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Integral inequalities of systems and the estimate for solutions of certain nonlinear two-dimensional fractional differential systems

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ABSTRACT

This paper generalizes the results for the constructions of explicit bounds and the qualitative properties for the solutions of certain two-dimensional fractional differential systems established in a recent paper of the authors. The main generalizations come from an elementary inequality and by means of the modification of Medveď's de-singular approach.

1. Introduction

In the last few decades, fractional differential equations have gained considerable importance and attention due to their applications in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, mechanics, chemistry, aerodynamics, and the electrodynamics of complex mediums, etc. The amount of literature on such differential equations and their applications is vast; see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3] and the papers of Daftardar-Gejji and Jafari [4], Diethelm [5], Lakshmikantham [6], Lakshmikantham and Vatsala [7,8], Lin et al. [9], Hsieh et al. [10], Wang et al. [11], Zhou et al. [12,13] and the references given therein.

Integral inequalities are indispensable for us in the qualitative study of various differential equations and integral equations. A fundamental one is Gronwall–Bellman's inequality [14,15]. Very recently, some efforts have been made to generalize it to weakly singular situations to cope with some problems of fractional differential equations. For example, [16] generalized a singular Gronwall inequality and then used it to investigate the dependence of the solution on the order and initial condition to a certain fractional differential equation with Riemann–Liouville fractional derivatives; Lazarević and Spasić [17] used Ye, Gao and Ding's result to study the stability of a class of fractional linear autonomous systems with a time delay described by the state space equation; Ma and Pečarić [18] gained a class of nonlinear integral inequalities and used them to obtain bounds of a class of fractional differential equation and integral equation involving Erdélyi–Kober fractional integral.

In this paper, we first establish a linear and nonlinear two-dimensional integral inequalities system; and then use these results and the modification of Medveď's de-singular method [19,20] to obtain the component-wise (not on some norm) upper bounds for solutions of a class of nonlinear two-dimensional systems of fractional differential equations. The uniqueness and continuous dependence of the solutions are also discussed here. Our results have generalized some very recently results obtained in [21].

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Throughout this article, R denotes the set of real numbers, $R_+ = [0, +\infty), I = [0, T), 0 < T \le +\infty$; $C^i(M, S)$ denotes the class of all i-times continuously differentiable defined on set M with ranges in the set $S (i = 1, 2, \ldots)$ and $C^0(M, S) = C(M, S)$.

2. Integral inequalities of systems

In this section, we cite and establish some useful results and the definitions in the discussion of our proof as follows:

Lemma 2.1 ([18]). *Let $c \ge 0, m \ge n \ge 0$ and $m \ne 0$, then*

$$c^{\frac{n}{m}} \le \frac{n}{m} K^{\frac{n-m}{m}} c + \frac{m-n}{m} K^{\frac{n}{m}}$$

for any $K > 0$.

Lemma 2.2. *Let u, v and $f_i \in C(I, R_+)$, $i = 1, 2$ with f_i be nondecreasing; $\varphi_{ij} \in C(I \times I, R_+)$ be nondecreasing in the variable t for every s fixed $(i = 1, 2)$. If*

$$\begin{cases} u(t) \le f_1(t) + \int_0^t [\varphi_{11}(t, s)u(s) + \varphi_{12}(t, s)v(s)] \mathrm{d}s, \\ v(t) \le f_2(t) + \int_0^t [\varphi_{21}(t, s)u(s) + \varphi_{22}(t, s)v(s)] \mathrm{d}s, \end{cases} \quad t \in I \tag{2.1}$$

then for $t \in I$, we have

$$\begin{cases} u(t) \le \left[f_1(t) + f_2(t) \int_0^t \varphi_{12}(t, s)\Phi_2(s)\mathrm{d}s\right] \exp\left\{\int_0^t \varphi_{11}(t, s)\mathrm{d}s \right. \\ \qquad \left. + \int_0^t \varphi_{12}(t, s)\Phi_2(s) \left(\int_0^s \varphi_{21}(s, \tau)\Phi_1(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right\}, \\ v(t) \le \left[f_2(t) + f_1(t) \int_0^t \varphi_{21}(t, s)\Phi_1(s)\mathrm{d}s\right] \exp\left\{\int_0^t \varphi_{22}(t, s)\mathrm{d}s \right. \\ \qquad \left. + \int_0^t \varphi_{21}(t, s)\Phi_1(s) \left(\int_0^s \varphi_{12}(s, \tau)\Phi_2(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right\}, \end{cases} \tag{2.2}$$

where $\Phi_i(t) := \exp \int_0^t \varphi_{ii}(t, s)\mathrm{d}s,\ i = 1, 2.$

Proof. By similar arguments as in [22, see pp. 234–236], we can obtain the conclusion of Lemma 2.2 directly. To save space, we omit the details here. □

Theorem 2.3. *Let u, v, a and $b \in C(I, R_+)$; $\varphi_{ij} \in C(I \times I, R_+)$ be nondecreasing in variable t for every s fixed $(i = 1, 2)$. If*

$$\begin{cases} u^{p_1}(t) \le a(t) + \int_0^t \left[\varphi_{11}(t, s)u^{q_{11}}(s) + \varphi_{12}(t, s)v^{q_{12}}(s)\right] \mathrm{d}s, \\ v^{p_2}(t) \le b(t) + \int_0^t \left[\varphi_{21}(t, s)u^{q_{21}}(s) + \varphi_{22}(t, s)v^{q_{22}}(s)\right] \mathrm{d}s, \end{cases} \quad t \in I \tag{2.3}$$

where p_i and q_{ij} $(i, j = 1, 2)$ are constants satisfying $p_1 \ge q_{1j} > 0$ and $p_2 \ge q_{2j} > 0$ $(j = 1, 2)$, then for $t \in I$, we have

$$\begin{cases} u(t) \le \left\{a(t) + \left[A(t) + B(t) \int_0^t \overline{\varphi}_{12}(t, s)\overline{\Phi}_2(s)\mathrm{d}s\right] \exp\left[\int_0^t \overline{\varphi}_{11}(t, s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \int_0^t \overline{\varphi}_{12}(t, s)\overline{\Phi}_2(s) \left(\int_0^s \overline{\varphi}_{21}(s, \tau)\overline{\Phi}_1(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right]\right\}^{1/p_1}, \\ v(t) \le \left\{b(t) + \left[B(t) + A(t) \int_0^t \overline{\Phi}_{21}(t, s)\overline{\Phi}_1(s)\mathrm{d}s\right] \exp\left[\int_0^t \overline{\varphi}_{22}(t, s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \int_0^t \overline{\varphi}_{21}(t, s)\overline{\Phi}_1(s) \left(\int_0^s \overline{\Phi}_{12}(s, \tau)\overline{\Phi}_2(\tau)\mathrm{d}\tau\right) \mathrm{d}s\right]\right\}^{1/p_2}, \end{cases} \tag{2.4}$$

where

$$A(t) = \int_0^t \left(\frac{q_{11}}{p_1} K^{\frac{q_{11}-p_1}{p_1}} a(s) + \frac{p_1 - q_{11}}{p_1} K^{\frac{q_{11}}{p_1}} \right) \varphi_{11}(t,s) \mathrm{d}s + \int_0^t \left(\frac{q_{12}}{p_2} K^{\frac{q_{12}-p_2}{p_2}} b(s) + \frac{p_2 - q_{12}}{p_2} K^{\frac{q_{12}}{p_2}} \right) \varphi_{12}(t,s) \mathrm{d}s, \tag{2.5}$$

$$B(t) = \int_0^t \left(\frac{q_{21}}{p_1} K^{\frac{q_{21}-p_1}{p_1}} a(s) + \frac{p_1 - q_{21}}{p_1} K^{\frac{q_{21}}{p_1}} \right) \varphi_{21}(t,s) \mathrm{d}s + \int_0^t \left(\frac{q_{22}}{p_2} K^{\frac{q_{22}-p_2}{p_2}} b(s) + \frac{p_2 - q_{22}}{p_2} K^{\frac{q_{22}}{p_2}} \right) \varphi_{22}(t,s) \mathrm{d}s, \tag{2.6}$$

$$\overline{\varphi}_{ij}(t,s) = \frac{q_{ij}}{p_j} K^{\frac{q_{ij}-p_j}{p_j}} \varphi_{ij}(t,s) \tag{2.7}$$

and $\overline{\Phi}_i(t) := \exp \int_0^t \overline{\varphi}_{ii}(t,s) \mathrm{d}s,\ i,j = 1,2.$

Proof. Setting

$$z_1(t) = \int_0^t \left[\varphi_{11}(t,s) u^{q_{11}}(s) + \varphi_{12}(t,s) v^{q_{12}}(s) \right] \mathrm{d}s \tag{2.8}$$

and

$$z_2(t) = \int_0^t \left[\varphi_{21}(t,s) u^{q_{21}}(s) + \varphi_{22}(t,s) v^{q_{22}}(s) \right] \mathrm{d}s, \tag{2.9}$$

then

$$u^{p_1}(t) \le a(t) + z_1(t)$$

and

$$u^{p_2}(t) \le b(t) + z_2(t)$$

or

$$u(t) \le (a(t) + z_1(t))^{1/p_1} \tag{2.10}$$

and

$$v(t) \le (a(t) + z_2(t))^{1/p_2}. \tag{2.11}$$

By Lemma 2.1 and (2.10)–(2.11), for any $K > 0$, we have

$$u^{q_{11}}(t) \le \frac{q_{11}}{p_1} K^{\frac{q_{11}-p_1}{p_1}} \left(a(t) + z_1(t) \right) + \frac{p_1 - q_{11}}{p_1} K^{\frac{q_{11}}{p_1}},$$
$$u^{q_{21}}(t) \le \frac{q_{21}}{p_1} K^{\frac{q_{21}-p_1}{p_1}} \left(a(t) + z_1(t) \right) + \frac{p_1 - q_{21}}{p_1} K^{\frac{q_{21}}{p_1}},$$
$$v^{q_{12}}(t) \le \frac{q_{12}}{p_2} K^{\frac{q_{12}-p_2}{p_2}} \left(a(t) + z_2(t) \right) + \frac{p_2 - q_{12}}{p_2} K^{\frac{q_{12}}{p_2}}$$

and

$$v^{q_{22}}(t) \le \frac{q_{22}}{p_2} K^{\frac{q_{22}-p_2}{p_2}} \left(a(t) + z_2(t) \right) + \frac{p_2 - q_{22}}{p_2} K^{\frac{q_{22}}{p_2}}.$$

Substituting the last relations into (2.8) and (2.9) we get

$$\begin{cases} z_1(t) \le A(t) + \int_0^t \left[\frac{q_{11}}{p_1} K^{\frac{q_{11}-p_1}{p_1}} \varphi_{11}(t,s) z_1(s) + \frac{q_{12}}{p_2} K^{\frac{q_{12}-p_2}{p_2}} \varphi_{12}(t,s) z_2(s) \right] \mathrm{d}s, \\ z_2(t) \le B(t) + \int_0^t \left[\frac{q_{21}}{p_1} K^{\frac{q_{21}-p_1}{p_1}} \varphi_{21}(t,s) z_1(s) + \frac{q_{22}}{p_2} K^{\frac{q_{22}-p_2}{p_2}} \varphi_{22}(t,s) z_2(s) \right] \mathrm{d}s, \end{cases} \tag{2.12}$$

where $A(t)$ and $B(t)$ are defined as in (2.5) and (2.6), respectively.

Obviously, the functions $A(t)$ and $B(t)$ are nondecreasing on I. An application of Lemma 2.2 to (2.12) and combing with (2.10)–(2.11) yield the desired bounds given in (2.4). □

Remark 2.1. We note that the nondecreasing condition for functions $a(t)$ and $b(t)$ are not needed in Theorem 2.3.

Definition 2.4 (*See [3]*)**.** The fractional derivative of order $0 < \alpha < 1$ of a function $f(x) \in C(R_+, R)$ is given by

$$D^{\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-t)^{-\alpha}f(t)\mathrm{d}t$$

provided that the right side is pointwise defined on R_+.

Definition 2.5 (*See [3]*)**.** The fractional primitive of order $\alpha > 0$ of a function $f : R_+ \to R$ is given by

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\mathrm{d}t$$

provided the right side is pointwise defined on R_+.

Lemma 2.6 (*See [23, p. 296]*)**.** *Let α, β, γ, k and p be positive constants. Then*

$$\int_0^t (t^{\alpha}-s^{\alpha})^{p(\beta-1)}s^{kp(\gamma-1)}\mathrm{d}s = \frac{t^{\theta}}{\alpha}B\left[\frac{kp(\gamma-1)+1}{\alpha}, p(\beta-1)+1\right], \quad t \in R_+,$$

where $B[\xi,\eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}\mathrm{d}s(\Re\xi > 0, \Re\eta > 0)$ and $\theta = p[\alpha(\beta-1)+k(\gamma-1)]+1$.

3. The estimate of fractional differential systems

In this section, we consider the boundedness, uniqueness and continuous dependence of the solutions of the nonlinear two-dimensional systems of fractional differential equations

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t), y(t)), & \text{(a)} \\ D^{\beta}y(t) = g(t, x(t), y(t)), & \\ D^{\alpha-1}x(t)|_{t=0} = \eta, \qquad D^{\beta-1}y(t)|_{t=0} = \zeta, & \text{(b)} \end{cases} \tag{3.1}$$

where D^{α} and D^{β} are the fractional derivative (in the sense of Riemann–Liouville) of order $0 < \alpha, \beta < 1, t \in I = [0, T)$, $0 < T \le +\infty, f$ and g are some continuous functions and $\eta, \zeta \in R$.

Firstly, we consider the case $\alpha \le \beta$.

Theorem 3.1. *If f and $g \in C(I \times R^2, R)$ and they satisfy condition* (H_1)*:*

$$\begin{cases} |f(t,x,y)| \le \varphi_{11}(t)|x|^{k_{11}} + \varphi_{12}(t)|y|^{k_{12}}, \\ |g(t,x,y)| \le \varphi_{21}(t)|x|^{k_{21}} + \varphi_{22}(t)|y|^{k_{22}}, \end{cases}$$

where $\varphi_{ij} \in C(I, R_+)$, $k_{ij} \in (0, 1]$ $(i, j = 1, 2)$ and $x, y \in R$. Assume that $\alpha \le \beta$. Then for any $K > 0$ we have

$$\begin{cases} |x(t)| \le 3^{1-\frac{1}{q}}t^{\alpha-1}\left\{\left(\dfrac{|\eta|}{\Gamma(\alpha)}\right)^q + \left[\widetilde{A}(t)+\widetilde{B}(t)\displaystyle\int_0^t \widetilde{\varphi}_{12}(t,s)\widetilde{\Phi}_2(s)\mathrm{d}s\right]\exp\left[\displaystyle\int_0^t \widetilde{\varphi}_{11}(t,s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \displaystyle\int_0^t \widetilde{\varphi}_{12}(t,s)\widetilde{\Phi}_2(s)\left(\int_0^s \widetilde{\varphi}_{21}(s,\tau)\widetilde{\Phi}_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\}^{1/q}, \\ |y(t)| \le 3^{1-\frac{1}{q}}t^{\beta-1}\left\{\left(\dfrac{|\zeta|}{\Gamma(\beta)}\right)^q + \left[\widetilde{B}(t)+\widetilde{A}(t)\displaystyle\int_0^t \widetilde{\varphi}_{21}(t,s)\widetilde{\Phi}_2(s)\mathrm{d}s\right]\exp\left[\displaystyle\int_0^t \widetilde{\varphi}_{22}(t,s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \displaystyle\int_0^t \widetilde{\varphi}_{21}(t,s)\widetilde{\Phi}_1(s)\left(\int_0^s \widetilde{\varphi}_{12}(s,\tau)\widetilde{\Phi}_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\}^{1/q}, \end{cases} \tag{3.2}$$

for $t > 0$, where

$$p = \frac{1+2\alpha}{1+\alpha}, \qquad q = \frac{1+2\alpha}{\alpha}, \tag{3.3}$$

$$\widetilde{A}(t) = K_{11}E_{11}(t)\int_0^t \varphi_{11}^q(\tau)\mathrm{d}\tau + K_{12}E_{12}(t)\int_0^t \varphi_{12}^q(\tau)\mathrm{d}\tau, \tag{3.4}$$

$$\widetilde{B}(t) = K_{21}E_{21}(t)\int_0^t \varphi_{21}^q(\tau)\mathrm{d}\tau + K_{22}E_{22}(t)\int_0^t \varphi_{22}^q(\tau)\mathrm{d}\tau, \tag{3.5}$$

$$\widetilde{\varphi}_{ij}(t,s) = 3^{q-1}k_{ij}K^{k_{ij}-1}E_{ij}(t)\varphi_{ij}^q(s), \qquad \widetilde{\Phi}_i(t) = \exp\int_0^t \widetilde{\varphi}_{ij}(t,s)\mathrm{d}s, \quad i,j=1,2,$$

$$K_{11} = k_{11}K^{k_{11}-1}\frac{3^{q-1}|\eta|^q}{\Gamma^q(\alpha)} + (1-k_{11})K^{k_{11}}, \tag{3.6}$$

$$K_{12} = k_{12}K^{k_{12}-1}\frac{3^{q-1}|\zeta|^q}{\Gamma^q(\beta)} + (1-k_{12})K^{k_{12}}, \tag{3.7}$$

$$K_{21} = k_{21}K^{k_{21}-1}\frac{3^{q-1}|\eta|^q}{\Gamma^q(\alpha)} + (1-k_{21})K^{k_{21}}, \tag{3.8}$$

$$K_{22} = k_{22}K^{k_{22}-1}\frac{3^{q-1}|\zeta|^q}{\Gamma^q(\beta)} + (1-k_{22})K^{k_{22}}, \tag{3.9}$$

$$E_{11}(t) = \frac{1}{\Gamma^q(\alpha)}t^{\theta_1}B^{q/p}[pk_{11}(\alpha-1)+1, p(\alpha-1)+1], \tag{3.10}$$

$$E_{12}(t) = \frac{1}{\Gamma^q(\alpha)}t^{\theta_2}B^{q/p}[pk_{12}(\beta-1)+1, p(\alpha-1)+1], \tag{3.11}$$

$$E_{21}(t) = \frac{1}{\Gamma^q(\beta)}t^{\theta_3}B^{q/p}[pk_{21}(\alpha-1)+1, p(\beta-1)+1], \tag{3.12}$$

$$E_{22}(t) = \frac{1}{\Gamma^q(\beta)}t^{\theta_4}B^{q/p}[pk_{22}(\beta-1)+1, p(\beta-1)+1], \tag{3.13}$$

$$\theta_1 = qk_{11}(\alpha-1) + \frac{q}{p}, \tag{3.14}$$

$$\theta_2 = qk_{12}(\beta-1) + \frac{q}{p}, \tag{3.15}$$

$$\theta_3 = qk_{21}(\alpha-1) + \frac{q}{p} \tag{3.16}$$

and

$$\theta_4 = qk_{22}(\beta-1) + \frac{q}{p}. \tag{3.17}$$

Proof. From the problems (3.1)(a)–(b) we can get the fractional integral system

$$\begin{cases} x(t) = \dfrac{\eta}{\Gamma(\alpha)}t^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-\tau)^{\alpha-1}f(\tau,x(\tau),y(\tau))\mathrm{d}\tau, \\ y(t) = \dfrac{\zeta}{\Gamma(\beta)}t^{\beta-1} + \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-\tau)^{\beta-1}g(\tau,x(\tau),y(\tau))\mathrm{d}\tau, \end{cases} \tag{3.18}$$

which is equivalent to the initial value problems (3.1)(a)–(b) (cf. [1, pp. 145–146]).

We derive from (H_1) and (3.18) that

$$\begin{cases} u(t) \le \dfrac{|\eta|}{\Gamma(\alpha)} + \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}\left[\displaystyle\int_0^t (t-\tau)^{\alpha-1}\tau^{k_{11}(\alpha-1)}\varphi_{11}(\tau)u^{k_{11}}(\tau)\mathrm{d}\tau \right. \\ \qquad \left. + \displaystyle\int_0^t (t-\tau)^{\alpha-1}\tau^{k_{12}(\beta-1)}\varphi_{12}(\tau)v^{k_{12}}(\tau)\mathrm{d}\tau\right], \\ v(t) \le \dfrac{|\zeta|}{\Gamma(\beta)} + \dfrac{t^{1-\beta}}{\Gamma(\beta)}\left[\displaystyle\int_0^t (t-\tau)^{\beta-1}\tau^{k_{21}(\alpha-1)}\varphi_{21}(\tau)u^{k_{21}}(\tau)\mathrm{d}\tau \right. \\ \qquad \left. + \displaystyle\int_0^t (t-\tau)^{\beta-1}\tau^{k_{22}(\beta-1)}\varphi_{22}(\tau)v^{k_{22}}(\tau)\mathrm{d}\tau\right], \end{cases} \tag{3.19}$$

where

$$u(t)=|x(t)|t^{1-\alpha},\qquad v(t)=|y(t)|t^{1-\beta}. \tag{3.20}$$

Using Hölder's inequality with the index $p=\frac{1+2\alpha}{1+\alpha}$ for $\frac{1}{p}+\frac{1}{q}=1$ in (3.19) we get

$$\begin{cases} u(t)\le \dfrac{|\eta|}{\Gamma(\alpha)}+\dfrac{t^{1-\alpha}}{\Gamma(\alpha)}\left[\left(\displaystyle\int_0^t (t-\tau)^{p(\alpha-1)}\tau^{pk_{11}(\alpha-1)}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\displaystyle\int_0^t \varphi_{11}^q(\tau)u^{qk_{11}}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\ \qquad\left.+\left(\displaystyle\int_0^t (t-\tau)^{p(\alpha-1)}\tau^{pk_{12}(\beta-1)}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\displaystyle\int_0^t \varphi_{12}^q(\tau)v^{qk_{12}}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right],\\ v(t)\le \dfrac{|\zeta|}{\Gamma(\beta)}+\dfrac{t^{1-\beta}}{\Gamma(\beta)}\left[\left(\displaystyle\int_0^t (t-\tau)^{p(\beta-1)}\tau^{pk_{21}(\alpha-1)}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\displaystyle\int_0^t \varphi_{21}^q(\tau)u^{qk_{21}}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right.\\ \qquad\left.+\left(\displaystyle\int_0^t (t-\tau)^{p(\beta-1)}\tau^{pk_{22}(\beta-1)}\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\displaystyle\int_0^t \varphi_{22}^q(\tau)v^{qk_{22}}(\tau)\mathrm{d}\tau\right)^{\frac{1}{q}}\right]. \end{cases} \tag{3.21}$$

Taking both sides of (3.21) the power q and using the elementary inequality $(a+b+c)^l\le 3^{l-1}(a^l+b^l+c^l)$ ($l\ge 1$, a, b and c are nonnegative constants), we get

$$\begin{cases} u^q(t)\le 3^{q-1}\left[\dfrac{|\eta|^q}{\Gamma^q(\alpha)}+\dfrac{t^{q(1-\alpha)}}{\Gamma^q(\alpha)}\left(\displaystyle\int_0^t (t-\tau)^{p(\alpha-1)}\tau^{pk_{11}(\alpha-1)}\mathrm{d}\tau\right)^{\frac{q}{p}}\left(\displaystyle\int_0^t \varphi_{11}^q(\tau)u^{qk_{11}}(\tau)\mathrm{d}\tau\right)\right.\\ \qquad\left.+\dfrac{t^{q(1-\alpha)}}{\Gamma^q(\alpha)}\left(\displaystyle\int_0^t (t-\tau)^{p(\alpha-1)}\tau^{pk_{12}(\beta-1)}\mathrm{d}\tau\right)^{\frac{q}{p}}\left(\displaystyle\int_0^t \varphi_{12}^q(\tau)v^{qk_{12}}(\tau)\mathrm{d}\tau\right)\right],\\ v^q(t)\le 3^{q-1}\left[\dfrac{|\zeta|^q}{\Gamma^q(\beta)}+\dfrac{t^{q(1-\beta)}}{\Gamma^q(\beta)}\left(\displaystyle\int_0^t (t-\tau)^{p(\beta-1)}\tau^{pk_{21}(\alpha-1)}\mathrm{d}\tau\right)^{\frac{q}{p}}\left(\displaystyle\int_0^t \varphi_{21}^q(\tau)u^{qk_{21}}(\tau)\mathrm{d}\tau\right)\right.\\ \qquad\left.+\dfrac{t^{q(1-\beta)}}{\Gamma^q(\beta)}\left(\displaystyle\int_0^t (t-\tau)^{p(\beta-1)}\tau^{pk_{22}(\beta-1)}\mathrm{d}\tau\right)^{\frac{q}{p}}\left(\displaystyle\int_0^t \varphi_{22}^q(\tau)v^{qk_{22}}(\tau)\mathrm{d}\tau\right)\right]. \end{cases} \tag{3.22}$$

By Lemma 2.6, (3.22) can be rewritten as

$$\begin{cases} u^q(t)\le 3^{q-1}\left[\dfrac{|\eta|^q}{\Gamma^q(\alpha)}+E_{11}(t)\displaystyle\int_0^t \varphi_{11}^q(\tau)u^{qk_{11}}(\tau)\mathrm{d}\tau+E_{12}(t)\displaystyle\int_0^t \varphi_{12}^q(\tau)v^{qk_{12}}(\tau)\mathrm{d}\tau\right],\\ v^q(t)\le 3^{q-1}\left[\dfrac{|\zeta|^q}{\Gamma^q(\beta)}+E_{21}(t)\displaystyle\int_0^t \varphi_{21}^q(\tau)u^{qk_{21}}(\tau)\mathrm{d}\tau+E_{22}(t)\displaystyle\int_0^t \varphi_{22}^q(\tau)v^{qk_{22}}(\tau)\mathrm{d}\tau\right], \end{cases} \tag{3.23}$$

where $E_{ij}(t)$ and θ_i ($i,j=1,2$) are defined as in (3.10)–(3.13) and (3.14)–(3.17), respectively.

Since

$$p(\alpha-1)+1=\frac{1+2\alpha}{1+\alpha}(\alpha-1)+1=\frac{2\alpha^2}{1+\alpha}>0, \tag{3.24}$$

$$pk_{11}(\alpha-1)+1\ge p(\alpha-1)+1=\frac{2\alpha^2}{1+\alpha}>0, \tag{3.25}$$

and

$$\theta_1=qk_{11}(\alpha-1)+\frac{q}{p}\ge q(\alpha-1)+\frac{q}{p}=\alpha q-1=2\alpha>0; \tag{3.26}$$

$$pk_{12}(\beta-1)+1\ge p(\beta-1)+1\ge p(\alpha-1)+1=\frac{2\alpha^2}{1+\alpha}>0 \tag{3.27}$$

and

$$\theta_2=qk_{12}(\beta-1)+\frac{q}{p}\ge q(\alpha-1)+\frac{q}{p}=2\alpha>0, \tag{3.28}$$

it follows that

$$0 < B[pk_{11}(\alpha-1)+1, p(\alpha-1)+1] < +\infty$$
$$0 < B[pk_{12}(\beta-1)+1, p(\alpha-1)+1] < +\infty$$

and the functions $E_{1j}(t)$ $(j = 1, 2)$ are nondecreasing on I. By a similar argument as in (3.24)–(3.28), we can conclude that

$$0 < B[pk_{21}(\alpha-1)+1, p(\beta-1)+1] < +\infty$$
$$0 < B[pk_{22}(\beta-1)+1, p(\beta-1)+1] < \infty$$

and the functions $E_{2j}(t)$ $(j = 1, 2)$ are also nondecreasing on I.

Now an application of Theorem 2.3 to (3.23) combined with (3.20) yields the desired bounds on (3.2). □

The next theorem deals with the uniqueness of the solutions of systems (3.1)(a)–(b).

Theorem 3.2. *If f and $g \in C(I \times R^2, R)$ and they satisfy the Lipschitz type condition* (H_2)

$$\begin{cases} |f(t,u_2,v_2) - f(t,u_1,v_1)| \le \phi_{11}(t)|u_2-u_1| + \phi_{12}(t)|v_2-v_1|, \\ |g(t,u_2,v_2) - g(t,u_1,v_1)| \le \phi_{21}(t)|u_2-u_1| + \phi_{22}(t)|v_2-v_1|, \end{cases}$$

where $\phi_{ij} \in C(I, R_+)$, then the systems (3.1) (a)–(b) have at most one solution.

Proof. Let $\varepsilon > 0$ be an arbitrary small real number and take $p = \frac{1+2\alpha}{1+\alpha}$ for $\frac{1}{p} + \frac{1}{q} = 1$. If systems (3.1)(a)–(b) have two solutions $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$, then we have

$$\begin{cases} x_i(t) = \dfrac{\eta}{\Gamma(\alpha)} t^{\alpha-1} + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} f(\tau, x_i(\tau), y_i(\tau)) \mathrm{d}\tau, \\ y_i(t) = \dfrac{\zeta}{\Gamma(\beta)} t^{\beta-1} + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-\tau)^{\beta-1} g(\tau, x_i(\tau), y_i(\tau)) \mathrm{d}\tau. \end{cases} \quad i = 1, 2. \tag{3.29}$$

From (3.29) and (H_2) we derive that

$$\begin{cases} x(t) \le \varepsilon + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} [\phi_{11}(\tau)x(\tau) + \phi_{12}(\tau)y(\tau)] \mathrm{d}\tau, \\ y(t) \le \varepsilon + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (t-\tau)^{\beta-1} [\phi_{21}(\tau)x(\tau) + \phi_{22}(\tau)y(\tau)] \mathrm{d}\tau, \end{cases} \tag{3.30}$$

where $x(t) = |x_2(t) - x_1(t)|$, $y(t) = |y_2(t) - y_1(t)|$. Taking similar procedures from (3.19)–(3.30) we obtain

$$\begin{cases} x^q(t) \le 2^{q-1}\varepsilon^q + \dfrac{2^{2q-2}}{\Gamma^q(\alpha)} B^{\frac{q}{p}}[1, \bar{\theta}_1] t^{\bar{\theta}_1 \frac{q}{p}} \displaystyle\int_0^t [\phi_{11}^q(\tau)x^q(\tau) + \phi_{12}^q(\tau)y^q(\tau)] \mathrm{d}\tau, \\ y^q(t) \le 2^{q-1}\varepsilon^q + \dfrac{2^{2q-2}}{\Gamma^q(\beta)} B^{\frac{q}{p}}[1, \bar{\theta}_2] t^{\bar{\theta}_2 \frac{q}{p}} \displaystyle\int_0^t [\phi_{21}^q(\tau)x^q(\tau) + \phi_{22}^q(\tau)y^q(\tau)] \mathrm{d}\tau, \end{cases} \tag{3.31}$$

where $\bar{\theta}_1 = p(\alpha-1)+1$ and $\bar{\theta}_2 = p(\beta-1)+1$.

Since

$$\bar{\theta}_1 = p(\alpha-1)+1 = \frac{1+2\alpha}{1+\alpha}(\alpha-1)+1 = \frac{2\alpha^2}{1+\alpha} > 0$$

and

$$\bar{\theta}_2 = p(\beta-1)+1 \ge p(\alpha-1)+1 = \bar{\theta}_1 > 0,$$

it follows that

$$0 < B[1, \bar{\theta}_1], \qquad B[1, \bar{\theta}_2] < +\infty$$

and the power functions $t^{\bar{\theta}_1 \frac{q}{p}}$ and $t^{\bar{\theta}_1 \frac{q}{p}}$ are nondecreasing.

Now using Lemma 2.2 to (3.31) it follows that

$$\begin{cases} x^q(t) \le 2^{q-1}\varepsilon^q \left[1 + E_1(t)\int_0^t \phi_{12}^q(t,s)\Psi_2(s)\mathrm{d}s\right] \exp\left\{E_1(t)\left[\int_0^t \phi_{11}^q(s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \int_0^t \phi_{12}^q(s)\Psi_2(s)\left(E_2(s)\int_0^s \phi_{21}^q(\tau)\Psi_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\}, \\ y^q(t) \le 2^{q-1}\varepsilon^q \left[1 + E_2(t)\int_0^t \phi_{21}^q(t,s)\Psi_1(s)\mathrm{d}s\right] \exp\left\{E_2(t)\left[\int_0^t \phi_{22}^q(s)\mathrm{d}s \right.\right. \\ \qquad \left.\left. + \int_0^t \phi_{21}^q(s)\Psi_1(s)\left(E_1(s)\int_0^s \phi_{12}^q(\tau)\Psi_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\}, \end{cases} \tag{3.32}$$

for $t \in I$, where $E_i(t) = \frac{2^{2q-2}}{\Gamma^q(\alpha)} B^{\frac{q}{p}}[1, \bar{\theta}_i] t^{\bar{\theta}_i \frac{q}{p}}$ and $\Psi_i(t) = \int_0^t E_i(t)\phi_{ii}^q(s)\mathrm{d}s,\ i = 1, 2$.

Letting $\varepsilon \to 0$, we derive from (3.32) that $x^q(t) = y^q(t) = 0$ and hence $x_1(t) = x_2(t), y_1(t) = y_2(t)$, which proves the uniqueness of the solutions of systems (3.1)(a)–(b). □

The following theorem investigates the continuous dependence of the solutions of systems (3.1)(a)–(b) on the initial value and the functions f and g. For this we consider the following variation system of (3.1)(a)–(b):

$$\begin{cases} D^\alpha x(t) = \bar{f}(t, x(t), y(t)) & \text{(a)} \\ D^\beta y(t) = \bar{g}(t, x(t), y(t)) & \\ D^{\alpha-1}x(t)|_{t=0} = \bar{\eta}, \qquad D^{\beta-1}y(t)|_{t=0} = \bar{\zeta} & \text{(b)} \end{cases} \tag{$\overline{3.1}$}$$

for $t \in I$, where $\bar{f}$ and $\bar{g} \in C(I \times R^2)$, $\bar{\eta}, \bar{\zeta} \in R$.

Theorem 3.3. *Assume that the hypotheses in Theorem 3.2 hold. Suppose that*

$$\begin{cases} |\eta - \bar{\eta}| + |\zeta - \bar{\zeta}| + \le \varepsilon \\ |f(t, \bar{x}, \bar{y}) - \bar{f}(t, \bar{x}, \bar{y})| + |g(t, \bar{x}, \bar{y}) - \bar{g}(t, \bar{x}, \bar{y})| \le \varepsilon \end{cases} \tag{3.33}$$

for $t \in I$ and $\bar{x}, \bar{y} \in R$, then

$$\begin{cases} |x(t) - \bar{x}(t)| \le 3^{1-\frac{1}{q}}\varepsilon t^{\alpha-1}\left[\left(\frac{\alpha+t}{\alpha\Gamma(\alpha)}\right)^q + \left(\frac{\beta+t}{\beta\Gamma(\beta)}\right)^q \int_0^t \widetilde{\phi}_{12}(s)\widetilde{\Psi}_2(s)\mathrm{d}s\right]^{1/q} \\ \qquad \times \exp\left\{\frac{1}{q}\left[\int_0^t \widetilde{\phi}_{11}(s)\mathrm{d}s + \int_0^t \widetilde{\phi}_{12}(s)\widetilde{\Psi}_2(s)\left(\int_0^s \widetilde{\phi}_{21}(\tau)\widetilde{\Psi}_1(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\}, \\ |y(t) - \bar{y}(t)| \le 3^{1-\frac{1}{q}}\varepsilon t^{\beta-1}\left[\left(\frac{\beta+t}{\beta\Gamma(\beta)}\right)^q + \left(\frac{\alpha+t}{\alpha\Gamma(\alpha)}\right)^q \int_0^t \widetilde{\phi}_{21}(s)\widetilde{\Psi}_1(s)\mathrm{d}s\right]^{1/q} \\ \qquad \times \exp\left\{\frac{1}{q}\left[\int_0^t \widetilde{\phi}_{22}(s)\mathrm{d}s + \int_0^t \widetilde{\phi}_{21}(s)\widetilde{\Psi}_1(s)\left(\int_0^s \widetilde{\phi}_{12}(\tau)\widetilde{\Psi}_2(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right]\right\} \end{cases} \tag{3.34}$$

for $t > 0$, where $\widetilde{\phi}_{ij}(t,s) = 3^{q-1}\widetilde{E}_{ij}(t)\phi_{ij}^q(s)$, $\widetilde{\Psi}_i(t) = \int_0^t \widetilde{\phi}_{ii}(\tau)\mathrm{d}\tau,\ i, j = 1, 2,$

$$\widetilde{E}_{11}(t) = \frac{1}{\Gamma^q(\alpha)} t^{\widetilde{\theta}_1} B^{q/p}[p(\alpha-1)+1, p(\alpha-1)+1], \tag{3.35}$$

$$\widetilde{E}_{12}(t) = \frac{1}{\Gamma^q(\alpha)} t^{\widetilde{\theta}_2} B^{q/p}[p(\beta-1)+1, p(\alpha-1)+1], \tag{3.36}$$

$$\widetilde{E}_{21}(t) = \frac{1}{\Gamma^q(\beta)} t^{\widetilde{\theta}_1} B^{q/p}[p(\alpha-1)+1, p(\beta-1)+1], \tag{3.37}$$

$$\widetilde{E}_{22}(t) = \frac{1}{\Gamma^q(\beta)} t^{\widetilde{\theta}_2} B^{q/p}[p(\beta-1)+1, p(\beta-1)+1], \tag{3.38}$$

$$\widetilde{\theta}_1 = q(\alpha-1) + \frac{q}{p}, \tag{3.39}$$

and

$$\widetilde{\theta}_2 = q(\beta-1) + \frac{q}{p}. \tag{3.40}$$

Proof. Let $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ be the solutions of (3.1)(a)–(b) and $(\overline{3.1})$(a)–(b), respectively. Then $(x(t), y(t))$ satisfies (3.1)(a)–(b), $(\bar{x}(t), \bar{y}(t))$ satisfies $(\overline{3.1})$(a)–(b). Hence

$$\begin{aligned} |x(t)-\bar{x}(t)| &\le \frac{|\eta-\zeta|}{\Gamma(\alpha)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,x(\tau),y(\tau)) - f(\tau,\bar{x}(\tau),\bar{y}(\tau))|\mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}|f(\tau,\bar{x}(\tau),\bar{y}(\tau)) - \bar{f}(\tau,\bar{x}(\tau),\bar{y}(\tau))|\mathrm{d}\tau \\ &\le \varepsilon\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{t^\alpha}{\alpha\Gamma(\alpha)}\right) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}[\phi_{11}(\tau)|x(\tau)-\bar{x}(\tau)| \\ &\quad + \phi_{12}(\tau)|y(\tau)-\bar{y}(\tau)|]\mathrm{d}\tau \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} |y(t)-\bar{y}(t)| &\le \frac{|\eta-\zeta|}{\Gamma(\beta)}t^{\beta-1} + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}|g(\tau,x(\tau),y(\tau)) - g(\tau,\bar{x}(\tau),\bar{y}(\tau))|\mathrm{d}\tau \\ &\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}|g(\tau,\bar{x}(\tau),\bar{y}(\tau)) - \bar{g}(\tau,\bar{x}(\tau),\bar{y}(\tau))|\mathrm{d}\tau \\ &\le \varepsilon\left(\frac{t^{\beta-1}}{\Gamma(\beta)} + \frac{t^\beta}{\alpha\Gamma(\beta)}\right) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}[\phi_{21}(\tau)|x(\tau)-\bar{x}(\tau)| \\ &\quad + \phi_{22}(\tau)|y(\tau)-\bar{y}(\tau)|]\mathrm{d}\tau \end{aligned} \tag{3.42}$$

by (H_2) and the assumption (3.33).

Rewriting (3.41) and (3.42) as

$$\begin{cases} \bar{u}(t) \le \varepsilon\left(\frac{1}{\Gamma(\alpha)} + \frac{t}{\alpha\Gamma(\alpha)}\right) + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\left[\int_0^t (t-\tau)^{\alpha-1}\tau^{\alpha-1}\phi_{11}(\tau)\bar{u}(\tau) + \int_0^t (t-\tau)^{\alpha-1}\tau^{\beta-1}\phi_{12}(\tau)\bar{v}(\tau)\mathrm{d}\tau\right], \\ \bar{v}(t) \le \varepsilon\left(\frac{1}{\Gamma(\beta)} + \frac{t}{\alpha\Gamma(\beta)}\right) + \frac{t^{1-\beta}}{\Gamma(\beta)}\left[\int_0^t (t-\tau)^{\beta-1}\tau^{\alpha-1}\phi_{21}(\tau)\bar{u}(\tau) + \int_0^t (t-\tau)^{\beta-1}\tau^{\beta-1}\phi_{22}(\tau)\bar{v}(\tau)\right]\mathrm{d}\tau, \end{cases} \tag{3.43}$$

where

$$\bar{u}(t) = |x(t)-\bar{x}(t)|t^{1-\alpha}, \qquad \bar{v}(t) = |y(t)-\bar{y}(t)|t^{1-\beta} \tag{3.44}$$

and then taking similar procedures from (3.19)–(3.23), we can obtain

$$\begin{cases} \bar{u}^q(t) \le 3^{q-1}\left[\varepsilon^q\left(\frac{\alpha+t}{\alpha\Gamma(\alpha)}\right)^q + \widetilde{E}_{11}(t)\int_0^t \phi_{11}^q(\tau)\bar{u}^q(\tau)\mathrm{d}\tau + \widetilde{E}_{12}(t)\int_0^t \phi_{12}^q(\tau)\bar{v}^q(\tau)\mathrm{d}\tau\right], \\ \bar{v}^q(t) \le 3^{q-1}\left[\varepsilon^q\left(\frac{\beta+t}{\beta\Gamma(\beta)}\right)^q + \widetilde{E}_{21}(t)\int_0^t \phi_{21}^q(\tau)\bar{u}^q(\tau)\mathrm{d}\tau + \widetilde{E}_{22}(t)\int_0^t \phi_{22}^q(\tau)\bar{v}^q(\tau)\mathrm{d}\tau\right], \end{cases} \tag{3.45}$$

for $t \in I$, where $\widetilde{E}_{ij}(t)$ $(i, j = 1, 2)$ is defined as in (3.35)–(3.40). Now a suitable application of Lemma 2.2 combined with (3.44) yields the desired estimate in (3.34). □

Remark 3.1. Evidently, from estimate (3.34), for any closed interval $J = [\delta, K] \subseteq I(\forall\delta > 0)$, the solutions $(x(t), y(t))$ of the systems (3.1)(a)–(b) depend continuously on the initial value and f, g.

Remark 3.2. If $\alpha > \beta$, by taking $p = \frac{1+2\beta}{1+\beta}$, we can get some similar results as in Theorems 3.1–3.3. To save space, the details are omitted here.

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