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A Note on Chow Groups and Intersection Multiplicity of Modules

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INTRODUCTION

Our main focus in this paper is to study Chow groups of a complete ramified regular local ring and Serre's conjecture on intersection multiplicity of modules on the same ring. The unramified case of both the problems were solved by Claborn and Fossum in [C-F] and by Serre in [S], respectively. Following Claborn and Fossum [C-F] and Fulton [F] we define the ith Chow group $A_i(R)$ for a noetherian Cohen–Macaulay local ring on dimension n as $\mathbb{Z}_i[R]/\text{Rat}_i[R]$, where $Z_i[R]$ is the free abelian group generated by the primes $[P]$ of R of height $n-i$ and $\text{Rat}_i[R]$ is the subgroup of $Z_i[R]$ generated by the cycles of the form $\Sigma l(R_{P_i}/(q+x)\,R_{P_i})\,[P_i]$, where q is a prime ideal of height $n-i-1$ and P_i ranges over the minimal primes of $R/(q+(x))$. Here $l(M)$ stands for the length of a module M. Fossum and Claborn proved in [C-F] that when R is a formal power series ring over a field k or a complete discrete valuation ring V, $A_i(R)=0$ for $i<n$, $n=$ Krull dimension of R. ($A_n(R)=\mathbb{Z}$ and $A_0(R)=0$ are obvious). The question is very much open when R is a complete ramified regular local ring. On the other hand, we state the multiplicity conjectures as follows: let R be a complete regular local ring and let M, N be two finitely generated modules on R such that $l(M\otimes_R N)<\infty$. We write $\chi_i^R(M,N)=\sum_{t=0}^{\dim R-i}(-1)^t\,l(\text{Tor}_{i+t}(M,N))$ for $i\geqslant 0$ and when $i=0$ we write $\chi^R(M,N)$ instead of $\chi_0^R(M,N)$ (we drop R from χ^R when there is no ambiguity). Serre proved in [S] that when R is a complete unramified regular local ring, $\chi(M,N)\geqslant 0$, and the sign of equality holds if and only if $\dim M+\dim N<\dim R$. He conjectured that the above result should hold in the ramified case too. Roberts [R] and Gillet and Soulé [G-S] independently proved the vanishing part of this conjecture, i.e.,

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$\chi(M, N)=0$ when $\dim M+\dim N<\dim R$. The positivity part of the conjecture, i.e., $\chi(M, N)>0$ when $\dim M+\dim N=\dim R$, is very much open. A special case was proved in [D2]. In this paper we study both problems over a complete ramified regular local ring R with maximal ideal m such that R/m is algebraically closed. (For the multiplicity problem without any loss of generality we can have the above assumption). We write $R=S[X_n]/(f)$, where $S=V[[X_1, ..., X_{n-1}]]$, V a complete discrete valuation ring and f is an Eisenstein polynomial.

First we prove (1.1) that $A_1(R)=0$. In (1.5) we show that when degree of f is a prime number, R/P is a normal domain and p (= the characteristic of $R/m, >0) \notin P$, then $[P]=0$ in $A_i(R)$ when R/P is not the integral closure of $S/P \cap S$ in its field of fractions; when the degree of f is not prime we show by inducting on the degree of f that such a P (viewing P as an ideal in $S[X_n]$) contains another Eisenstein polynomial $\tilde{f}$ with degree of $\tilde{f}=$ degree of f and $[P]=0$ in $A_i(\tilde{R})$, where $\tilde{R}=S[X_n]/(\tilde{f})$.

In Section 2, we show (2.1) that supposing Serre's conjecture holds for prime ideals of height less than i, if P is a prime ideal of height i such that $S/P \cap S$ is a normal domain, the conjecture then holds for R/P and R/q with $l(R/P+q)<\infty$. This led me to investigate the following question: Is it true, given a prime ideal P in $K[[X_1, ..., X_n]]$ for $n>1$, that we can make a change of variables so that $K[[X_1, ..., X_{n-1}]]/P \cap K[[X_1, ..., X_{n-1}]]$ is a normal domain? We find out in (2.3) that an affirmative answer will force P to be a complete intersection. In (2.4) we reduce the conjecture to the following case: $p \notin P$, $P \subset (x_1, ..., x_{n-1})R$ and R/P is normal (x_i is the image of X_i in R). Then we establish (2.5) that when the degree of f is prime, the conjecture holds good in the case when R/P is not the integral closure $S/P \cap S$ in its field of fractions. When the degree of f is not prime, by inducting on the degree of f we show that P (actually the lift of P in $S[X_n]$) contains another Eisenstein polynomial $\tilde{f}$ of the same degree as that of f and, viewing P as prime ideal in $\tilde{R}=S[X_n]/\tilde{f}$, the conjecture holds for $\tilde{R}/P$, $\tilde{R}/q$ for any prime ideal q of $\tilde{R}$ such that $l(\tilde{R}/P+q)<\infty$.

In Section 3 we study Serre's conjecture from the point of view of depth. It is known that when M, N are Cohen–Macaulay, and $\dim M+\dim N=\dim R$, $\chi(M, N)=l(M \otimes_R N)$. The higher Tors get into the expression of $\chi(M, N)$ because of the lack of Cohen–Macaulayness in M and N, and from the computational point, the less the number of Tors the better the chance of proving the conjecture. For the positivity part, changing the pairs of modules without changing the sign of χ is the most difficult part. We have the following result (3.1): To prove positivity it is enough to consider M, N such that depth $M=\dim M-1$ and depth $N=\dim N-1$; i.e., we have to prove $l(M \otimes_R N)-l(\operatorname{Tor}_1^R(M, N))+l(\operatorname{Tor}_2^R(M, N))$ is positive. In (3.2) we study the same question when M is Cohen–Macaulay. A special case of this situation was proved in [D2]. Here we recall that

χ_i-conjecture of Serre for $i>0$: over a regular local ring R, $\chi_i \geqslant 0$ and sign of equality holds if and only if $\mathrm{Tor}^R_{i+t}(M, N)=0\ \forall t \geqslant 0$. Lichtenbaum [L2] and Hochster [H2] proved the χ_i conjecture for unramified complete regular local ring; the ramified case is still open.

We prove in (3.2) that when M is Cohen–Macaulay, $\chi(M, N)$ is positive if χ_2 is non-negative for pairs of modules whose sum of dimensions is less than that of R.

One final comment: Malliavin proved in [M] Serre's conjecture for the pair of modules M, N such that $pM=0$ and $pN=0$. Her proof involves a spectral sequence argument which can be used to prove the more general case: If $g \in R$ be such that $gM=0$, $gN=0$ and the conjecture holds good for M, N over $S[X_n]/(\tilde{g})$, where $\eta(\tilde{g})=g(\eta : S[X_n] \rightarrow R$, the natural surjection) then the conjecture holds good for M, N over R. We use this observation in (2.6).

Throughout this work all rings are commutative neotherian with the identity element and all modules are finitely generated. For a complete ramified regular local ring (R, m) we will write $R=S[X_n]/(f)$, where f is an Eisenstein polynomial in $S[X_n]$, $S=V[[X_1, ..., X_{n-1}]]$ power series ring over V in $n-1$ variables, V is a complete discrete valuation ring with maximal ideal of V generated by the prime $p>0$ which is also the characteristic of R/m. R/m is assumed to be algebraically closed. We write x_i to denote the image of X_i in R. For any local ring A, m_A will denote the maximal ideal of A and we will drop this A from m_A when there is no ambiguity. We abbreviate Cohen–Macaulay as C–M, projective dimension as Pd, dimension of M as dim M, discrete valuation ring as d.v.r., and height of P as htP.

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We note the following two results, the first one is from [F] and the second one is an easy exercise:

Result 1. If $f: A \subsetneq B$ be a ring homomorphism of C-M local rings such that B is a finite A-module via f, then we obtain a sequence of group homomorphisms which we denote by $f_*: A_i(B) \rightarrow A_i(A)$ prescribed by $f_*[P]=f^{-1}(P)][R(P) : R(q)]$, where $q=f^{-1}(P)$ and $(R(P) : R(q))$ denotes the degree of the extension B_P/pB_P over Aq/qAq.

Result 2. If P is a complete intersection prime ideal of height i in a C-M local ring A then $[\mathrm{P}]=0$ in $A_{n-i}(A)$.

1.1. THEOREM. *Let R be a complete regular local ring of dimension n. Then $A_1(R)=0$.*

Proof. Let P be a prime ideal of height $n-1$. Let us assume that P is not a complete intersection. Write $\bar{R}=R/P$, then $\bar{R}$ is a complete local domain of dimension one and $\bar{R}$ is not a d.v.r. Let W be the integral closure of $\bar{R}$ in its field of fractions. Since R/m is algebraically closed $R/m \simeq W/m_w$. Since W is a d.v.r., this implies that we can write W as $R[[y]]/q$. First let us assume that P does not contain any element of $m-m^2$, $m=$ maximal ideal of R. Let $x_1, ..., x_n$ generate m. Then from $\bar{R} \subsetneq W=R[[y]]/q$ we derive that $\exists$ units $\lambda_1, ..., \lambda_n$ in $R[[y]]$ and integers $t_1, ..., t_n$ such that $x_1-\lambda_1 y^{t_1}, ..., x_n-\lambda_n y^{t_n}$ belong to q. Since q is a prime ideal of height n in $R[[y]]$, $q=(x_1-\lambda_1 y^{t_1}, ..., x_n-\lambda_n y^{t_n})$.

Let us denote the complete regular local ring $R[[y]]/(x_1-\lambda_1 y^{t_1})$ by B and let $\tilde{q}$ denote the image of q in B. Now consider the natural injection $i\colon R \to B$. It is easy to see that i is a finite, flat map and $i_*[\tilde{q}]=[P]$. But since $[\tilde{q}]=0$ in $A_1(B)$, this implies that $[P]=0$ in $A_1(R)$.

Now if P contains some non-zero $x \in m-m^2$ we consider $[\bar{P}]$, the class of image of P in $\bar{R}=R/xR$. If $\bar{P}$ does not contain any element of $m_{\bar{R}}-m_{\bar{R}}^2$, we obtain by the above method (or by induction on dim R) $[\bar{P}]=0$ in $A_1(\bar{R})$ and hence, via $R \xrightarrow{\eta} R/xR$, the natural surjection, we obtain $[P]=0$ in $A_1(R)$. If $\bar{P}$ contains some non-zero element of $m_{\bar{R}}-m_{\bar{R}}^2$ we repeat this process again and, after a finite number of such steps, we obtain the desired result.

1.2. *Remark.* Let P be any prime ideal of R which is contained in a prime ideal of q of R of height $n-1$ such that R/q is not a d.v.r. Then we can construct $i\colon R \to B$ a finite flat map, B complete regular local as in the proof of (1.1), and see that $\exists$ a prime ideal P' of B with ht $P=$ ht P', $i^{-1}(P')=P$, and P' is contained in the prime ideal of $(w_2, ..., w_n)$ of B, where $w_i=$ image of $x_i-\lambda_i y^{t_i}$ in B for $i=2, ..., n$ and $w_i \in m_B-m_B^2$. Here $B=V[[w_2, ..., w_n]][Y]/(g)$, where g is an Eisenstein polynomial in $V[[w_2, ..., w_n]][Y]$. Let me remind the reader that the assertion above is a consequence of going down theorem.

1.3. Proposition. *Let R be a complete ramified regular local ring of dimension n. Let P be a prime ideal of R such that $p \notin P$. Then there are two possibilities: either* (i) $P \subset (y_1, ..., y_{n-1})R$ *and* $R \simeq V[[y_1, ..., y_{n-1}]][X_n]/(f(X_n))$ *or* (ii) $\exists$ *finite flat extension* $i\colon R \to B$ *as described in* (1.1) *such that* $\exists P'$ *in* spec B, $P' \subset (w_2, ..., w_n)B$, $i^{-1}(P')=P$, *and* $B=V[[w_2, ..., w_n, y]]/(g)$, *where g is an Eisenstein polynomial in y in* $V[[w_2, ..., w_n]][y]$.

Proof. We can always find a prime ideal q of ht $n-1$ such that $P \subset q$ and $p \notin q$. There are two cases to consider: Case 1. R/q is a d.v.r.; Case 2. It is not so.

Case 1. Since R/q is a d.v.r. $\exists y_1, ..., y_{n-1} \in m-m^2$ such that $q=(y_1, y_2, ..., y_{n-1})$. As $p \notin q$, $p, y_1, ..., y_{n-1}$ form a system of parameters of R.

Hence we can find $x_n \in m - m^2$ such that $R \simeq S[X_n]/(f)$ ([N]), where f is an Eisenstein polynomial in $S[X_n]$. (This is done by sending X_i to y_i for $1 \leqslant i \leqslant n-1$ and X_n to x_n.) It is clear that $P \subset (y_1, ..., y_{n-1})\,R$.

Case 2. In this case R/q is a complete local domain of dimension one, but not a d.v.r. Then by the method described in (1.1) and (1.2) we obtain a flat local extension $i: R \to B$ such that (a) $\exists$ a prime ideal $P' \subset (w_2, ..., w_n)\,B$ such that $i^{-1}(P') = P$ and (b) $B = T[y]/(g)$, where g is an Eisenstein polynomial in $T[y]$, where $T = V[[w_2, ..., w_n]]$ as described in (1.2).

1.4. *Remark.* The above construction tells us that when $p \notin P$, ht $P = i$, $1 < i < n-1$, in order to show that $[P] = 0$, modulo some torsion, it suffices to show that $[P] = 0$ when $P \subset (x_1, ..., x_{n-1})\,R$ for any complete ramified regular local ring R. On the other hand, for the question of intersection multiplicity, since $R \to B$ is finite faithfully flat and B is a complete ramified regular local ring, it is enough to prove the positivity part of the conjecture [S] for any pair R/P, R/q with $P \subset (x_1, ..., x_{n-1})\,R$ and $l(R/P + q) < \infty$.

1.5. THEOREM. *Let R be a complete ramified regular local ring of dimension n. Let P be a prime ideal of height i in R such that $p \notin P$, $P \subset (x_1, ..., x_{n-1})\,R$, and R/P is normal. Then the following holds*: (1) *if the degree of f is prime, then $[P] = 0$ in $A_i(R)$ unless R/P is the integral closure of $S/P \cap S$ in its field of fractions.* (2) *We assume that for any Eisenstein polynomial $g(X_n)$ with the degree of $g(X_n) <$ the degree of $f(X_n)$, the Chow groups vanish over $S[X_n]/(g(X_n))$. Then if the degree of f is not prime and R/P is not the integral closure of $S/P \cap S$ in its field of fractions, we obtain another Eisenstein polynomial $\tilde{f}$ in $S[X_n]$ such that $\tilde{f} \in \tilde{P}$, where $\tilde{P}$ is the pullback of P in $S[X_n]$, the degree of $\tilde{f} =$ the degree of f and $[\tilde{P}/\tilde{f}] = 0$ in $A_i(\tilde{R})$, where $\tilde{R} = S[X_n]/(\tilde{f})$.*

Proof. Write $\bar{S} = S/P \cap S$, $\bar{R} = R/P$, $\bar{T} =$ integral closure of $\bar{S}$ in its field of fractions. Since $\bar{T}$ is normal, $f(X_n)$ breaks into monic irreducible factors $g(X_n)\,g_2(X_n) \cdots g_r(X_n)$ in $\bar{T}[X_n]$. Suppose P corresponds to $g(X_n)$. Then we have the commutative diagram

$$\begin{array}{ccc} \bar{S} & \overset{\alpha}{\hookrightarrow} & \bar{T} \\ {\scriptstyle\beta}\downarrow & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle\delta} \\ \bar{R} = R/P & \overset{\gamma}{\hookrightarrow} & \dfrac{\bar{T}[X_n]}{(g(X_n))}, \end{array} \qquad (*)$$

where α, β, γ, δ are injective maps. Since the field of fractions of

$\bar{T}[X_n]/(g(X_n))$ and R/P are the same and $\bar{T}[X_n]/(g(X_n))$ is integral over R/P, the assumption that R/P is normal implies γ is an isomorphism. Moreover, this induces a map $\psi\colon \bar{T} \to \bar{R}$ such that $\psi \circ \alpha = \beta$ and $\gamma \circ \psi = \delta$.

Now tensoring $(*)$ with $S/(x_1, ..., x_{n-1})\,S$ over S we obtain the commutative diagram

$$\begin{array}{ccc} V & \overset{\tilde{\alpha}}{\hookrightarrow} & \bar{T}/(x_1, ..., x_{n-1})\,\bar{T} = W \\ \downarrow{\scriptstyle \tilde{\beta}} & \swarrow{\scriptstyle \tilde{\psi}} & \downarrow{\scriptstyle \tilde{\delta}} \\ \dfrac{V[X_n]}{(\bar{f}(X_n))} & \overset{\tilde{\gamma}}{\hookrightarrow} & \dfrac{W[X_n]}{(\bar{g}(X_n))} \end{array}, \qquad (**)$$

where $W = \bar{T}/(x_1, ..., x_{n-1})\,\bar{T}$ is a d.v.r. (This follows from the fact that $\dim W = 1$, $\tilde{\delta}$ is flat, and $W[X_n]/(\bar{g}(X_n))$ is a d.v.r. as it is isomorphic to $V[X_n]/(\bar{f}(X_n))$. $\bar{R}/(x_1, ..., x_{n-1})\,\bar{R} = V[X_n]/(\bar{f}(X_n))$, and $\bar{f}$, $\bar{g}$ denote the images of f and g in $V[X_n]$ and $W[X_n]$, respectively; $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are induced by α, β, γ, δ after tensoring with $S/(x_1, ..., x_{n-1})$ over S in $(*)$. Since f is an Eisenstein polynomial, $V[X_n]/\bar{f}(X_n)$ is a d.v.r. and hence, $W[X_n]/(\bar{g}(X_n))$ is also a d.v.r. Let $s =$ degree of $g(X_n)$ and $r =$ rank of W over V. Then $rs =$ the degree of f. (When we write f in $\bar{S}[X_n]$, we mean the image of f in $\bar{S}[X_n]$).

(1) *Degree of f Is Prime*

In this case either $r = 1$ or $s = 1$. If $r = 1$, since $\bar{S}$ is a complete local domain and $\bar{T}$ is a finitely generated $\bar{S}$ module integral over $\bar{S}$ such that $\bar{S}/(x_1, ..., x_{n-1})\,\bar{S} = \bar{T}/(x_1, ..., x_{n-1})\,\bar{T}$, we have $\bar{S} = \bar{T}$; i.e., $\bar{S}$ is normal. Now as the image of f in $\bar{S}/(x_1, ..., x_{n-1})\,\bar{S}$ is irreducible, normality of $\bar{S}$ implies that the image of f is prime in $\bar{S}[X_n]$. Hence $P = (P \cap S)\,R$. Since $S \xrightarrow{i} R$ is flat and $[P \cap S] = 0$ in $A_i(S)$ we obtain $[P] = 0$ in $A_i(R)$. If $s = 1$ this implies the degree of $g(X_n) = 1$ and hence, $\bar{R} = \bar{T}$; i.e., $\bar{R}$ is the integral closure of $\bar{S}$ in its field of fractions.

(2) *Degree of f Is Not Prime*

Since $\bar{T}/(x_1, ..., x_{n-1})\,\bar{T}$ is a d.v.r. and a module-finite extension of V and $V/(p)$ is algebraically closed, it follows that $\bar{T} = \bar{S}[y]/\bar{q} \simeq S[y]/q$, where $q = \eta^{-1}(\bar{q})$, $\eta : S[y] \to \bar{S}[y]$, the natural surjection. Now $\bar{T}/(x_1, ..., x_{n-1})\,\bar{T} = W$ is a d.v.r., i.e., $S[y]/q + (x_1, ..., x_{n-1})\,S[y]$ is a d.v.r. and $(**)$ implies that $W = V[y]/(h(y))$, where $h(y)$ is an Eisenstein polynomial of degree r. Thus in $S[y]$, $q + (x_1, ..., x_{n-1}) = (x_1, ..., x_{n-1}, h(y))$. Hence $\exists \lambda_1, ..., \lambda_{n-1} \in S[y]$ such that $h(y) + \Sigma\lambda_i x_i \in q$. Write $\tilde{h}_1(y) = h(y) + \Sigma\lambda_i x_i$; note that $\tilde{h}_1(y)$ has a unit multiple $\tilde{h}(y)$ which is also an Eisenstein polynomial of degree r and r is less than the degree of $f(x_n)$.

Let T denote $S[y]/(\tilde{h}(y))$; then T is a complete ramified regular local ring. By induction on the degree of f it follows that $[\tilde{q}]=0$ in $A_i(T)$, where $\tilde{q}=$ *image* of q in (T). From the commutativity of $(*)$ we see that if we write $\psi(\bar{y})=\bar{b}_0+\bar{b}_1X_n+\cdots+\bar{b}_dX_n^d$, $\bar{b}_i\in\bar{S}$, then $\bar{y}-\bar{b}_0-\bar{b}_1X_n-\cdots-\bar{b}_dX_n^d\in(g(X_n))$ (here $\bar{y}$, $\bar{b}_i$, etc. denote the image of y, b_i, etc. in $\bar{T}$, $\bar{S}$, respectively). If $d<s$ that would imply $\bar{b}_1=\cdots=\bar{b}_d=0$ and $\bar{y}=\bar{b}_0$ which in turn would imply $\bar{T}=\bar{S}$; i.e., $\bar{S}$ is normal. In this case we are done by arguments similar to those in (1). If $d\geqslant s$, then from $(**)$ it follows that $\bar{y}-\bar{b}_0\in m_{\bar{T}}-m_{\bar{T}}^2$ and this forces $\bar{y}-\bar{b}_0-\bar{b}_1X_n-\cdots-\bar{b}_dX_n^d$ to be a unit times $g(X_n)$ in $\bar{T}[X_n]$; and as $g(X_n)$ is monic of degree s, $\bar{b}_s$ becomes a unit. Let us write $\tilde{g}(X_n)$ for $\bar{y}-\bar{b}_0-\bar{b}_1X_n-\cdots-\bar{b}_dX_n^d$. Then $\bar{T}[X_n]/(\tilde{g}(X_n))=S[y,X_n]/(\tilde{h}(y)\,\tilde{g}(X_n))=S[X_n]/(\tilde{f}(X_n))$, where $\tilde{f}(X_n)$ is obtained from $\tilde{h}(y)$, by replacing y with $b_0+b_1X_n+\cdots+b_dX_n^d$. By applying Weierstrass's preparation theorem, if necessary, we can assume $\tilde{f}(X_n)$ is a monic polynomial. It is easy to check that the degree of $\tilde{f}(X_n)=rs=$ the degree of $f(X_n)$ and $\tilde{f}(X_n)$ is an Eisenstein polynomial (as $\tilde{h}(y)$ is Eisenstein). Morever, from $(*)$ it is clear that if η denotes the natural surjection $\eta: S[X_n]\to R$, $\tilde{P}$ denotes $\eta^{-1}(P)$, then $R/P=S[X_n]/\tilde{P}$, and $\tilde{P}$ contains $\tilde{f}(X_n)$. Write $\tilde{R}$ for $S[X_n]/(\tilde{f})$. Since $T\xrightarrow{j}T[X_n]/(\tilde{g}(X_n))$ is flat and $q\tilde{R}=\tilde{P}/(f(X_n))$ it follows that $[\tilde{P}/(\tilde{f}(X_n))]=0$ in $A_i(\tilde{R})$.

2

In this section we state that Serre's conjecture (or simply the conjecture) holds for P, P a prime ideal in R, to mean that given any other prime ideal q of R such that $l(R/P+q)<\infty$, $\chi(R/P, R/q)$ satisfies the conjecture.

2.1. THEOREM. *Let R be a ramified complete regular local ring of dimension n. Suppose P is a prime ideal of R of height i such that $S/P\cap S$ is normal. If the conjecture holds for prime ideals of height less than i in any complete ramified regular local ring of the same dimension, then the conjecture holds for P.*

Proof. Without any loss of generality can assume $1<i<n-1$ [D1]. Write $\bar{S}$ for $S/P\cap S$ and $\bar{R}$ for R/P. Since $\bar{S}$ is normal, we can write $f(X_n)=\bar{g}(X_n)\,\bar{h}(X_n)$, $g(X_n)$, $h(X_n)$ monic polynomials in $S[X_n]$ so that $\bar{g}(X_n)$ is irreducible in $\bar{S}[X_n]$ and $R/P=\bar{S}[X_n]/(\bar{g}(X_n))$. This implies that $f(X_n)-g(X_n)\,h(X_n)\in(P\cap S)\,S[X_n]$; i.e., if we write $f(X_n)-g(X_n)\,h(X_n)=\alpha(X_n)=\alpha_0+\alpha_1X_n+\cdots+\alpha_{t-1}X_n^{t-1}$, all $\alpha_i\in P\cap S$. Moreover, since f is Eisenstein and the constant terms of $g(X_n)$ and $h(X_n)$ are in m_s, $\alpha_0\in m_s-m_s^2$. This implies that $T=S[X_n]/(\alpha(X_n))$ is a complete regular local ring of dimension n. Moreover, the fact that the ideal $(f(X_n), g(X_n))$

is same as the ideal $(\alpha(X_n), g(X_n))$ it follows that $R/(g) = T/(g)$, where by g we denote the image of $g(X_n)$ in R and T simultaneously. Note that $\bar{S}[X_n] = S[X_n]/(P \cap S)\, S[X_n]$ is a cyclic T-module and the height of the image of $(P \cap S)\, S[X_n]$ in T is $i-1(\alpha(X_n) \in (P \cap S)\, S[X_n])$; moreover, $g(X_n)$ is a non-zero-divisor on T and on $\bar{S}[X_n]$. Since $R/P = \bar{S}[X_n]/(g(X_n))$, this implies that R/P is of finite projective dimension over $R/(g)$.

Let q be any prime ideal of R such that $l(R/P+q) < \infty$. We have two cases to consider.

Case 1: $g \in q$. Now since R/P has finite projective dimension over $R/(g)$, by a standard spectral sequence argument for change of rings $R \to R/(g)$, we have $\chi^R(R/P, R/q) = \chi^{R/(g)}(R/P, R/q) - \chi^{R/(g)}(R/P, R/q) = 0$.

Case 2: $g \notin q$. In this case $\chi^R(R/P, R/q) = \chi^{R/(g)}(R/P, R/(q, g))$. Since $R/P = \bar{S}[x_n]/(g(x_n))$, g is a non-zero-divisor on T and $R/(g) = T/(g)$. We have $\chi^T(\bar{S}[X_n], R/(q, g)) = \chi^{R/(g)}(R/P, R/(q, g))$. But on T, codimension of $\bar{S}[X_n]$ is $i-1$ and hence the conjecture is valid by our assumption for $\bar{S}[X_n]$ and $R/(q, g)$. Thus the conjecture holds for P.

2.2. *Remark.* The argument used in the proof above establishes the following: Let P be a prime ideal of height two in R such that $S/P \cap S$ is normal. Then P is a complete intersection, if $P \neq (P \cap S)\, R$.

The main point to note here is that $\operatorname{ht}(P \cap S)\, T = 1$ if $\operatorname{ht} P = 2$ and as T is regular local, $(P \cap S)T$ is principal.

2.3. The above result prompted me to ask the following question: Given a power series ring R_n over K in n variables $X_1, ..., X_n$, $n > 1$, where K is either a field or a complete d.v.r. and a prime ideal P of R, can we make a change of variables so that $R_{n-1}/P \cap R_{n-1}$ is the normal domain? The following proposition dashes any hope for an affirmative answer.

PROPOSITION. *Let K be an infinite field. The assumption that given a prime ideal P in $R_n = K[[X_1, ..., X_n]]$ for $n > 1$, a change of variables could be found so that $R_{n-1}/P \cap R_{n-1}$ would be normal, would force P to be a complete intersection. In particular when $n = 3$, for any* ht 2 *prime P if $\exists$ a change of co-ordinates such that $P \cap k[[X_1, X_2]]$ is normal, then P is a complete intersection.*

Proof. We first note that in the proof of the previous theorem we have shown that if $P_{n-1} = P \cap R_{n-1}$ is normal (i.e., R_{n-1}/P_{n-1} is normal) and P contains a monic polynomial in $R_{n-1}[X_n]$, then $P = (P_{n-1}, g_n(X_n))$, where $g_n(X_n)$ is a monic polynomial in $R_{n-1}[X_n]$. The assumption that P contains a monic polynomial is no restriction, since this can easily be achieved by making a linear change of variables.

Now suppose ht $P=i$. By repeating the above method for $i-1$ times, we see that P_{n-i+1}, will be a ht 1 prime ideal ($P_n=P$) in R_{n-i+1} and hence principal. Thus P becomes a complete intersection.

2.4. *Remark.* While studying the conjecture for the pair R/P, R/q with $l(R/P+q)<\infty$, we can assume that (i) R/P is normal and (ii) in case $p\notin P$, $P\subset(x_1, ..., x_{n-1})R$. If $\bar{R}=R/P$ is not normal let $\bar{T}$ be the integral closure of $\bar{R}$. We can write $\bar{T}=R[y_1, ..., y_r]/P'$. Then via $R\xrightarrow{i}R[y_1, ..., y_r]=T$, the natural injection, we see $i^{-1}(P')=P$ and $\chi^R(R/P, R/q)$ is positive, zero, or negative, according as $\chi^T(T/p', T/qT)$ is positive, zero, or negative. To see this last point one can use either the vanishing part of the conjecture or use induction on dim $R-$ dim $R/P-$ dim R/q. The second assertion that when $p\notin P$, one can assume $P\subset(x_1, ..., x_{n-1})R$, can be established by the method of (1.3) and considering $P[[y]]$, $R[[y]]$ instead of P and R.

2.5. THEOREM. *Let R be a complete ramified regular local ring of dimension n. Let P be a prime ideal of R of* ht *i such that $P\subset(x_1, ..., x_{n-1})R$ and R/P is normal. Then* (i) *if the degree of f is prime, the conjecture holds for P unless R/P is the integral closure of $S/P\cap S$ in its field of fractions.* (ii) *We assume that for any Eisenstein polynomial $g(X_n)$ with degree of $g(X_n)<$ the degree of $f(X_n)$, the conjecture holds over $S[X_n]/[g(X_n))$. Then if the degree of f is not prime and R/P is not the integral closure of $S/P\cap S$ in its field of fractions, the lift $\tilde{P}$ of P in $S[X_n]$, contains another Eisenstein polynomial $\tilde{f}(X_n)$ with the degree of $\tilde{f}(X_n)=$ the degree of $f(X_n)$ such that over $\tilde{R}=S[X_n]/(\tilde{f})$, the conjecture holds for $\tilde{P}/(\tilde{f})$.*

Proof. The crux of the proof has already been worked out in (1.5). Write $P'=P\cap S$.

(i) As pointed out in (1.5), in this case supposing R/P is not the integral closure of $S/P\cap S$, we have $P=P'R$. Since $S\subsetneq R$ is a finite flat extension, $\chi^S(S/P, R/q)=\chi^R(R/P, R/q)$ and since the conjecture is valid in S, we are done.

(ii) In this case, again it follows from (1.5) that $\exists T$, a ramified complete regular local ring of the form $S[y]/(\tilde{h}(y))$, where $\tilde{h}(y)$ is an Eisenstein polynomial whose degree is less than that of f and a prime ideal P' of ht i in T such that in $\tilde{R}=T[X_n]/(\tilde{g}(X_n))=S[X_n]/(\tilde{f}(X_n))$, $P'\tilde{R}=\tilde{P}/(\tilde{f}(X_n))$. We refer to Theorem 1.5 for notations. As the conjecture holds good for P' (by induction on the degree of f as the degree of $\tilde{h}<$ the degree of f) and $T\to\tilde{R}$ is a finite flat map, the conjecture holds for $\tilde{P}/(\tilde{f}(X_n))$ in $\tilde{R}$.

2.6. *Remark.* Since $\tilde{R}=T[X_n]/(g(X_n))=S[X_n]/(\tilde{f}(X_n))$, where $g(X_n)$, $\tilde{f}(X_n)$ are Eisenstein polynomials in $T[X_n]$ and $S[X_n]$, respectively (as in

(2.5)), and the conjecture holds for $\tilde{P}/(\tilde{f}(X_n))$ in $\tilde{R}$, it follows by the method worked out in [M] that if q is a prime ideal in R such that it contains the image of $\tilde{f}(X_n)$ in R and $l(R/P+q)<\infty$, then $\chi(R/P, R/q)$ satisfies the conjecture. Thus in order to prove Serre's conjecture for R/P, R/q with conditions as in (ii) of (2.5) over $R=S[X_n]/(f)$, we are reduced to the following case: If $\tilde{P}$ is the lift of P in $S[X_n]$, we can assume P contains two distinct Eisenstein polynomials $f(X_n)$ and $\tilde{f}(X_n)$ of the same degree such that over $\tilde{R}=S[X_n](\tilde{f}(X_n))$ the conjecture holds good for $\tilde{P}/(\tilde{f}(X_n))$ in $\tilde{R}$ and the image of $\tilde{f}(X_n)$ in R is not in q. (This resembles the following situation: to resolve $\chi(M, N)$ when p is a n.z.d. on M and $pN=0$.) In this case I am unable to prove the conjecture in the following situation: Let us lift the map $\bar{T}\xrightarrow{\psi}\bar{R}$ to a map $\tilde{\psi}: S[Y]\to R$, $\tilde{\psi}(Y)=b_0+b_1X_n+\cdots+b_dX_n^d$, and let $u(y)$ denote the monic irreducible polynomial generating ker $\tilde{\psi}$; I still do not know how to resolve the conjecture when the degree of $u(y)=$ the degree of $f(X_n)$.

3

In this section our main goal is to cut down the number of Tors in the expression $\chi(M, N)$ as much as possible. For positivity one cannot change modules by short exact sequences in the usual way as one can for the vanishing part because the method fails to guarantee the sign of $\chi(M, N)$, while changing M (or N) by M' (or N'). This makes the positivity part very difficult to deal with via short exact sequences. We know that the higher the depth of M and N, the less the number of Tors [S]. So starting with arbitrary M, N such that $l(M\oplus N)<\infty$ and $\dim M+\dim N=n$, $n=\dim R$, it would be best if we could form another pair M', N' with $l(M'\otimes N')<\infty$, $\dim M'+\dim N'=n$, and both M' and N' are C-M so that $\chi(M, N)$, $\chi(M', N')$ have the same sign. For in such a case, $\chi(M', N')=l(M'\otimes N')>0$ and this would imply $\chi(M, N)>0$. Unfortunately I cannot reduce to the best possible case yet. What I have is the following.

3.1. THEOREM. *Let (R, m) be a regular local ring. In order to prove the positivity part it is enough to assume that depth $M=\dim M-1$, depth $N=\dim N-1$ when $l(M\otimes_R N)<\infty$ and* $\dim M+\dim N=\dim R$.

Proof. Let $\dim M=r$ and $\dim N=s$ and $I=\operatorname{ann}_R M$. Since $l(M\otimes_R N)<\infty$ and $r+s=n$ it follows that I contains an R-sequence $\{x_1, ..., x_s\}$ which forms a system of parameters for N. We write $\bar{R}=R/(x_1, ..., x_s)$. If $Pd_{\bar{R}}M<\infty$, from the exact sequence $0\to\bar{R}^d\to M\to H\to 0$, where $\dim H<\dim M$ and $d=\operatorname{rank}_{\bar{R}} M$ (i.e., rank $\bar{S}^{-1}M$, where S

denotes the set of non-zero-divisors of $\bar{R}$; this rank is well defined as finite projective dimension of M forces each M_p to have the same rank over $\bar{R}_p$ for each minimal prime ideal of P of $\bar{R}$), we see that $\chi(M, N) = d\chi(\bar{R}, N) > 0$. So we assume $Pd_{\bar{R}}\, M = \infty$. By Theorem (4.26) in [A-B] we obtain an exact sequence

$$0 \to L \to \bar{R}^t \oplus M \to Q \to 0, \tag{1}$$

where Q is a module with $Pd_{\bar{R}}\, Q < \infty$ and L is a maximal C-M module over $\bar{R}$. If dimension $Q = \dim \bar{R}$, let $d' = \operatorname{rank}_{\bar{R}} Q$. Then we obtain the exact sequence

$$0 \to \bar{R}^{d'} \to Q \to Q' \to 0, \tag{2}$$

where Q' is a module of finite projective dimension over $\bar{R}$ and $\dim Q' < \dim \bar{R}$. Now from

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & \bar{R}^{d'} & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & \bar{R}^t \oplus M & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & Q' & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

we obtain the following short exact sequence:

$$0 \to L \oplus \bar{R}^{d'} \to \bar{R}^t \oplus M \to Q' \to 0.$$

This shows that we can assume $\dim Q < \dim \bar{R}$ in (1) and hence $\chi(Q, N) = 0$. From (1) then we obtain $\chi(M, N) = -\chi(\bar{R}^t, N) + \chi(L, N)$. On the other hand, since rank of $L \geqslant t$ we obtain the following short exact sequence:

$$0 \to \bar{R}^t \to L \to M' \to 0.$$

This gives us $\chi(M', N) = \chi(L, N) - \chi(\bar{R}^t, N)$. Thus $\chi(M, N) = \chi(M', N)$, where depth $M' \geqslant \dim M - 1$.

Now repeating the same process with N we obtain an N' such that $\chi(M', N') = \chi(M', N)$ and depth $N' \geqslant \dim N' - 1$. Thus the desired result follows.

3.2. We now try to understand $\chi(M, N)$, when M is C-M. A special case of this situation, when p is a non-zero-divisor on M and $p^t N = 0$ was proved in [D2]. The following theorem connects the positivity part with the χ_2-conjecture.

THEOREM. *Let R be a regular local ring of dimension n and M be a C-M module over R. Then for any module N with $l(M \otimes N) < \infty$ and* $\dim M + \dim N = \dim R$, *$\chi(M, N)$ is positive if for any pair of modules M', N', with $l(M' \otimes N') < \infty$ and* $\dim M' + \dim N' < \dim R$, *the χ_2-conjecture holds. Here we can assume M' is C-M. (The χ_i-conjecture is stated in the Introduction.)*

Proof. Without any loss of generality can assume depth $N = \dim N - 1$ (3.1) and hence $\chi(M, N) = l(M \otimes N) - l(\mathrm{Tor}_1(M, N))$. (We drop R from tensor product, Ext, Tor for convenience of typing.) Let $\dim M = r$, $\dim N = s$.

We know the spectral sequence $\{\mathrm{Ext}^i(\mathrm{Tor}_j(M, N), R)\}_{i+j=t}$ and $\{\mathrm{Ext}^i(M, \mathrm{Ext}^j(N, R))\}_{i+j=t}$ converge to the same limit. Since $l(\mathrm{Tor}_j(M, N)) < \infty$, $\mathrm{Ext}^i(\mathrm{Tor}_j(M, N), R) = 0$ for $i \neq n$ and when $i = n$, it is 0 for $j > 1$. Since $PdM = s = \dim N$ and depth $N = s - 1$, $\mathrm{Ext}^i(M, \mathrm{Ext}^j(N, R)) = 0$ for $i > s$ and when $i \leqslant s$ it is zero for $j < r$. Thus the spectral sequence gives rise to the exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}^{s-2}(M, \mathrm{Ext}^{r+1}(N, R)) \to \mathrm{Ext}^s(M, \mathrm{Ext}^r(N, R) \\ &\to \mathrm{Ext}^n(M \otimes N, R) \to \mathrm{Ext}^{s-1}(M, \mathrm{Ext}^{r+1}(N, R)) \to 0. \end{aligned} \tag{1}$$

Denote $\mathrm{Ext}^s(M, R)$ by $\tilde{M}$ and note that $\tilde{M}$ is a C–M module. Then (1) shows

$$\begin{aligned} l(M \otimes N) &= l(\mathrm{Ext}^n(M \otimes N, R)) \geqslant l(\mathrm{Ext}^{s-1}(M, \mathrm{Ext}^{r+1}(N, R)) \\ &= l(\mathrm{Tor}_1(\tilde{M}, \mathrm{Ext}^{r+1}(N, R))). \end{aligned}$$

On the other hand, $\mathrm{Ext}^n(\mathrm{Tor}_1(M, N), R) \simeq \mathrm{Ext}^s(M, \mathrm{Ext}^{r+1}(N, R))$.

This implies $l(\mathrm{Tor}_1(M, N)) = l(\tilde{M} \otimes \mathrm{Ext}^{r+1}(N, R))$. We note that $\dim \mathrm{Ext}^{r+1}(N, R) < \dim N$. Hence

$$\begin{aligned} \chi(M, N) &= l(M \otimes N) - l(\mathrm{Tor}_1(M, N)) \\ &\geqslant l(\mathrm{Tor}_1(\tilde{M}, \mathrm{Ext}^{r+1}(N, R)) - l(\tilde{M} \otimes \mathrm{Ext}^{r+1}(N, R)) \\ &= \chi_2(\tilde{M}, \mathrm{Ext}^{r+1}(N, R)). \end{aligned}$$

So if $\chi_2(\tilde{M}, \mathrm{Ext}^{r+1}(N, R)) > 0$, we are done. Suppose $\chi_2(\tilde{M}, \mathrm{Ext}^{r+1}(N, R)) = 0$. By the χ_2-conjecture we know that this implies $\mathrm{Tor}_2(\tilde{M}, \mathrm{Ext}^{r+1}(N, R)) = 0$. Then it follows from (1) that

$$l(M \otimes N) = l(\mathrm{Tor}_1(\tilde{M}, \mathrm{Ext}^{r+1}(N, R))) + l(\tilde{M} \otimes \mathrm{Ext}^r(N, R)).$$

Hence

$$\begin{aligned}\chi(M, N) &= l(\tilde{M} \otimes \mathrm{Ext}^r(N, R)) + \chi_2(\tilde{M} \otimes \mathrm{Ext}^{r+1}(N, R)) \\ &= l(\tilde{M} \otimes \mathrm{Ext}^r(N, R)) > 0.\end{aligned}$$

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