The Center-Focus Problem and Reversibility

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Received April 5, 2000; revised April 5, 2000

In this paper we discuss the relation between time-reversibility and the center/focus problem. We show the following: for any analytic planar system \(X\), if \(j^1X\), the one-jet of \(X\), is conjugated to \((-y, x)\) then \(X\) is analytically time-reversible if and only if it is a center; if \(j^1X\) is conjugated to \((y, 0)\), then some sufficient and necessary conditions for \(X\) to have a center are given. In particular, the reversibility of certain types of polynomial vector fields is studied.

\(2001\) Academic Press

Key Words: center; involution; normal form; reversibility; vector field.

1. INTRODUCTION AND MAIN RESULTS

Reversibility is one of the interesting concepts that has been proved to be very useful in the qualitative theory of differential equations. It asks for conditions under which the system considered admits reversibility. In this paper we shall discuss this problem for certain classes of analytic vector fields on \(\mathbb{R}^2\) having a singularity at 0 of center-focus type.

**Definition 1.1.** A vector field \(X\) is said to be \(\phi\)-time-reversible if

\[
\phi(p)\cdot X(p) = -X(\phi(p)), \quad p \in \mathbb{R}^2, 0,
\]

where \(\phi\) is an involution (a diffeomorphism such that \(\phi^{-1}=\phi\)). \(X\) is called time-reversible if it is \(\phi\)-time-reversible for some involution \(\phi\).

A vector field \(X\) is said to be \(w\)-\(\phi\) reversible if there exists a \(\phi\)-time reversible vector field \(\bar{X}\) such that \(X\) is orbitally equivalent to \(\bar{X}\).

It is clear that time reversibility implies \(w\)-reversibility, but not vice versa.

Note that a vector field \(Y\) which is orbitally equivalent to a \(w\)-\(\phi\) reversible system \(X\) does not mean that \(Y\) is necessarily \(\phi\)-time reversible. In other

\(^1\)Supported by Fapesp-Brazil under Grant 97/10735-3 and PRONEX 76.97.1080/00.
\(^2\)Supported by Fapesp-Brazil under Grant 97/00695-4.
words, $w$-reversibility of $X$ only means the existence of $\phi$-time reversible vector fields in the set of all systems which are orbitally equivalent to $X$.

**Example 1.1.** The vector field given by the differential equation $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ is time-reversible with respect to the involution $\phi_0(x, y) = (-x, y)$ if and only if $f$ is even and $g$ is odd in $x$.

Not every involution has such a simple form as in the example. However, it is well known, due to the Montgomery–Bochner Theorem (see [8]), that $\phi$ is $C^\infty$-conjugated to the germ of the involution

$$\phi_0 : (x, y) \mapsto (-x, y).$$

The concept of reversibility is intrinsically linked to a given involution. Of primary importance in the study of time-reversible systems is the symmetry property with respect to $\text{Fix}(\phi)$, the fixed point set of the involution. For example, if $X$ is a $\phi$-time-reversible vector field and $u(t)$ is a solution of $X$ then $\phi(u(-t))$ is also a solution of the system. An immediate corollary of this observation is that if an orbit of a reversible vector field meets the fixed point set at two distinct points then it is necessarily a symmetric periodic orbit. In other words, reversibility implies certain geometric symmetries. This fact motivates us to consider the inverse problem: if a system possesses certain symmetries, then how about its reversibility? In this paper we try to discuss this problem, restricting ourselves primarily to two dimensional systems having a center at the singular point. It is worth pointing out that we are trying to establish the relation between time-reversibility and center, contrasting the known results concerning $w$-reversibility and center (see, for example [7, 12]).

Given a vector field $X$ on $\mathbb{R}^2$, one sees that if $X$ has a center at 0 then $j^1X$, the 1-jet of $X$, must be necessarily equivalent to one of the following three forms:

$$L_1 = (-y, x), \quad L_2 = (y, 0), \quad L_3 = (0, 0).$$

Vector fields having an $L_3$ linear part can be arbitrarily degenerated and will not be considered here. In this paper we shall mainly study the first two cases, i.e., vector fields whose linear parts are equivalent to $L_1$ or $L_2$. This means that by taking a linear change of coordinates we can put $X$ into the form

$$\dot{x} = -y + f(x, y), \quad \dot{y} = x + g(x, y),$$

or

$$\dot{x} = y + f(x, y), \quad \dot{y} = g(x, y),$$

or...
respectively, where \( f \) and \( g \) are smooth or analytic functions without linear terms. Throughout the paper we assume that this step has been taken.

In what follows we shall discuss these two cases separately.

1.1. Vector Fields with an \( L_1 \) Linear Part

Let \( X \) be an analytic vector field with an \( L_1 \) linear part. Then \( X \) is either a focus or a center at 0, and under a formal change of coordinates \( \Phi \) it can be reduced to the Birkhoff normal form:

\[
\begin{align*}
\dot{x} & = -y + S_1(r^2) x - S_2(r^2) y \\
\dot{y} & = x + S_2(r^2) x + S_1(r^2) y,
\end{align*}
\]

where \( S_1 \) and \( S_2 \) are formal series of \( r^2 = x^2 + y^2 \). We call \( \hat{X} \) the formal normal form of \( X \). \( \hat{X} \) has a center at 0 if and only if \( S_1 \neq 0 \). If \( S_1 \) is different from 0, i.e., \( \hat{X} \) is a focus, then it is known that there exists a smooth (\( C^\infty \)) normalization between \( X \) and \( \hat{X} \) (see [2, 11]). In this case the original system \( X \) is a focus at 0. On the other hand, if \( S_1 = 0 \), i.e., \( \hat{X} \) has a center at 0, then it is clear that \( \hat{X} \) is \( \phi_0 \)-time-reversible. Recall that in this case the normalization is always analytic (see [3]), therefore the original vector field \( X \) is also a center. Moreover, it is time-reversible with respect to the involution \( \Phi^{-1} \circ \phi_0 \circ \Phi \). Since any reversible system with an \( L_1 \) linear part is always a center, we have the following conclusion.

**Theorem 1.** Any analytical vector field with an \( L_1 \) linear part is time-reversible if and only if it is a center.

Recall that the result above may be not true for smooth (\( C^\infty \)) vector fields, due to the possible existence of flat terms.

1.2. Vector Fields with an \( L_2 \) Linear Part

It is well known that any vector field with an \( L_2 \) linear part can be normalized, via formal changes of coordinates, to a system of the form (see [10])

\[
X: \quad \dot{x} = y + f(x), \quad \dot{y} = g(x)
\]

where \( f \) and \( g \) are formal series having no linear parts. Below we assume that in (7) neither \( f \) nor \( g \) are identically 0. Then they can be expanded in the form

\[
f(x) = x^m(a_0 + a_1 x + \cdots), \quad a_0 \neq 0,
\]

and

\[
g(x) = x^k(b_0 + b_1 x + \cdots), \quad b_0 \neq 0.
\]
Since we are interested in vector fields having a center at 0, therefore we shall only consider those systems which have no characteristic lines. To this end, we find it is more convenient to make use of the term Monodromy conditions. Throughout the paper we assume that the Monodromy conditions are satisfied. System (7) with \( f \) and \( g \) given by (8) and (9) is said to satisfy the Monodromy conditions if and only if one of the following two conditions is satisfied:

(i) \( b_0 < 0, k = 2n - 1, m > n, (n \geq 2) \);

(ii) \( b_0 < 0, k = 2n - 1, m = n, ma_0^2 + 4b_0 < 0, (n \geq 2) \).

The negativity of \( b_0 \) allows us to rescale it to \(-1\). Therefore we always assume that

\[
g(x) = -x^{2n-1} + b_1 x^{2n} + \cdots. \tag{10}
\]

Remark 1.1. The Monodromy conditions exclude the possibility of \( g(x) \equiv 0 \). In the rest of the paper we also assume that \( f(x) \) is not 0. Otherwise \( X \) is always time-reversible with respect to \( \phi_1 = (x, -y) \), and it has a center at 0 if and only if \( b_0 < 0 \) and \( k \) is odd in (8) (see [1]).

In this paper, we shall depart from normal form (7), assuming that it is analytic. For any system of the form (7), we observe that there is a close relation between reversibility, \( w \)-reversibility and the conditions of being a center. For example, the results of the present paper imply that system (7) has a center at 0 if and only if it is \( w \)-\( \phi_0 \) reversible, since any analytic system (7) satisfying the Monodromy conditions is analytically orbitally equivalent (see Theorem 2) to a system of the form

\[
\bar{X}: \quad \dot{x} = y + h(x), \quad \dot{y} = -x^{2n-1}, \tag{11}
\]

and the latter has a center if and only if it is \( \phi_0 \)-time reversible (see [1]). Observe that the \( \phi_0 \)-time reversibility of system (11) precisely indicates that the function \( h(x) \) is even. Therefore we are interested in studying:

(I) the regularity of the normalization between (7) and (11); the relation between the function \( h \) and functions \( f \) and \( g \);

(II) time-reversibility of (7).

As to the first question, we shall show that the normalization between (7) and (11) is actually analytic. This means that certain dynamical properties, say, center/focus, can be preserved. In other words, we shall prove that there exist an analytic function \( F \) and an analytic change of coordinates \( \Phi \)
bringing the vector field (7) multiplied by $F$ to (11). This will be done after the proof of the following theorem. We define

$$G(x) = \int_0^x g(s) \, ds. \tag{12}$$

**Theorem 2.** Let $\bar{X}$ and $X$ be given, respectively, by (11) and (7) where $f$ is from (8) and $g$ from (10). Let $F$ be a formal series and $\Phi$ a formal change of coordinates reducing $F \cdot X$ ($X$ multiplied by $F$) to $\bar{X}$. Then

(i) $F$ and $\Phi$ can take the forms $F = F(x), F(0) = 1$, and $\Phi = (\phi_1(x), y)$, respectively. Moreover, $j^1\phi_1(x) = x, G(\phi_1(x)) = -\frac{1}{2}x^{2n}$.

(ii) $h(x) = f(\phi_1(x))$.

(iii) $\phi(x)$ is odd if and only if $g(x)$ is odd.

In this paper, we shall prove the following

**Theorem 3.** Let $X$ be an analytic vector field satisfying the Monodromy conditions and having the form (7). Then it has a center at 0 if and only if there exists a germ at 0 of an analytic function $\xi(x)$, $j^1\xi(x) = x$, such that

$$G(\xi(x)) = -\frac{1}{2n}x^{2n}, \quad f(\xi(-x)) = f(\xi(x)). \tag{13}$$

One sees from Theorem 3 that a necessary condition for system (7) to have a center is that the leading degrees of $f(x)$ and $G(x)$ must be even.

Theorem 3 corrects some mistakes restated in [6] (originally from [5]). In [6], it says that system (7) has a center if and only if the equation $\Phi(f(x)) = G(x)$ has an analytic solution. One can see that the necessary condition is not true if one takes $G(x) = -x^4$ and $f(x) = x^4$, since in this case the corresponding system has a center but the equation $\Phi(f(x)) = G(x)$ has no analytic solutions. It turns out that the sufficient condition stated there is also in doubt, since one can find examples such that even the Monodromy conditions are violated.

We feel that in general to give a complete answer to the second question posed previously is much less trivial. In this paper we shall study some special cases, and in Sections 3 and 4 we shall treat with some detail polynomial vector fields having lower degrees, showing the complexity of the problem.

The following notation is of convenience.

Let $\xi(x)$ be a formal series with an even degree of leading term, namely, $\xi(x) = c_0x^{2n} + c_1x^{2n+1} + c_2x^{2n+2} + \cdots$, where $c_0 \neq 0$. Denote by $\tau(\xi)$ the minimal possible number $k$ such that $c_{2k-1} \neq 0$. If $\xi(x)$ is an even function then $\tau(\xi) \equiv \infty$. 

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**Center-Focus and Reversibility**

241
Example 1.2. Let \( \xi(x) = x^4 + x^6 + x^7 \), then \( \tau(\xi) = 2 \).
Let \( \eta(x) = x^{10} + ax^{12} + x^{15} \), then \( \tau(\eta) = 3 \).

Note that for any given function \( \xi(x) \) the value of \( \tau(\xi) \) does not depend on \( p \).

Recall that the leading degrees of \( G(x) \) and \( f(x) \) are necessarily even if \( X \) has a center. Therefore \( \tau(G) \) and \( \tau(f) \) are well defined. In this paper, we prove the following.

**Theorem 4.** Let \( X \) be given by (7). In terms of (8) and (12), the following statements hold.

1. If \( \tau(f) \neq \tau(G) \), then \( X \) has a focus at 0.
2. If one of \( \tau(f) \) and \( \tau(G) \) is infinity, then \( X \) has a center at 0 if and only if the other one is also equal to infinity.

Observe that Moussu’s theorem concerning vector field (11) is a special case of Theorem 4, since in (11), \( \tau(G) = \infty \). Also Moussu’s theorem deals with the \( w \)-reversibility, not time-reversibility of the original system. The following corollary establishes certain relation between the two terms.

**Corollary 1.1.** If \( X \) is \( w \)-\( \phi_0 \)-reversible and one of \( \tau(f) \) and \( \tau(G) \) is infinity, then \( X \) is \( \phi_0 \)-time reversible.

2. PROOFS

**Proof of Theorem 2.** (i) To obtain the normal form (11), we look for a function \( F \) and a change of coordinates \( \Phi \) having forms \( F = F(x) \) and \( \Phi = (\phi(x), y) \), respectively. Then one sees that \( F(x) \) and \( \phi(x) \) must satisfy the following relations

\[
F(\phi(x)) = \phi'(x), \quad g(\phi(x)) F(\phi(x)) = -x^{2n-1},
\]

or equivalently,

\[
F(\phi(x)) = \phi'(x), \quad G(\phi(x)) = -\frac{1}{2n} x^{2n}.
\]

Since the leading term of \( G \) is of degree \( 2n \), the existence of the analytic solution \( \phi(x) \) of the form \( \phi(x) = x + o(x) \) is clear by virtue of the implicit
function theorem. It follows from (14) that $F(x)$ is well defined. Moreover, $F(0) \neq 0$ due to the fact that $\phi'(0) = 1$.

(ii) Notice that after the multiplication $F$ and the transformation $\Phi$ the function $f(x)$ in (7) is transformed to $f(\phi_1(x)) F(\phi_1(x))/\phi_1'(x)$. Therefore the relation $h(x) = f(\phi_1(x))$ holds.

(iii) First it is easy to show that if $\phi_1(x)$ is odd then $g(x)$ is odd. To see this point, one considers the relation $\Phi(x) = \Phi_1(x)$.

To show the reverse of the statement, let $F(x) = 1 + \mu_1 x + \mu_2 x^2 + \cdots$ and $\phi_1(x) = x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots$. With this notation, the equation (15) is equivalent to a homologic system of the following equations:

$$
\begin{align*}
\alpha_{2n+1}^2 + \alpha_1 &= 0 \\
\alpha_{2n+2}^2 &= \left(\alpha_3 + \frac{2n-1}{2}\alpha_2^2\right) + \alpha_1 \alpha_2 + \alpha_2 \frac{1}{2n+2} = 0 \\
\vdots & \quad \vdots \\
x^{2n+k} = U_k(\alpha_2, ..., \alpha_{k+1}, b_1, ..., b_k) &= 0 
\end{align*}
$$

(16)

for $k = 1, 2, ...$, where $U_k$ is a function of $\alpha_2, ..., \alpha_{k+1}$ and $b_1, ..., b_k$. It is linear in $b_1, ..., b_k$ and nonlinear in $\alpha_2, ..., \alpha_{k+1}$. More precisely, it has the following form:

$$
U_k = \frac{1}{2n+k} b_k + u^{(k)}(b_{k-1} + \cdots + u^{(k)}_{k-1} b_{k-1} - u^{(k)}_k),
$$

(17)

where

$$
\begin{align*}
u^{(k)}_j &= \nu_j^{(k)}(\alpha_2, ..., \alpha_{j+1}) \\
&= \alpha_{j+1} + \sum_{p=2}^j \sum_{i_p + \cdots + i_{p-1} = j-p} l^{(k)}_{i_1, ..., i_p} \alpha_{i_1} \cdots \alpha_{i_p},
\end{align*}
$$

(18)

for $j = 1, ..., k$, where $l^{(k)}_{i_1, ..., i_p}$ are constant. Now assume that $b_{2k-1} = 0$, i.e., assume that $g(x)$ is odd. From the first relation of (16), we know $\alpha_2 = 0$. We shall use induction to show that $\alpha_{2k} = 0$ for all $k = 1, 2, ...$. Suppose that
\[ x_{2k} = u_{2k-1}^1(x_2) b_{2k+2} + u_{2k-6}^2(x_2, x_3, x_4) b_{2k-4} + \cdots \]

\[ + u_{2k-3}^2(x_2, \ldots, x_{2k-2}) b_2 + \sum_{p=2}^{2k-1} \sum_{i_1 + \cdots + i_p = 2k + p - 1} f_{i_1, \ldots, i_p}^{(2k-1)} x_{i_1} \cdots x_{i_p}. \]

(19)

We claim that \( x_{2k} = 0 \).

To prove the above claim we need only to show that all \( u_{2k-1}^{(2k-1)} \) \((i = 1, \ldots, k - 1)\) and the last summation are 0.

From (18) we know that \( u_{2k-1}^{(2k-1)} = \sum_{p=1}^{2k-1} \sum_{i_1 + \cdots + i_p = 2k - 1} f_{i_1, \ldots, i_p}^{(2k-1)} x_{i_1} \cdots x_{i_p} \). Notice that in this summation each term \( x_{i_1} \cdots x_{i_p} \) must contain at least one \( x_j \) with an even subscript \( i_s \). In fact, since \( i_1 + \cdots + i_p = 2k - 1 + p \), if \( p \) is odd then \( i_1 + \cdots + i_p \) is even, clearly there is at least one even \( i_s \); similar arguments can be applied to the case that \( p \) is even, in this case \( i_1 + \cdots + i_p \) is odd. Therefore \( u_{2k-1}^{(2k-1)} = 0 \). By the same arguments we can show in the last summation in (19) each term contains at least one \( x_j \) with an even subscript, therefore \( x_{2k} = 0 \). We prove that \( \phi_1(x) \) is odd.

We have shown that (7) is orbitally equivalent to (11) in which \( h(x) \) can be expressed in terms of \( f(x) \) and \( g(x) \), namely, the second statement of the theorem. However, to decide the types of singularity of (11) from that of (7) we need the analyticity of \( F \) and \( \Phi \). This is based on the following statement whose proof is easy and is omitted here.

**Proposition 2.1.** The change of coordinates \( \Phi(x) \) and the multiplication \( F(x) \) stated above are analytic.

**Proof of Theorem 3.** Theorem 3 follows from Theorem 2. In fact, in reducing a system having a form of (7) to its orbital normal form (11), we have \( h(x) = f(\phi_1(x)) \). Recall that (11) has a center at 0 if and only if it is \( \phi_0 \)-time reversible, i.e., \( h(x) \) is even. Equivalently, system (7) has a center at 0 if and only if \( f(\phi_1(x)) \) is even. Therefore, one can take function \( \phi_1(x) \) as the function \( \zeta(x) \) in Theorem 3.

**Proof of Theorem 4.** To prove the theorem it is enough to show the following: If one of \( f(x) \) or \( G(x) \) is even, then \( X \) has a center at 0 is equivalent to say the other function is also even. The first statement follows from the proof of the second.
Assume that \( g(x) \) is odd. Then \( \phi_1(x) \) is odd, by the third statement of Theorem 2. We know in this case \( X \) has a center if and only if \( h(x) = f(\phi_1(x)) \) is even. Therefore we need to prove the evenness of \( f(x) \). Expand \( h(x) \) as follows:

\[
h(x) = f(\phi_1(x)) = x^m(d_0 + d_1 x + \cdots)
\]

where the coefficients \( d_j \) can be described as

\[
d_0 = a_0 \\
d_1 = a_1 + a_2 a_0 \\
\vdots \\
d_k = a_k + w_{k, 1}(x_2) d_{k-1} + w_{k, 2}(x_2, x_3) a_{k-2} + \cdots + w_{k, k}(x_2, \ldots, x_{k+1}) a_0,
\]

where

\[
w_{k, j}(x_2, \ldots, x_{j+1}) = \sum_{p=1}^{j} \sum_{i_1 + \cdots + i_j = j+p} w_{i_1, \ldots, i_j}(x_2) \cdots x_{j+1}, \quad j = 1, \ldots, k,
\]

where \( w_{i_1, \ldots, i_j} \) are constants.

It is clear that \( m \) is even if \( h(x) \) is even. It remains to show that under the assumption of oddness of \( \phi_1(x) \), \( d_{2k-1} = 0 \) is equivalent to \( a_{2k-1} = 0 \) for \( k = 1, 2, \ldots \).

In fact, since \( \phi_1(x) \) is odd, we have \( x_{2k} = 0 \), \( k = 1, 2, \ldots \), consequently, \( d_1 = a_1 \) due to (21). More generally, from (21) we know that

\[
d_{2k+1} = a_{2k+1} + w_{2k+1, 1}(x_2) a_{2k} + w_{2k+1, 2}(x_2, x_3) a_{2k-2} + \cdots + w_{2k+1, 2k+1} a_0,
\]

where

\[
w_{2k+1, j} = \sum_{p=1}^{2j+1} \sum_{i_1 + \cdots + i_p = 2j+1+p} f_{i_1, \ldots, i_p}(x_2) \cdots x_{j+1} \]

We shall show that the right side of (24) is 0. In fact if \( p \) is odd, i.e., the number of \( x \) in the product \( x_{i_1} \cdots x_{i_p} \) is odd, then at least one \( i_s \) is even since \( i_1 + \cdots + i_p = 2j+1+p \) is even, it follows that \( x_{i_s} = 0 \), and consequently the term \( x_{i_1} \cdots x_{i_p} \) is 0. If \( p \) is even, then \( i_1 + \cdots + i_p = 2j+1+p \) is odd, the same arguments say that at least one \( i_s \) in \( i_1, \ldots, i_p \) is even. Therefore \( x_{i_1} \cdots x_{i_p} \) is 0, too. Thus we prove \( d_{2k+1} = a_{2k+1} \), namely, \( f(x) \) is even.

On the other hand, if \( f(x) \) is even then, clearly, \( X \) is a center, this is because in this case \( X \) is \( \phi_0 \)-time reversible.
The proof of the case that \( f(x) \) is even and \( X \) has a center imply \( g(x) \) is odd can be given in a similar way. Therefore we have proved the theorem.

3. SOME POLYNOMIAL VECTOR FIELDS

In this section we shall study some applications of the results obtained. To obtain the detailed calculation, we make use of the software “axiom”.

Example 3.1. Consider the following cubic system studied in [4]:

\[
X: \begin{cases}
    \dot{x} = -y \\
    \dot{y} = x + a_1 x^2 + a_2 x y + a_3 y^2 + a_4 x^3 + a_5 x^2 y + a_6 x y^2.
\end{cases} \tag{25}
\]

This is a vector field having an \( L_1 \) linear part. It is proved in [4] that \( X \) has a center if one of the following three conditions is satisfied.

1. \( a_2 = a_5 = 0 \);
2. \( a_1 = a_3 = a_5 = 0 \);
3. \( a_4 = a_5 = a_6 = 0, a_1 + a_3 = 0 \).

In other words, if one of these three conditions is satisfied then, by Theorem 1, \( X \) is time-reversible with respect to some involution \( \phi \). It is easy to see that in the first two cases we can take \( \phi \) to be \( (x, -y) \) and \( (-x, y) \), respectively. In the third case, if \( a_2 = 0 \), then \( X \) is time-reversible with respect to \( \phi = (x, -y) \). If \( a_2 \neq 0 \), then there is no linear involution \( \phi \) such that \( X \) is \( \phi \)-time-reversible. To nonlinear involution can be found jet-by-jet. In fact, for any fixed number \( k \) it is possible to obtain an explicit expression of \( j^k \phi \). Take, for example, \( a_1 = -a_3 = 1 \) and replace \( a_2 \) by \( c \) in (25), namely, \( X \) has a form

\[
X: \begin{cases}
    \dot{x} = -y \\
    \dot{y} = x + x^2 + c x y - y^2.
\end{cases}
\]

Then one can solve the equations \( \phi \cdot \phi = id \) and \( \phi X = -X(\phi) \) and obtain, say, the 3-jet of \( \phi \)

\[
j^3 \phi = \left( \begin{array}{c}
    x - \frac{7}{2} c x y - \frac{5}{2} c^2 x^3 + \left( \frac{7}{4} c + b \right) x^2 y + \frac{5}{4} c^2 x y^2 + \left( \frac{1}{4} c + b \right) y^3 \\
    -y - \frac{3}{2} c (x^2 - y^2) + bx^3 + \frac{5}{8} c^2 x^2 y + \left( \frac{1}{4} c + b \right) x^2 y^2 - \frac{5}{8} c^2 y^3
\end{array} \right);
\]

where \( b \) is a parameter which can be fixed in determining a higher jet of the involution. Indeed, one can see the following equalities:

\[
j^3(\phi \cdot \phi) = id, \quad j^3(\phi X) = -j^3 X(\phi).
\]
EXAMPLE 3.2. Consider the following polynomial vector field of degree 4.

\[
\begin{align*}
X: \quad \dot{x} &= y + a_1 x^2 + a_2 x^3 + a_3 x^4 \\
\dot{y} &= b_1 x^2 + b_2 x^3 + b_3 x^4,
\end{align*}
\]  

(26)

where \(a_i\) and \(b_i\) are parameters such that at least one \(a_i\) and at least one \(b_i\) are different from 0.

Assume that \(X\) satisfies the Monodromy conditions, i.e., \(b_1 = 0, b_2 = -1\) (up to a rescaling), and \(a_i^2 < 2\). We have the following.

PROPOSITION 3.2. Vector field (26) has a center if and only if it is \(\phi_0\)-time reversible.

Proof. The validity of the statement in the case \(b_3 = 0\) is trivial. It remains to prove that if \(b_3 \neq 0\) then \(X\) can never be a center.

Let \(b_3 \neq 0\). If \(a_1 = 0\) then \(a_2 = 0\), since the leading degree of \(f(x)\) in (7) must be even. In this case for any \(a_3\) the system can not be a center at 0, due to the first statement of Theorem 4.

Below we assume that \(a_1 \neq 0\). Replace \(b_3\) by \(b\) and consider the orbital normal form of (26), namely, we reduce it to a form

\[
\begin{align*}
\dot{x} &= y + \hat{a}_1 x^2 + \hat{a}_2 x^3 \cdots, \\
\dot{y} &= -x^3
\end{align*}
\]

(27)

where \(\hat{a}_i\) are functions of \(b, a_1, a_2,\) and \(a_3\). We shall in what follows prove that if \(b \neq 0\) then \(\tau(\hat{a}_1 x^2 + \cdots) \leq 4\), namely, at least one of \(\hat{a}_2, \hat{a}_4, \hat{a}_6\) and \(\hat{a}_8\) is different from 0.

To normalize (26), we first solve for \(\phi(x)\) from the equation \(\phi''(x) + \frac{4b}{5} \phi^3(x) = x^4\), which is derived from \(G(\phi(x)) = -x^4/4\), and then substitute it to the function \(a_1 \phi(x)^2 + a_2 \phi^3(x) + a_3 \phi^4(x)\). With some direct calculation, we can explicitly write the first few terms of \(\phi\),

\[
\phi(x) = x + A_2 x^2 + A_3 x^3 + \cdots,
\]

where

\[
A_2 = \frac{1}{5} b, \quad A_3 = \frac{7}{50} b^2, \quad A_4 = \frac{16}{125} b^3, \quad A_5 = \frac{463}{62500} b^4, \quad A_6 = \frac{462}{3125} b^5, \cdots.
\]

Consequently,

\[
a_1 \phi(x)^2 + a_2 \phi^3(x) + a_3 \phi^4(x) = \hat{a}_1 x^2 + \hat{a}_2 x^3 + \cdots,
\]

where \(\hat{a}_2 = a_2 + \frac{7}{5} a_1 b\). If \(a_2 \neq -\frac{7}{5} a_1 b\), then \(X\) is a focus, otherwise, i.e., \(a_2 = -\frac{7}{5} a_1 b\) then we have \(\hat{a}_4 = \frac{4b}{5} a_3 + \frac{18}{5} a_1 b^3 - \frac{6}{5} a_1 b^2\). If \(\hat{a}_4 \neq 0\) then \(X\) is a focus, otherwise, \(a_3 = -\frac{4b}{5} (3b - 5) a_1 b\). In this case, \(\hat{a}_6 = -\frac{11}{375} (7b - 75) a_1 b^4\). Since \(a_1 b \neq 0\), therefore if \(b \neq \frac{25}{7}\) then \(X\) is a focus,
otherwise, $\hat{a}_8 = -\frac{14096753734375}{52944132} a_1$ which is never equal to 0. Therefore $X$ is not a center.

This example shows the coincidence of time-reversibility and the conditions to be a center. The following example implies that the situation is not always so simple for systems having higher degrees.

**Example 3.3.** Consider the following polynomial vector field of degree 5

$$
\begin{align*}
\dot{x} &= y + P(x), \\
\dot{y} &= Q(x),
\end{align*}
$$

(28)

where $P = a_1 x^2 + a_2 x^3 + a_4 x^4 + a_4 x^5$, $Q(x) = -x^3 + b_3 x^4 + b_3 x^5$. We assume that $a_2^3 < 2$, for the sake of Monodromy conditions.

We shall prove the following

**Proposition 3.3.** Vector field (28) has a center if and only if one of the following conditions holds.

(i) $X$ is $\phi_0$-time reversible, i.e, $b_2 = a_2 = a_4 = 0$, or

(ii) there is a parameter $b$ such that $b_2 = -\frac{1}{2} b_2$, $b_3 = -\frac{1}{2} b^2$, $a_2 = a_1 b$, i.e.,

$$P(x) = a_1 (x^2 + bx^3), \quad \int_0^x Q(x) \, dx = -\frac{1}{4} (x^4 + bx^3)^2. \quad (29)$$

**Proof.** The sufficiency of these conditions is not hard to check. We shall in what follows prove that the conditions are also necessary. We distinguish two cases: (1) $b_2 = 0$; (2) $b_2 \neq 0$. In the first case, $\pi \int_0^x Q(x) \, dx = \infty$, due to therefore $a_2 = a_4 = 0$, by Theorem 4, namely, $X$ is $\phi_0$-time reversible. In the second case, we shall show that necessarily $\int_0^x Q(x) \, dx$ has a form $-\frac{1}{4} (x^2 + bx^3)^2$ and $P(x) = a_1 (x^2 + bx^3)^2$. Note that the second case implies that $a_1 \neq 0$ unless $P(x)$ is identically equal to 0.

Let $b_2 \neq 0$. Reduce the system to its orbitally equivalent normal form

$$
\begin{align*}
\dot{x} &= y + \tilde{P}(x), \\
\dot{y} &= -x^3,
\end{align*}
$$

(30)

where $\tilde{P}(x) = P(\phi(x))$ and $\phi$ is from the equation $\int_0^x Q(x) \, dx = -\frac{1}{4} x^4$, and study the evenness of $\tilde{P}(x)$. We shall prove that if $b_2 \neq 0$ then $\pi \tilde{P}(x) = \infty$ unless both $\int_0^x Q(x) \, dx$ and $P(x)$ are functions of $(x^2 + bx^3)$, i.e., $\int_0^x Q(x) \, dx = -\frac{1}{4} (x^2 + bx^3)^2$ and $P(x) = a_1 (x^2 + bx^3)$.

Consider the solution of the equation

$$
\phi^4 + B\phi^5 + C\phi^6 = x^4,
$$

(31)
where $B = -\frac{4}{7} b_2$ and $C = -\frac{2}{7} b_3$. This equation is derived from the relation 
\[ \int f(x) Q(x) \, dx = -\frac{1}{4} x^4. \]
With some straightforward calculation one gets the first few terms of the solution
\[ \phi(x) = x + A_2 x^2 + A_3 x^3 + \cdots, \]
where
\[
A_2 = -\frac{1}{2} B, \quad A_3 = -\frac{1}{4} C + \frac{7}{32} B^2, \quad A_4 = \frac{1}{8} BC - \frac{1}{2} B^3,
\]
\[
A_5 = \frac{9}{32} C^2 - \frac{117}{128} B^2 C + \frac{663}{2048} B^4, \quad A_6 = -\frac{35}{32} BC^2 + \frac{105}{256} B^3 C - \frac{231}{512} B^5,
\]
\[
A_7 = -\frac{15}{64} C^3 + \frac{1111}{2048} B^2 C^2 - \frac{24095}{65536} B^4 C + \frac{83261}{1048576} B^6
\]
\[
A_8 = \frac{3}{8} BC^3 - \frac{13}{4} B^2 C^2 + \frac{21}{4} B^3 C - B^7,
\]
\[
A_9 = \frac{1547}{4096} C^4 - \frac{46675}{8192} B^2 C^3 + \frac{112157}{8192} B^4 C^2 - \frac{2467665}{65536} B^6 C + \frac{13083315}{8388608} B^8
\]
\[
A_{10} = \frac{-3003}{512} BC^4 + \frac{15015}{512} B^3 C^3 - \frac{153153}{4096} B^5 C^2 + \frac{1838567}{16777216} B^7 C - \frac{323323}{131072} B^9, \ldots
\]

Now we can analyze the evenness of the function $\tilde{P}(x) := P(\phi(x))$. Denoting $\tilde{P}(x) = \tilde{a}_1 x^2 + \tilde{a}_2 x^3 + \cdots$, we shall show that the equalities $\tilde{a}_2k = 0$, $k = 1, 2, \ldots, 5$, are already enough to determine that $C = B^3/4$.

It is easy to see that $\tilde{a}_1 = a_1$ and $\tilde{a}_2 = a_2 - \frac{1}{2} B a_1$. If $a_2 \neq \frac{1}{2} B a_1$ then the term $x^3$ exists in $\tilde{P}(x)$, namely, $X$ is a focus. Let $a_2 = \frac{1}{2} B a_1$, then one finds that the relation $\tilde{a}_4 = 0$ is equivalent to $a_4 = B a_3 + (-\frac{1}{4} BC + \frac{1}{2} B^3) a_1$ for any $a_1$. Assuming that this relation holds, we pass to the coefficient $\tilde{a}_6$. We have $\tilde{a}_6 = \frac{1}{2} B C a_3 + (-\frac{1}{4} BC^2 + \frac{1}{2} B^2 C - \frac{1}{16} B^4) a_1$. The similar reasoning, if $\tilde{a}_6 \neq 0$, then the system has a focus, otherwise the relation that $\tilde{a}_6 = 0$ results in $C a_3 = (\frac{1}{16} B^2 - \frac{1}{2} B^2 C - \frac{1}{2} C^2) a_1$. Collecting above facts we obtain
\[ \tilde{a}_8 = \frac{-1872 B^4 C^2 + 1352 B^4 C - 221 B^7}{3072} a_1. \]

Observe the relation $\tilde{a}_8 = 0$ can stand only if $C = \frac{8}{7} b_3$ or $C = \frac{12}{7} b_3$. Keeping this fact, we go one step further by calculating
\[ \tilde{a}_{10} = \left( \frac{855}{576} B^2 C^3 - \frac{1001}{512} B^2 C^2 + \frac{2185}{1024} B^3 C - \frac{9177}{32768} B^7 \right) a_1. \]

Now $\tilde{a}_{10} = 0$ is equivalent to $C = \frac{8}{7} b_3$ or $1440 C^2 - 1748 B^3 C + 483 B^4 = 0$. It is easy to see that $\tilde{a}_7 = 0$ and $\tilde{a}_{10} = 0$ can not stand simultaneously unless $C = \frac{8}{7} b_3$, since $B \neq 0$ as assumed. Therefore we prove the proposition.
While it is true that any vector field with the following form has a center 
\[ \dot{x} = y + f(\zeta(x)), \quad \dot{y} = \zeta'(x), \]
where \( \zeta(x) \) and \( f \) are any analytic functions such that the Monodromy conditions are satisfied, it is far from trivial to give efficient necessary conditions detecting if a given system has a center. In this section, basing on the examples of the previous section, we shall briefly discuss this problem with respect to certain types of polynomial vector fields.

By Theorem 4 we know that the center/focus problem of the system 
\[ \dot{x} = y + f(x), \quad \dot{y} = g(x), \]
where \( g(x) = x^{2n-1} + \ldots \), is equivalent to the evenness of the function \( f(x) \), where \( f(x) \) satisfies \( \int_0^{2\pi} g(x) \, dx = -\frac{1}{2n} x^{2n} \).

Therefore the problem is closely related to the solution of the last equation. In what follows we consider two kinds of polynomial vector fields with

1. \[ g(x) = x^{2n-1} + bx^{2n}, \quad n \geq 2; \]
2. \[ g(x) = x^{2n-1} + bx^{2n} + cx^{2n+1}, \quad n \geq 2. \]

In the first case the relation \( \int_0^{2\pi} g(x) \, dx = -\frac{1}{2n} x^{2n} \) gives rise to 
\[ \phi(x)^{2n} + \tilde{b} \phi^{2n+1}(x) = x^{2n}, \]
where \( \tilde{b} = -\frac{2n}{2n+1} b \). If \( b \neq 0 \) then \( \phi(x) \) can never be polynomial. Consequently, \( f(\phi(x)) \) is not a polynomial, and its evenness is not easy to check. Nevertheless according to the Hilbert theorem, it depends only on the finite number of the so-called “focus numbers,” which is determined by the coefficients of \( f(x) \) and \( b \). We pose the following questions:

**Problem 1.** Let \( f(x) = a_0 x^{2p} + \ldots + a_s x^{2p+s}, \quad a_0 \neq 0, \quad g(x) = -x^{2n-1} + bx^{2n}, \quad n \geq 2. \) Assume that the Monodromy conditions are satisfied. Prove or disprove the following: Vector field (7) has a center if and only if

(i) \( b = 0 \) and \( f(x) \) is even, i.e., \( X \) is \( \phi_0 \)-time reversible; or

(ii) \( b \neq 0, \) \( f(x) \) is a function of \( \int_0^{2\pi} g(x) \, dx, \) i.e., there is a function \( \eta \) such that \( f(x) = \eta(\int_0^{2\pi} g(x) \, dx). \)

Moreover, if \( X \) is not a center then \( \tau(h(x)) \leq (s+2), \) where \( h(x) \) is defined as in (11).

Note that the second statement implies that \( b \neq 0, \) \( p \geq n \) and it is divisible by \( n. \) In the second case, we ask the following problem.
Problem 2. \( f(x) = a_0 x^{2p} + \cdots + a_s x^{2p+s}, \ a_0 \neq 0, \ g(x) = -x^{2n-1} + bx^{2n} + cx^{2n+1}, \ n \geq 2. \) Assume that the Monodromy conditions are satisfied. Prove or disprove the following.

(i) if \( b = 0, \) then \( X \) has a center if and only if it is \( \phi_0 \)-time reversible.

(ii) if \( b \neq 0 \) and \( c = -\frac{(n+1)}{2n+1} b^2 \) (i.e., \( G = -\frac{1}{2n+1}(x^n - \frac{n}{n+1}hx^{n+1})^2), \) then \( X \) has a center if and only if \( f(x) \) is a function of \( x^n - \frac{n}{n+1}hx^{n+1}. \)

(iii) if \( b \neq 0 \) and \( c \neq -\frac{n+1}{2n+1} b^2 \) then \( X \) has a center if and only if \( f(x) \) is a function of \( \int g(x) \, dx. \)

Moreover, if \( X \) is not a center, then \( \tau(h(x)) \leq s + 1. \)

Note that the second case implies that \( 2p, \) the leading degree of \( f \) can be less than \( 2n, \) the leading degree of \( G, \) however, in such a case \( 2p \) can only be equal to \( n. \) In other words, the system is not a center if \( n/(2p) \) is not an integer.

ACKNOWLEDGMENT

We are very grateful to the referee for pointing out a simpler proof of (iii) of Theorem 2 and some suggestive remarks.

REFERENCES