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Derivation of the errors involved in interpolation and their application to numerical quadrature formulae

A. Chakrabarti *, Hamsapriye

Department of Mathematics, Indian Institute of Science, Bangalore - 560 012, India

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Abstract

Simple methods are presented to derive closed-form expressions for the errors involved in the Lagrange interpolation formula. As applications of this formula for the error in the interpolation, the corresponding errors in the quadrature formulae are also taken up. Few examples are considered for numerical experiments. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In many text books on numerical analysis, (see [2, 3]), the closed form for the error in Lagrange's interpolation formula, involving a given function $f(x)$, for $x \in [a,b]$, is derived by writing $f(x)$ as

$$f(x) = L_n(x) + E_n(x), \tag{1.1}$$

where $L_n(x)$ is the nth-order interpolation polynomial based on the nodal points $x_0, x_1, \ldots, x_n$, with $x_0 = a < x_1 < x_2 < \cdots < x_n = b$ and $E_n(x)$ is the error involved, which vanishes at the nodal points x_j, $j = 0, 1, \ldots, n$ and is sought in the form

$$E_n(x) = \pi_{n+1}(x) r(x), \tag{1.2}$$

in which $\pi_{n+1}(x)$ denotes the product function as defined by

$$\pi_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \tag{1.3}$$

and $r(x)$ is an unknown function to be determined.

* Corresponding author. alok@math.iisc.ernet.in.

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Then defining a function $F(x)$ as

$$F(x) = f(x) - L_n(x) - \pi_{n+1}(x)r(x'), \tag{1.4}$$

for $x' \in (a,b)$, $x' \neq x_j, j=1,2,\dots,n-1$, so that $F(x_j)=0=F(x')$, for $j=0,1,\dots,n$ and by a repeated application of Rolle's theorem, $(n+1)$ times, we easily obtain that

$$r(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}, \tag{1.5}$$

for some $\xi \in (a,b)$ and $f \in C^{n+1}([a,b])$.

An alternative approach (see [1]) to derive the above form of E_n, as given by (1.2) and (1.5), is to look upon the error as a solution of an $(n+1)$th-order ordinary differential equation (ODE), with $(n+1)$ conditions prescribed. We have used the idea of Green's function technique, applicable to this ODE to obtain a closed form expression for $E_n(x)$, without actually constructing the Green's function. The details of this derivation are given in Section 2.

In both the above descriptions of the error formula, the function $f(x)$ has been assumed to possess $(n+1)$ derivatives and thereby error bounds can be obtained by assuming that $f^{(n+1)}(x)$ is bounded by a constant M_n, say, over the interval $[a,b]$. Then from relations (1.2) and (1.5), we get

$$|E_n(x)| \leqslant \frac{M_n}{(n+1)!} |\pi_{n+1}(x)|. \tag{1.6}$$

There are cases, where the function $f(x)$ may not be differentiable $(n+1)$ times. In other words, the function $f(x)$ may be differentiable only p times, say, for $p \leqslant n$, or may not be differentiable at all, in which case the bound as given by (1.6) cannot be used. In this paper, for the former case, we have given the error bound, by first recognizing the error $E_n(x)$ in the form

$$E_n(x) = \pi_{n+1}(x) f(x, x_0, \dots, x_n) \tag{1.7}$$

and then utilizing some interesting properties of the nth-order divided difference $f(x,x_0,\dots,x_n)$, which can be easily proved. In the later case, that is, if f is not at all differentiable, but may just satisfy Lipschitz condition over $[a,b]$, with the Lipschitz constant L, (i.e. $|f(x) - f(y)| < L|x-y|$, $\forall x,y \in (a,b)$), we have given a computable error bound. Also in some cases, where f may satisfy a special condition, called as "G-condition", (see [4]), we are still able to give an error bound and a detailed description about this has been given in Section 3. Before we proceed for any further study, we define the term "G-condition", as below.

Definition. A function $f(x)$ is said to satisfy the G-condition, if $\forall x,y,z \in (a,b)$, $\exists\ G \in \mathbb{R}$ such that:

$$|f(x)(y-z) + f(y)(z-x) + f(z)(x-y)| < G|(x-y)(y-z)(z-x)|. \tag{1.8}$$

A numerical example has been taken up to clarify the situation.

In Section 4, we have used the ideas of Section 3, to get the error bounds in the various quadrature formulae. A numerical example is given in this context, for clarity purposes.

2. The Green's function approach

From expression (1.1), we have that

$$E_n(x) = f(x) - L_n(x). \tag{2.1}$$

Now, it is clear that the operator D^{n+1}, with $D := \mathrm{d}/\mathrm{d}x$, representing the ordinary differential operator, annihilates the Lagrange polynomial $L_n(x)$, i.e. $D^{n+1}L_n(x)=0$. Thus, operating by D^{n+1}, on both sides of relation (2.1) and noting that $f(x_j) = L_n(x_j)$, for $j = 0, 1, \ldots, n$, we arrive at the relations

$$\begin{aligned} D^{n+1}E_n(x) &= f^{(n+1)}(x), \\ E_n(x_j) &= 0. \end{aligned} \tag{2.2}$$

Using the idea of the well-known Green's function technique, we can express $E_n(x)$ in the closed form as given by

$$E_n(x) = \int_a^b G_n(x,s) f^{(n+1)}(s)\,\mathrm{d}s, \quad x \in [a,b], \tag{2.3}$$

where $G_n(x,s)$ is the Green's function of the operator D^{n+1}, satisfying the conditions:

(i) $D^{n+1}G_n(x,s) = \delta(x-s)$,
(ii) $G(x_j,s) = 0, j = 0, 1, \ldots, n$.

On the assumption that $G_n(x,s)$ maintains the same sign throughout the interval (a,b), we can express (2.3), in its simplified form, by using the mean value theorem of integral calculus (see [2, 3]), as

$$E_n(x) = \psi_n(x) f^{(n+1)}(\xi), \quad \xi \in (a,b), \tag{2.4}$$

where $\psi_n(x)$ is defined as

$$\psi_n(x) = \int_a^b G_n(x,s)\,\mathrm{d}s \tag{2.5}$$

and satisfies the property that (cf. (i) and (ii), above)

$$\begin{aligned} D^{n+1}\psi_n(x) &= 1, \\ \psi_n(x_j) &= 0. \end{aligned} \tag{2.6}$$

The system in relations (2.6) can be solved, by assuming $\psi_n(x)$ in the form

$$\psi_n(x) = c_0 + c_1 x + \cdots + c_n x^n + \frac{x^{n+1}}{(n+1)!}. \tag{2.7}$$

We find that

$$\psi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}\ . \tag{2.8}$$

Using relation (2.8) in relation (2.4), we get

$$E_n(x) = f^{(n+1)}(\xi)\frac{\pi_{n+1}(x)}{(n+1)!}, \quad \xi \in (a,b)\ . \tag{2.9}$$

We could as well stop at relation (2.8), (after constructing the function $\psi_n(x)$) and define a function $G_{\tilde{x}}(x)$, for $\tilde{x} \in (a,b)$, $\tilde{x} \neq x_j$, $j = 0, 1, \ldots, n$, such that

$$G_{\tilde{x}}(x) = E_n(\tilde{x})\psi_n(x) - E_n(x)\psi_n(\tilde{x}). \tag{2.10}$$

Now, it is clear that $G_{\tilde{x}}(x_j) = 0$, $G_{\tilde{x}}(\tilde{x}) = 0$ and hence by a repeated application of Rolle's theorem, one arrives at the relation that

$$D^{n+1}G_{\tilde{x}}(\tilde{\xi}) = 0, \quad \tilde{\xi} \in (a,b). \tag{2.11}$$

Using relations (2.11) in (2.10) and using the relations as given by (2.2) and (2.6), we easily derive that

$$E_n(\tilde{x}) = f^{(n+1)}(\tilde{\xi})\psi_n(\tilde{x}), \tag{2.12}$$

which is true for every $\tilde{x} \in (a,b)$. Thus, by using the relation in (2.8), we get the relation as given by (2.9).

3. An approach for weaker classes of functions

Normally, one may be interested in knowing the amount of error incurred in the interpolation, even before interpolating the given data. In such cases, one uses the estimates of the error, that are available. One such estimate of the error is given by relation (1.6), which is based on a very strong condition that the function $f(x) \in C^{n+1}([a,b])$. Now, as is clear, always we may not have a data, which has the above property. Suppose the function is differentiable p times, for $p \leqslant n$, then still an error bound can be obtained using the following art:

From relation (1.7), we get

$$|E_n(x)| = |f(x, x_0, \ldots, x_n)||\pi_{n+1}(x)|. \tag{3.1}$$

By the principle of mathematical induction, it can be easily proved that the nth-order divided difference $f(x, x_0, \ldots, x_n)$ can be expressed in the form as given by

$$\begin{aligned} f(x, x_0, \ldots, x_n) = & \frac{f(x, x_0, \ldots, x_{p-2}, x_{p-1})}{(x_{p-1} - x_p)(x_{p-1} - x_{p+1}) \cdots (x_{p-1} - x_n)} \\ & + \frac{f(x, x_0, \ldots, x_{p-2}, x_p)}{(x_p - x_{p-1})(x_p - x_{p+1}) \cdots (x_p - x_n)} + \cdots \\ & + \frac{f(x, x_0, \ldots, x_{p-2}, x_n)}{(x_n - x_{p-1})(x_n - x_p) \cdots (x_n - x_{n-1})}, \end{aligned} \tag{3.2}$$

for any fixed $p \geqslant 1$. For example, we can express $f(x, x_0, x_1, x_2, x_3)$ as equal to

$$f(x, x_0, x_1, x_2, x_3) = \frac{f(x, x_0, x_1)}{(x_1 - x_2)(x_1 - x_3)} + \frac{f(x, x_0, x_2)}{(x_2 - x_1)(x_2 - x_3)} + \frac{f(x, x_0, x_3)}{(x_3 - x_1)(x_3 - x_2)},$$

by fixing $p = 2$ and $n = 3$.

It is interesting to note that the above representation of the nth-order divided difference is not unique. Since f is differentiable p times, we may express (3.2) in the form

$$f(x,x_0,\dots,x_n)=\frac{1}{p!}\left[\frac{f^{(p)}(\xi_1)}{(x_{p-1}-x_p)(x_{p-1}-x_{p+1})\cdots(x_{p-1}-x_n)}+\frac{f^{(p)}(\xi_2)}{(x_p-x_{p-1})(x_p-x_{p+1})\cdots(x_p-x_n)}+\cdots+\frac{f^{(p)}(\xi_{n-p+1})}{(x_n-x_{p-1})(x_n-x_p)\cdots(x_n-x_{n-1})}\right], \tag{3.3}$$

for some $\xi_1,\xi_2,\dots,\xi_{n-p+1}\in(a,b)$. Now, assuming that, $f^{(p)}(x)$ is bounded on $[a,b]$, by a constant, say, M_p, we get,

$$|E_n(x)|\leqslant\frac{M_p}{p!}\left[\sum_{x_{p-1},x_p,\dots,x_n}\frac{1}{|x_{p-1}-x_p||x_{p-1}-x_{p+1}|\cdots|x_{p-1}-x_n|}\right]|\pi_{n+1}(x)|. \tag{3.4}$$

Since, there is no unique way of representing the nth-order divided difference integers of sum of pth order difference ($p<n$), for each such representation we get a bound for the error.

Example 1. We consider a special example

$$f(x)=x|x|,\quad x\in[-1,1]. \tag{3.5}$$

It is clear that, $f(x)$ is once differentiable and $|f'(x)|\leqslant 2$. By fitting a third-order Newton's interpolation formula, we get

$$f_3(x)=\frac{3x^3+x}{4}\ . \tag{3.6}$$

For some arbitrary point, say, $x=\frac{1}{2}$, the exact error in the interpolation formula reads as $|E_3(\frac{1}{2})|=0.03125$. Using the bound as given by relation (3.4), we get $|E_3(\frac{1}{2})|\leqslant 0.9375$, ($M_1=2$).

We mention at this stage, that the following theorems have been proved in [4]:

Theorem A. *If a function $f(x)$ satisfies the conditions:*

(i) $f^{(k)}\in C([a,b])$, $k=0,1,\dots,n$,

(ii) $f^{(n)}$ *satisfies the Lipschitz condition with the constant M_n, on $[a,b]$, then the bound $|E_n(x)|$ is given to be*

$$|E_n(x)|\leqslant\frac{M_n|\pi_{n+1}(x)|}{(n+1)!} \tag{3.7}$$

and

Theorem B. *If a function $f(x)$ satisfies the conditions:*

(i) $f^{(k)}\in C([a,b])$, $k=0,1,\dots,n-1$,

(ii) $f^{(n-1)}$ *satisfies the G-condition on* $[a,b]$, *then the error* $|E_n(x)|$ *has the bound, as given to be*

$$|E_n(x)| \leqslant \frac{2G\ |\pi_{n+1}(x)|}{(n+1)!}. \tag{3.8}$$

It is clear that, none of the bounds given by (3.7) or (3.8), are applicable for the above example as in (3.5).

It may so happen in practice, that f need not be differentiable at all. In such cases the bounds that are studied below, may come to the rescue. We assume that the given function satisfies the G-condition, on the interval $[a,b]$. From the relation (1.8), we derive that

$$|f(x,y,z)| \leqslant G, \tag{3.9}$$

where $f(x,y,z)$ is the well-known second-order divided difference of $f(x)$. From relation (3.2), we get for $p=2$, that

$$\begin{aligned} f(x,x_0,\ldots,x_n) = {} & \frac{f(x,x_0,x_1)}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)} \\ & + \frac{f(x,x_0,x_2)}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} + \cdots \\ & + \frac{f(x,x_0,x_n)}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}. \end{aligned} \tag{3.10}$$

Using relation (3.10) in (3.1) and assuming that $f(x)$ satisfies the G-condition, we derive that

$$\begin{aligned} |E_n(x)| & \leqslant G|\pi_{n+1}(x)| \left[\sum_{x_1,x_2,\ldots,x_n} \frac{1}{|x_1-x_2||x_1-x_3|\cdots|x_1-x_n|} \right], \quad n \geqslant 2 \\ & \leqslant G|\pi_2(x)|, \quad n=1. \end{aligned} \tag{3.11}$$

If we assume that f satisfies the Lipschitz condition in (a,b), with the Lipschitz constant L, we derive that

$$\begin{aligned} |E_n(x)| & \leqslant L|\pi_{n+1}(x)| \left[\sum_{x_0,x_1,\ldots,x_n} \frac{1}{|x_0-x_1||x_0-x_2|\cdots|x_0-x_n|} \right], \quad n \geqslant 2 \\ & \leqslant 2L\frac{|\pi_2(x)|}{|x_1-x_0|}, \quad n=1, \end{aligned} \tag{3.12}$$

using a relation similar to that of (3.10), for $p=1$.

For the sake of completeness of the present discussion, we give the error bounds for the case when the $(n+1)$ nodes x_i $(i=0,1,\ldots,n)$ are equidistant, i.e. $x_i = x_0 + ih$.

From relation (3.4), we obtain

$$|E_n(x)| \leqslant \frac{M_p h^p}{p!} \left[\sum_{q=1}^{(n-p+1)/2} \frac{2}{(q-1)!(n-p+2-q)!} + \frac{1}{[((n-p+1)/2)!]^2} \right] |\omega_{n+1}(s)|, \tag{3.13}$$

for $n - p + 1$ even and

$$|E_n(x)| \leqslant \frac{2M_p h^p}{p!}\left[\sum_{q=1}^{(n-p+2)/2}\frac{1}{(q-1)!(n-p+2-q)!}\right]|\omega_{n+1}(s)| \ , \tag{3.14}$$

for $n - p + 1$ odd, wherein we have defined $s = (x - x_0)/h$ and that

$$\omega_{n+1}(s) = s(s-1)\cdots(s-n).$$

Relations (3.11) give rise to the following results:
For even n, we derive that

$$\begin{aligned}|E_n(x)| &\leqslant 2Gh^2|\omega_{n+1}(s)|\left[\sum_{p=1}^{n/2}\frac{1}{(p-1)!(n-p)!}\right], \quad n\geqslant 4,\\ &\leqslant 2Gh^2|\omega_3(s)|, \quad n=2,\end{aligned} \tag{3.15}$$

For odd n, we derive that

$$\begin{aligned}|E_n(x)| &\leqslant Gh^2|\omega_{n+1}(s)|\left[\sum_{p=1}^{(n-1)/2}\frac{2}{(p-1)!(n-p)!}+\frac{1}{[(n-1)/2!]^2}\right], \quad n\geqslant 3,\\ &\leqslant Gh^2|\omega_2(s)|, \ n=1.\end{aligned} \tag{3.16}$$

Similarly, for the case when n is even, we obtain from relation (3.12) that

$$|E_n(x)| \leqslant Lh|\omega_{n+1}(s)|\left[\sum_{p=1}^{n/2}\frac{2}{(p-1)!(n-p+1)!}+\frac{1}{[(n/2)!]^2}\right], \quad n\geqslant 2 \tag{3.17}$$

and for the case when n is odd, we obtain

$$|E_n(x)| \leqslant 2Lh|\omega_{n+1}(s)|\left[\sum_{p=1}^{(n+1)/2}\frac{1}{(p-1)!(n-p+1)!}\right], \quad n\geqslant 1. \tag{3.18}$$

For the example considered in relation (3.5), we see that relation (3.11) gives a bound $|E_3(\frac{1}{2})| \leqslant 0.46875$ and the relation in (3.4) gives a bound, which is twice of this. Thus, it is clear that if the function $f(x) = x|x|$ is approximated by a third-order polynomial, or even higher, the previously known error bounds are not applicable, because of the fact that the given function is not differentiable p times , for $p \geqslant 2$, at the origin, in the interval $[-1, 1]$, or in any such interval which contains the origin. It is clear that the function $f(x)$ satisfies Lipschitz condition with the constant $L = 2$. In this example G is computed to be 1, by a direct application of the following theorem:

Theorem C. *If $\phi'(x)$ exists, $\phi : [a, b] \longrightarrow \mathbb{R}$ and $\phi'(x)$ satisfies Lipschitz condition with the constant L, then $\phi(x)$ satisfies G-condition, with $G = L/2$.*

Example 2. Consider the function

$$f(x) = |x|, x \in [-1, 1],$$

suppose that we fit a 4th-order Newton's forward interpolation polynomial as given by the relation

$$f_4(x) = \frac{7x^2 - 4x^4}{3}.$$

Now, for some arbitrary point, say for $x = \frac{3}{8}$, the exact error is calculated to be $|E_4(\frac{3}{8})| = 0.0732$. We observe that the function $f(x)$ satisfies Lipschitz condition, with the Lipschitz constant $L = 1$, for $x \in [-1, 1]$. Using the error bound as given by relation (2.18), we obtain $|E_4(\frac{3}{8})| \leqslant 0.3759$, which is much more than the exact error.

We must mention here that for the function $f(x) = |x|$, the previously known error bounds are not applicable, because of the fact that $f(x)$ is not differentiable at the origin in $[-1, 1]$.

In the next section, we have carried forward these ideas to the well-known Newton–Cotes formula (NCF), where we deal with equidistant points $x_j = a + jh$, $j = 0, 1, \ldots, n$, with $b = a + nh$.

4. Error estimates in the quadrature formulae

The error E_n^Q in the quadrature formulae, based on an $(n + 1)$ point interpolation formula, is generally estimated by integrating over the given interval, the corresponding error estimate of the interpolation formula. Thus, on integrating the relations as given by (3.4), (3.11) and (3.12) over $[a, b]$, we get the estimates

$$|E_n^Q| \leqslant \frac{M_p}{p!}\left[\sum_{x_{p-1}, x_p, \ldots, x_n} \frac{1}{|x_{p-1} - x_p||x_{p-1} - x_{p+1}| \cdots |x_{p-1} - x_n|}\right] \int_a^b |\pi_{n+1}(x)|\, \mathrm{d}x, \tag{4.1}$$

$$\begin{aligned} |E_n^Q| &\leqslant G\left[\sum_{x_1, x_2, \ldots, x_n} \frac{1}{|x_1 - x_2||x_1 - x_3| \cdots |x_1 - x_n|}\right] \int_a^b |\pi_{n+1}(x)|\, \mathrm{d}x, \quad n \geqslant 2 \\ &\leqslant G \int_a^b |\pi_2(x)|\, \mathrm{d}x, \quad n = 1 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} |E_n^Q| &\leqslant L\left[\sum_{x_0, x_1, \ldots, x_n} \frac{1}{|x_0 - x_1||x_0 - x_2| \cdots |x_0 - x_n|}\right] \int_a^b |\pi_{n+1}(x)|\, \mathrm{d}x, \quad n \geqslant 2 \\ &\leqslant \frac{2L}{|x_1 - x_0|} \int_a^b |\pi_2(x)|\, \mathrm{d}x, \quad n = 1, \end{aligned} \tag{4.3}$$

corresponding to the cases when the given function $f(x)$ is such that (i) $f(x) \in C^p([a, b])$, for $p \leqslant n$, (ii) satisfies G-condition over $[a, b]$, (iii) satisfies the Lipschitz condition, with the constant L over

$[a,b]$. For the sake of completeness, we give estimates (4.1), (4.2) and (4.3), when x_j's are such that $x_j = x_0 + jh$, $h = (b-a)/n$, for $j = 0,1,\dots,n$. Thus, from relation (4.1), we get

$$|E_n^Q| \leqslant \frac{M_p h^p}{p!}\left[\sum_{q=1}^{(n-p+1)/2} \frac{2}{(q-1)!(n-p+2-q)!} + \frac{1}{[((n-p+1)/2)!]^2}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad (4.4)$$

for $n-p+1$ even and

$$|E_n^Q| \leqslant \frac{2M_p h^p}{p!}\left[\sum_{q=1}^{(n-p+2)/2} \frac{1}{(q-1)!(n-p+2-q)!}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad (4.5)$$

for $n-p+1$ odd.

From relation (4.2), we get

$$\begin{aligned} |E_n^Q| &\leqslant 2Gh^2\left[\sum_{p=1}^{n/2} \frac{1}{(p-1)!(n-p)!}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad n \geqslant 4 \\ &\leqslant 2Gh^2 \int_a^b |\omega_3(s)|\,\mathrm{d}x, \quad n = 2, \end{aligned} \quad (4.6)$$

for n even and

$$\begin{aligned} |E_n^Q| &\leqslant Gh^2\left[\sum_{p=1}^{(n-1)/2} \frac{2}{(p-1)!(n-p)!} + \frac{1}{[((n-1)/2)!]^2}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad n \geqslant 3 \\ &\leqslant Gh^2 \int_a^b |\omega_2(s)|\,\mathrm{d}x, \quad n = 1, \end{aligned} \quad (4.7)$$

for n odd.

Similarly, the relation in (4.3) gives, for n odd,

$$|E_n^Q| \leqslant 2Lh\left[\sum_{p=1}^{(n+1)/2} \frac{1}{(p-1)!(n-p+1)!}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad n \geqslant 1 \quad (4.8)$$

and for n even,

$$|E_n^Q| \leqslant Lh\left[\sum_{p=1}^{n/2} \frac{2}{(p-1)!(n-p+1)!} + \frac{1}{[(n/2)!]^2}\right] \int_a^b |\omega_{n+1}(s)|\,\mathrm{d}x, \quad n \geqslant 2. \quad (4.9)$$

Example 3. We consider the same function $f(x)$ as before, i.e., $f(x) = x|x|$, $x \in [-1,1]$, giving

$$I = \int_{-1}^{1} f(x)\,\mathrm{d}x = 0.$$

Also, if we fit a third-order Newton's forward interpolation formula and integrate over $[-1,1]$, we get

$$I_3 = \int_{-1}^{1} f_3(x)\,\mathrm{d}x = 0$$

which is equal to the actual value. That is, if we employ Simpson's ($\frac{3}{8}$)th rule we get $I_3 = 0$. Also, for this function, it can be easily verified that $G=1$. The error estimates as given by relations (4.5), ($n=3$, $p=1$) and the first of relations (4.7), (i.e. the error estimate in Simpson's ($\frac{3}{8}$)th rule), gives, respectively,

$$|E_3^Q| \leqslant \frac{16}{15}, \tag{4.10}$$

$$|E_3^Q| \leqslant \frac{8}{15}. \tag{4.11}$$

5. Conclusions

We have given a new approach to derive the error term (in a closed form), in the classical Lagrange interpolation formula. We have also been able to give error bounds for the class of functions, which need not be $(n+1)$ times differentiable, as it has been required earlier. In short, we have given the error bounds for those class of functions, which are only p times differentiable, for $p \leqslant n$. We have also been able to give error bounds, for even those classes of functions, which are not differentiable at all. Though the error bounds may be much larger than the actual error incurred, in practice, it is observed that the data available to us, may not say anything about the differentiability aspects of the function, in that range. In such cases our bounds are the only ones to provide some idea about the accuracy of the interpolation formula used for approximating the given data. We have also examined the error estimates, for such weaker class of functions, in the case of quadrature formulae. A special numerical example is dealt with, showing the practical utility of such error bounds.

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