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On Grade and Formal Connectivity for Weakly Normal Varieties

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1. INTRODUCTION

A weakly normal variety X can be characterized by the fact that any variety which is birationally homeomorphic to X is actually isomorphic to X. Various other characterizations of weakly normal varieties exist but to date there is no characterization from the point of view of local cohomology. It is well known that the vanishing of the first local cohomology groups with supports in the singular locus is a criterion for normality and it is natural to ask if there is a similar criterion for weak normality.

In [7] this author and J. V. Leahy defined a generic type singularity for weakly normal varieties called the multicross. Briefly, a point $y \in X$ is a multicross for X if the point y on X is analytically isomorphic to a point y' on X' where X' is the union of linear subspaces of affine space all meeting transversally along a common linear subspace. For a weakly normal variety X the complement Z of the set of multicrosses is a closed subset of the singular locus of X and has codimension at least two [7], Theorem 3.8]. We say a variety is C-weakly normal if the first local cohomology groups with supports in Z are identically zero. By a Hartogs-like result [7], Corollary 3.11] it was shown that a C-weakly normal variety is weakly normal but there are simple examples of weakly normal varieties that are not C-weakly normal [7], Example 4.5].

In this paper we give, within the class of weakly normal varieties, a topological criterion for the vanishing of the first local cohomology groups with supports in an arbitrary closed subset of the singular locus. Our techniques also enable us to prove the converse of Hartshorne's depth-connectivity result [5], Proposition 2.1] in the case of a reduced complete seminormal local ring with rational normalization.

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2. Preliminaries and Notation

By ring we mean a commutative ring with identity. If A is a ring, we let $R(A)$ denote the Jacobson radical of A. We let $\Omega(A)$ denote the maximal spectrum of A and $\mathrm{Spex}(A)$ denote $\mathrm{Spec}(A) - \Omega(A)$. If $(A, \mathfrak{m})$ is a noetherian local ring and M is an A-module, we let $\hat{M}$ denote the $\mathfrak{m}$-adic completion of M. If A is a ring, $x \in \mathrm{Spec}(A)$ corresponds to the prime ideal $\mathfrak{p}_x$ and M is an A-module, we let $M_x = S^{-1}M$ where $S = A - \mathfrak{p}_x$. For $m \in M$ we let m_x denote its image in M_x. We let $\kappa(x)$ denote $A_x/\mathfrak{p}_x A_x$.

If $\mathfrak{a}$ is an ideal of A, we let $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \mathfrak{a}^n m = 0 \text{ for some } n\}$. If A is noetherian, $\Gamma_{\mathfrak{a}}(\cdot)$ is a left exact functor from the category of A-modules to itself. We denote its right derived functors by $H^i_{\mathfrak{a}}(\cdot)$. For more details one can consult [4].

Let A be a ring and B be an overring integral over A. The seminormalization of A in B, denoted ${}^+_BA$ is defined by

$${}^+_BA = \{b \in B \mid b_x \in A_x + R(B_x), \forall x \in \mathrm{Spec}(A)\}.$$

If B is the normalization of A, then we call ${}^+_BA$ the seminormalization of A and denote it by ${}^+A$. A ring A is said to be seminormal if $A = {}^+A$.

For some of the fundamental results on seminormality the reader can consult Traverso's original paper [9] or [6]. We now recall Traverso's notion of gluing.

For a point $x \in \mathrm{Spec}(A)$ we define ${}^+_xA$ by

$${}^+_xA = \{b \in B \mid b_x \in A_x + R(B_x)\}.$$

We say that ${}^+_xA$ is obtained by gluing B over x and that A is its own gluing in B over x if $A = {}^+_xA$. The glued ring ${}^+_xA$ is characterized by that fact that is the largest subring A' of B containing A such that

(i) There is precisely one point $x' \in \mathrm{Spec}(A')$ lying over x, and

(ii) the canonical inclusion $\kappa(x) \to \kappa(x')$ is an isomorphism.

Suppose that A is a reduced noetherian ring with finite normalization B. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ denote the associated primes of B/A indexed so that $ht\mathfrak{p}_1 \leqslant \cdots \leqslant ht\mathfrak{p}_r$. Let $x_1, \ldots, x_r$ denote the corresponding points of $\mathrm{Spec}(A)$ and let ${}^+_kA$ denote the ring obtained from B by gluing over x_k for $k = 1, \ldots, r$. Let $B^0 = B$ and define $B^i = (B^{i-1} \cap {}^+_iA)$ for $i = 1, \ldots, r$. Then $B^i = \bigcap_{j \leqslant i} {}^+_jA$ and we have the following structure theorem.

Theorem 2.1 [6, Theorem 1.13]. *With notation and hypotheses as above, if A is seminormal but is not normal then*:

(i) *B^i is obtained from B^{i-1} by gluing over x_i, $i = 1, \ldots, r$.*

(ii) $\operatorname{Ass}_A(B^i/A) = \{\mathfrak{p}_{i+1},..., \mathfrak{p}_r\}$, $i = 1,..., r$.

(iii) $A = B^r < \cdots < B^0 = B$.

We now recall some of Hartshorne's results relating depth and connectivity.

PROPOSITION 2.2 [5, Proposition 2.1]. *Let A be a noetherian local ring. If* depth $A \geqslant 2$, *then* Spex(A) *is connected.*

THEOREM 2.3 [5, Theorem 2.2]. *Let $(X, \mathcal{O}_X)$ be a connected, locally noetherian scheme and let Y be a closed subset of X such that for each $y \in Y$, the local ring $\mathcal{O}_{X,y}$ has depth at least two. Then $X - Y$ is connected.*

We now turn our attention to weakly normal varieties over a fixed algebraically closed field k of characteristic O. When we use the term variety we assume that the underlying topological space is the set of closed points of a reduced, separated scheme of finite type over k.

Let U be an open subset of a variety $(X, \mathcal{O}_X)$. A k-valued function on U is said to be c-regular if it is continuous and is regular on the nonsingular points of U. Let $\mathcal{O}_X^c$ denote the sheaf of c-regular functions on X.

We say that X is weakly normal at $x \in X$ if $\mathcal{O}_{X,x} = \mathcal{O}^c_{X,x}$. X is said to be weakly normal if $\mathcal{O}_X = \mathcal{O}^c_X$.

We recall that an affine variety is weakly normal if and only if its affine coordinate ring is seminormal [6, Theorem 2.2 and (2.7)]. Another useful fact about weakly normal varieties is the following result.

PROPOSITION 2.4. *Let X be a weakly normal variety and suppose that $f: Y \to X$ is a birational morphism and is a homeomorphism of the underlying topological spaces. Then f is an isomorphism of varieties.*

Proof. It suffices to see that $f^{-1}: X \to Y$ is a morphism of varieties. Let V be any nonempty open subset of Y and let $\varphi \in \Gamma(V, \mathcal{O}_Y)$. Then $\varphi \circ f^{-1}$ is a continuous k-valued function on $f(V)$ and is regular on some dense open subset of $f(V)$, hence is regular on $f(V)$ [6, Proposition 2.6]. Thus f^{-1} is a morphism and f is an isomorphism of varieties.

3. MAIN THEOREM

Before we state our main result we need to establish some terminology.

DEFINITION 3.1. Let X be a variety and Y be a closed subvariety with ideal sheaf $\mathcal{J}_Y$. We say that X is formally Y-connected if for all points $y \in Y$, $\operatorname{Spec}(\mathcal{O}_{X,y})^\wedge - V(\mathcal{J}_{Y,y})^\wedge$ is connected.

THEOREM 3.2. *Let X be a weakly normal variety and $Y \subseteq S(X)$ be a closed subvariety with ideal sheaf $\mathcal{I}_Y$. Then X is formally Y-connected if and only if $H^1_{Y \cap U}(U, \mathcal{O}_X|_U) = 0$ for every open subset U of X.*

Proof. Since both properties are local in nature it suffices to assume that X is affine. Let $A = \Gamma(X, \mathcal{O}_X)$ and $I = \Gamma(X, \mathcal{I}_Y)$. Since $Y \subseteq S(X)$ we know that $\operatorname{ht} I = \operatorname{codim}(Y, X) \geqslant 1$ so that $\Gamma_I(A) = 0$.

Suppose that $O = H^1_Y(X, \mathcal{O}_X) \cong H^1_I(A)$. Then grade $I \geqslant 2$ and hence $\operatorname{grade}(IA_y)^\wedge \geqslant 2$ for every point $y \in Y$. Therefore $\operatorname{Spec}(\hat{A}_y) - V(\hat{I}_y)$ is connected for all $y \in Y$ by (2.3).

Conversely, assume that X is formally Y-connected. Suppose to the contrary that $H^1_I(A) \neq 0$. Then grade $I = 1$ so that there exists an A-regular element $a \in I$ such that I is contained in the set of zero divisors for A/aA. Thus $I \subseteq \mathfrak{p}$ for some associated prime $\mathfrak{p}$ of A/aA. Let B denote the normalization of A. We claim that $\mathfrak{p} \in \operatorname{Ass}_A(B/A)$. If $\operatorname{ht} \mathfrak{p} = 1$, then $A_\mathfrak{p}$ is not normal (as $Y \subseteq S(X)$) and hence $\mathfrak{p}$ is a minimal prime of B/A. If $\operatorname{ht} \mathfrak{p} \geqslant 2$, then depth $A_\mathfrak{p} = 1$ implies that $\mathfrak{p} \in \operatorname{Ass}_A(B/A)$ [3, Theorem 5.6].

Let $S = \{z \in \operatorname{Spec}(A) \mid \mathfrak{p}_z \in \operatorname{Ass}_A(B/A)$ and $\operatorname{ht} \mathfrak{p}_z < \operatorname{ht} \mathfrak{p}\}$ and let $A' = \bigcap_{z \in S} {}^{+}_{z}A$. Then $\mathfrak{p}$ is a minimal prime for A'/A by (2.1) so that we may choose an element $f \in A - \mathfrak{p}$ such that

$$\operatorname{Ass}_{A_f}(A'_f/A_f) = \{\mathfrak{p}A_f\}.$$

Let X' be the affine variety determined by A' and let $\rho: X'_f \to X_f$ be the induced morphism. Then ρ is finite and birational but is not an isomorphism. By (2.4) there exists a point $x \in X_f$ such that $\rho^{-1}(x) = \{x_1, ..., x_d\}$ where $d \geqslant 2$. We note that $x \in V(\mathfrak{p}A_f) \subseteq Y \cap X_f$.

Let $R = \hat{A}_x$ and $R' = (A'_x)^\wedge$. Then R' is complete in the $(m_{x_1} \cap \cdots \cap \mathfrak{m}_{x_d})$-adic topology. Hence $R' \cong R'_1 \times \cdots \times R'_d$ where $R'_i = (A'_{x_i})^\wedge$ for $i = 1, ..., d$. Since $(R: R') = (A_x : A'_x)^\wedge = \mathfrak{p}R$ we know that $\operatorname{Spec}(R') - V(\mathfrak{p}R')$ and $\operatorname{Spec}(R) - V(\mathfrak{p}R)$ are homeomorphic. Hence $\operatorname{Spec}(R') - V(IR')$ and $\operatorname{Spec}(R) - V(IR)$ are homeomorphic. However the former is the disjoint union of the nonempty closed subsets $\operatorname{Spec}(R'_1) - V(IR'_1), ..., \operatorname{Spec}(R'_d) - V(IR'_d)$, contradicting the fact that $\operatorname{Spec}(R) - V(IR)$ is connected. Hence grade $I \geqslant 2$ as asserted.

EXAMPLE 3.3. Let U, X, Y, Z, W be transcendentals over k and let $A = k[U, X, Y, Z, W]/Q_1 \cap Q_2$ where $Q_1 = (X^2Z - W^2, U - W)$ and $Q_2 = (U, X, Z)$. Let u, x, y, z and w denote the images of U, X, Y, Z and W, respectively, in A. Then A is weakly normal [6, Example 3.7] and its normalization $B = k[x, y, w/x] \times k[z, w]$ is the product of two polynomial rings. Then $(A: B) = (x, u, w) = \mathfrak{p}$ is a prime ideal of A and as an ideal of B

is $(x) \times k[z, w] \cap k[x, y, w/x] \times (w)$. Let p be the point in Spec(A) corresponding to $\mathfrak{m}_p = (u, x, y, z, w)$, let $\mathfrak{m}_{p_1} = (x, y, w/x) \times k[z, w]$ and $\mathfrak{m}_{p_2} = k[x, y, w/x] \times (z, w)$. Finally let $R = \hat{A}_p, R' = \hat{B}_p \cong \hat{B}_{p_1} \times \hat{B}_{p_2}$. Then Spec $R - V(\mathfrak{p}R)$ is disconnected as it is homeomorphic to

$$\operatorname{Spec}(R') - V(\mathfrak{p}R') \cong \operatorname{Spec}(\hat{B}_{p_1}) - V(\mathfrak{p}\hat{B}_{p_1}) \sqcup \operatorname{Spec}(\hat{B}_{p_2}) - V(\mathfrak{p}\hat{B}_{p_2}).$$

Also grade $\mathfrak{p} = 1$ since $\mathfrak{p} = (A : (1, 0))$.

Let A be a noetherian ring and let B denote its normalization. A is said to have rational normalization if $A/\mathfrak{n} \cap A = B/\mathfrak{n}$ for every maximal ideal $\mathfrak{n}$ of B.

For a complete local ring with rational normalization the techniques of (3.2) yield the following result (cf. [1, Theorem 3.1).

PROPOSITION 3.4. *Let $(A, \mathfrak{m}, k)$ be a reduced complete noetherian local ring with rational normalization B. Assume that A is seminormal of dimension at least two. Then* depth $A \geqslant 2$ *if and only if* Spex(A) *is connected.*

Proof. If depth $A \geqslant 2$, then Spex(A) is connected by Hartshorne's result (2.2).

Assume that Spex(A) is connected but that depth $A < 2$. Since A is complete, B is a finite A-module [8, Corollary 2, p. 234]. Then $\mathfrak{m} \in \operatorname{Ass}_A(B/A)$ as in the proof of (3.2).

Let $S = \{z \in \operatorname{Spec}(A) | \mathfrak{p}_z \in \operatorname{Ass}_A(B/A) - \{m\}\}$ and set $A' = \bigcap_{z \in S} {}^{+}_{z}A$. Then A is its own gluing in A' over $\mathfrak{m}$ and $\operatorname{Ass}_A(A'/A) = \{m\}$ by (2.1).

Since A has rational normalization we know that $A/\mathfrak{m} = A'/\mathfrak{m}'$ for each maximal ideal $\mathfrak{m}'$ of A'. Thus by Traverso's description of gluing, $\Omega(A') = \{\mathfrak{m}_1, ..., \mathfrak{m}_d\}$ where $d \geqslant 2$. Thus $A' \cong A'_1 \times \cdots \times A'_d$ where $A'_j = (A'_{\mathfrak{m}_j})^{\wedge}$ and Spex(A') is the disjoint union of the nonempty closed subset Spex(A'_1),..., Spex(A'_d).

However Spex(A') and Spex(A) are homeomorphic since $A_f = A'_f$ for all $f \in \mathfrak{m}$. This is a contradiction so that depth $A \geqslant 2$.

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