}_+$ is a constant and Eq. (4.4) implies that q contains the terms

$$d_1V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^TV(x,x^T)[h,h^T]=d_1\left(\sum_{j=0}^{N-1}\ell_j^TV_{N-j}\right)^T\left(\sum_{j=0}^{N-1}\ell_j^TV_{N-j}\right).\tag{4.8}$$

Then, since the degree of q is $2N$ and the degree of each border vector monomial in V_{N-j} is $N-j$, it follows that the degree of each nc analytic polynomial in ℓ_j^T is at most j.

Lemma 4.3 implies that q contains some term of the form

$$\alpha^2x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^Th_{i_1}x_{i_2}\cdots x_{i_N}\tag{4.9}$$

with α a nonzero real constant. This implies that the vector V_N contains the monomial $h_{i_1}x_{i_2}\cdots x_{i_N}$ as an entry. Without loss of generality, assume this monomial is first in lexicographic order. Then,

$$e_1^TV(x,x^T)[h,h^T]=e_1^TV_N=h_{i_1}x_{i_2}\cdots x_{i_N}.$$

As in the proof of Lemma 4.4, if M represents the middle matrix of q, then $e_1^TMe_1=\alpha^2$ and, after one step in the LDL^T algorithm, we see that $\alpha^2=d_1$. Then, by Theorem 3.3(P2), q also contains the Levi-differentially wed terms

$$\begin{aligned}&d_1x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^Tx_{i_1}h_{i_2}x_{i_3}\cdots x_{i_N}\\&d_1x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^Tx_{i_1}x_{i_2}h_{i_3}\cdots x_{i_N}\\&\vdots\\&d_1x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^Tx_{i_1}x_{i_2}\cdots x_{i_{N-1}}h_{i_N}.\end{aligned}$$

Since q contains these terms and the term in (4.9), Lemma 4.6 implies that $\mathcal{F}(x,h)$ contains the term

$$h_{i_1}x_{i_2}x_{i_3}\cdots x_{i_N}\tag{4.10}$$

and the terms

$$\begin{aligned}&x_{i_1}h_{i_2}x_{i_3}\cdots x_{i_N}\\&x_{i_1}x_{i_2}h_{i_3}\cdots x_{i_N}\\&\vdots\\&x_{i_1}x_{i_2}\cdots x_{i_{N-1}}h_{i_N}.\end{aligned}$$

Hence, $\mathcal{F}(x,h)$ contains all 1-differentially wed monomials to $h_{i_1}x_{i_2}x_{i_3}\cdots x_{i_N}$ as terms. Proposition 4.10 implies that $\mathcal{F}(x,h)$ contains the nc directional derivative of $x_{i_1}x_{i_2}x_{i_3}\cdots x_{i_N}$.

Now we pick any other term in $\mathcal{F}(x,h)$ and show that $\mathcal{F}(x,h)$ contains all other 1-differentially wed monomials to it and that they all occur with the same coefficient. Suppose $\mathcal{F}(x,h)$ contains the term

$$\gamma x_{s_1}\cdots x_{s_k}h_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}.$$

We already showed that $\mathcal{F}(x,h)$ contains the monomial in (4.10), $h_{i_1}x_{i_2}\cdots x_{i_N}$, as a term so $\mathcal{F}(x,h)^T$ must contain the monomial $x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^T$ as a term. This implies that $d_1\mathcal{F}(x,h)^T\mathcal{F}(x,h)$ contains the terms

$$d_1x_{i_N}^T\cdots x_{i_2}^Th_{i_1}^T(h_{i_1}x_{i_2}\cdots x_{i_N}+\gamma x_{s_1}\cdots x_{s_k}h_{\beta_1}x_{\beta_2}\cdots x_{\beta_{N-j}}).$$

Hence, q contains the term

$$d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1}\cdots x_{s_k} h_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}$$

and Theorem 3.3(P2) implies that q contains the Levi-differentially wed terms

$$\begin{gathered}
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T h_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1} h_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1} x_{s_2}\cdots h_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} h_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
d_1\gamma x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T x_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots h_{\beta_{N-j}}.
\end{gathered}$$

Since q contains all of these terms with $x_{i_N}^T\cdots x_{i_2}^T h_{i_1}^T$ on the left, Lemma 4.6 implies $\mathcal{F}(x,h)$ must contain the terms

$$\begin{gathered}
\gamma h_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\gamma x_{s_1} h_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
\gamma x_{s_1} x_{s_2}\cdots h_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\gamma x_{s_1} x_{s_2}\cdots x_{s_k} h_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}\\
\gamma x_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} h_{\beta_2}\cdots x_{\beta_{N-j}}\\
\vdots\\
\gamma x_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots h_{\beta_{N-j}}.
\end{gathered}$$

All of these terms in $\mathcal{F}(x,h)$ have the same coefficient, γ, and the monomials are 1-differentially wed to each other. Thus, Proposition 4.10 implies that they sum to the nc directional derivative of

$$\gamma x_{s_1} x_{s_2}\cdots x_{s_k} x_{\beta_1} x_{\beta_2}\cdots x_{\beta_{N-j}}.$$

Hence, we have shown that $\mathcal{F}(x,h)=(Le_1)^T V(x,x^T)[h,h^T]$ is an nc directional derivative, where, without loss of generality, we assumed that $\mathcal{F}(x,h)$ was nc analytic.

Now we have that

$$\mathcal{F}(x,h) := (Le_1)^T V(x,x^T)[h,h^T]$$

is the nc directional derivative of some nc analytic or nc antianalytic polynomial. Suppose, without loss of generality, that $\mathcal{F}(x,h)$ is the nc directional derivative of some nc analytic polynomial, $\mathcal{F}(x)$. Then, $\mathcal{F}(x,h)$ is nc analytic and

$$d_1\mathcal{F}(x,h)^T\mathcal{F}(x,h) = d_1 V(x,x^T)[h,h^T]^T(Le_1)(Le_1)^T V(x,x^T)[h,h^T]$$

is the nc complex hessian of the nc polynomial

$$d_1\mathcal{F}(x)^T\mathcal{F}(x).$$

Hence, for any $n\geq 1$, any $X\in\mathcal{G}$, and any $H\in(\mathbb{R}^{n\times n})^g$, we have

$$d_1\mathcal{F}(X,H)^T\mathcal{F}(X,H)\succeq 0. \quad \square$$

4.5. Constant D result

In this subsection, we show that for an nc symmetric polynomial, p, that is nc plush on an nc open set, the matrix $D(x,x^T)$ in Eq. (4.3) has no dependence on x or x^T and is actually a positive semidefinite constant real matrix. First, we require a helpful lemma.

Lemma 4.12. *If p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$, then its nc complex hessian, q, can be written as in Eq.* (4.3)

$$q = V(x,x^T)[h,h^T]^T L(x,x^T) D(x,x^T) L(x,x^T)^T V(x,x^T)[h,h^T]$$

where $D(x,x^T)$ is a diagonal matrix of nc rationals and $D(X,X^T)\succeq 0$ for all $X\in\mathcal{G}$.

Proof. This follows immediately from Theorem 4.1. □

Theorem 4.13. *Suppose p is an nc symmetric polynomial that is nc plush on an nc open set, $\mathcal{G}$. Let $2N$ denote the degree of its nc complex hessian, q. Then q can be written as in Eq. (4.3)*

$$q = V(x, x^T)[h, h^T]^T L(x, x^T)D(x, x^T)L(x, x^T)^T V(x, x^T)[h, h^T]$$

where $D(x, x^T) = \operatorname{diag}(d_1, d_2, \ldots, d_{\mathcal{N}})$ is a positive semidefinite constant real matrix (i.e., $d_i \in \mathbb{R}_{\geq 0}$ for all $i = 1, \ldots, \mathcal{N}$) and $L(x, x^T)$ is a unit lower triangular matrix of nc polynomials.

Proof. Lemma 4.12 implies $D(X, X^T) \succeq 0$ for every $X \in \mathcal{G}$. This means $d_i(X, X^T) \succeq 0$ for every $X \in \mathcal{G}$ and every $i = 1, \ldots, \mathcal{N}$. It remains to show that each d_i is a nonnegative constant real number. We proceed by induction.

First, write the nc complex hessian, q, as in Eq. (4.4)

$$q = \sum_{i=1}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i)d_i(x, x^T)(Le_i)^T V(x, x^T)[h, h^T].$$

Lemma 4.4 shows $d_1 \in \mathbb{R}_+$ is a constant, Le_1 contains nc polynomial entries, and Proposition 4.11 shows that

$$d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T]$$

is the nc complex hessian for some nc polynomial that is nc plush on $\mathcal{G}$. Since nc differentiation is linear, we know that the difference of two nc complex hessians is an nc complex hessian. This implies that

$$\begin{aligned} \widetilde{q} &:= q - d_1 V(x, x^T)[h, h^T]^T (Le_1)(Le_1)^T V(x, x^T)[h, h^T] \\ &= \sum_{i=2}^{\mathcal{N}} V(x, x^T)[h, h^T]^T (Le_i)d_i(x, x^T)(Le_i)^T V(x, x^T)[h, h^T] \end{aligned}$$

is an nc complex hessian. Since $d_i(X, X^T) \succeq 0$ for all $X \in \mathcal{G}$ and for all i, we have that $\widetilde{q}$ is the nc complex hessian for an nc symmetric polynomial that is nc plush on $\mathcal{G}$. Therefore, we inductively continue this same process and subtract off appropriate terms to conclude that each d_i is a nonnegative constant real number. □

Now we give a partial list of references. For a complete list, see [1].

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