View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

ELSEVIER
Discrete Mathematics 239 (2001) 1–12
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

On majority domination in graphs

Tara S. Holm

Department of Mathematics, Massachusetts Institute of Technology, 77, Massachusetts Avenue, 2-251, Cambridge, MA 02139, USA

Received 8 October 1996; revised 4 May 2000; accepted 10 July 2000

Abstract

A majority dominating function on the vertex set of a graph $G=(V,E)$ is a function $g: V \to \{1,-1\}$ such that $g(N[v]) \geqslant 1$ for at least half of the vertices v in V. The weight of a majority dominating function is denoted as $g(V)$ and is $\sum g(v)$ over all v in V. The majority domination number of a graph is the minimum possible weight of a majority dominating function, and is denoted as $\gamma_{\mathrm{maj}}(G)$. We determine the majority domination numbers of certain families of graphs. Moreover, we show that the decision problem corresponding to computing the majority domination number of an arbitrary disjoint union of complete graphs is NP-complete.
© 2001 Elsevier Science B.V. All rights reserved.

MSC: primary: 05C35; secondary: 05C85; 05C70

Keywords: Majority domination; NP-completeness; External problems; Graph algorithms

1. Introduction

Let $G=(V,E)$ be a finite graph and v be a vertex in V. The *open neighborhood* of v is defined to be the set of vertices adjacent to v in G, and is denoted as $N(v)$. Further, the *closed neighborhood* of v is defined by $N[v]=N(v) \cup \{v\}$. The closed neighborhood of a set of vertices S is denoted as $N[S]$ and is $\cup N[s]$ over all vertices s in S. Given a set S and a function g on S, we will use $g(S)$ to denote $\sum g(s)$ over all elements s in S. A dominating set of a graph is a set $S \subseteq V$ such that $N[S]=V$. The *domination number* of a graph G is the minimum size of a dominating set. We can express this concept in terms of a dominating function $g: V \to \{0,1\}$. If $g(N[v]) \geqslant 1$ for all v in V, we call g a *dominating function*. Notice that the set S of elements with the property $g(v)=1$ is a dominating set in this circumstance. We can equivalently define the domination number of G to be $\gamma(G)=\min\{g(V): g$ is a dominating function on $V.\}$

E-mail address: tsh@math.mit.edu (T.S. Holm).

0012-365X/01/$ - see front matter © 2001 Elsevier Science B.V. All rights reserved.
PII: S0012-365X(00)00370-8

The concept of domination in graphs can be extended. Consider a function $g: V \to \{1, -1\}$. We say that g is a *signed dominating function* on V if for each v in V, $g(N[v]) \geqslant 1$. The *total weight* of such a function is $g(V)$, and the *signed domination number* of a graph G, denoted $\gamma_s(G)$, is the minimum total weight of such a function. The signed domination number has been determined for many graphs [4,2].

A *majority dominating function* is defined in [1,2] as a function $g: V \to \{1, -1\}$ with the property that $g(N[v]) \geqslant 1$ for at least half of all vertices v in V. Note that every signed dominating function is also a majority dominating function. The *majority domination number* of a graph G, denoted $\gamma_{\text{maj}}(G)$, is the minimum total weight of such a function; that is, $\gamma_{\text{maj}}(G) = \min\{g(V) : g \text{ is majority dominating on } V\}$.

2. Complete multipartite graphs

Broere et al. [1] determined the majority domination numbers of many families of graphs. In particular, they gave the majority domination number of a general complete bipartite graph.

Theorem 1 (Broere et al. [1]). *Given two integers $n \geqslant m \geqslant 2$,*

$$\gamma_{\text{maj}}(K_{m,n}) = \begin{cases} 2 - n & \text{for } m \text{ even}, \\ 3 - n & \text{for } m \text{ odd}. \end{cases}$$

Using a different counting technique, we greatly simplify their proof. In addition, our technique is useful in determining the majority domination numbers of many other graphs. We will use $\bar{G}$ to denote the graph theoretic complement of G. Recall that the *join* $G_1 + G_2$ of two disjoint graphs G_1 and G_2 is the graph obtained from the union of G_1 and G_2 by adding all edges with one vertex in G_1 and the other in G_2.

Theorem 2. *Suppose $n \geqslant m \geqslant 2$. If G is a graph of order m and $H = \bar{K}_n + G$, then*

$$\gamma_{\text{maj}}(H) = \begin{cases} 2 - n & \text{for } m \text{ even}, \\ 3 - n & \text{for } m \text{ odd}. \end{cases}$$

Proof. Suppose first that $n > m$. Let V_n and V_m denote the vertices of $\bar{K}_n$ and G, respectively. Suppose g is a majority dominating function on $K_{m,n}$ such that $g(V) = \gamma_{\text{maj}}(H)$. Let v be an element of the V_n such that $g(N[v]) \geqslant 1$. Such a v exists as a result of the pigeonhole principle if $n > m$. Since $g(N[v]) \geqslant 1$, we get

$$g(V) = g(N[v]) + g(V_n \backslash \{v\}) \geqslant 1 - (n-1) = 2 - n.$$

Note that since $g(V)$ must have the same parity as $n + m$, when m is odd we necessarily have $g(V) \geqslant 3 - n$. This type of parity argument is often useful, but details will henceforth be omitted. Thus, $\gamma_{\text{maj}}(H)$ is at least the desired amount.

Next, consider the case that $n = m$. If there is an element v of V_n such that $g(N[v]) \geqslant 1$, then the proof is finished, by the above argument. Otherwise, a similar argument applies. In this case, we must have an element $v \in V_m$ with $g(N[v]) \geqslant 1$. Furthermore, $g(V - N[v]) \geqslant -(m-1)$, so

$$\gamma_{\text{maj}}(H) = g(V) = g(N[v]) + g(V - N[v]) \geqslant 1 - (m-1) = 2 - m = 2 - n,$$

proving that $\gamma_{\text{maj}}(H)$ is at least the above amount.

Finally, consider the function g defined by

$$g(v) = \begin{cases} 1 & \text{for } \lceil (m+2)/2 \rceil \text{ vertices } v \text{ in } V_m, \\ -1 & \text{otherwise.} \end{cases}$$

This is a majority dominating function on H and $g(V)$ is the desired value. Hence, $\gamma_{\text{maj}}(H)$ is at most the desired amount. The result now follows. □

Theorem 1 now follows as a corollary of this theorem. We will now state an additional corollary. We will use $K_{i^{(j)}}$ to denote $K_{\underbrace{i,\, i,\, \ldots,\, i}_{j}}$.

Corollary 1. *For integers $n \geqslant m \geqslant 2$,*

$$\gamma_{\text{maj}}(K_{1^{(m)},n}) = \begin{cases} 2 - n & \text{for } m \text{ even}, \\ 3 - n & \text{for } m \text{ odd}. \end{cases}$$

Proof. Notice that $K_{1^{(m)},n} = \bar{K}_n + K_{1^{(m)}}$. □

We can also determine the majority domination number of this graph when $n < m$.

Theorem 3. *For integers $m > n \geqslant 1$,*

$$\gamma_{\text{maj}}(K_{1^{(m)},n}) = \begin{cases} 1 & \text{for } n + m \text{ odd}, \\ 2 & \text{for } n + m \text{ even}. \end{cases}$$

Proof. Let G, V, V_n, and V_m be defined as in the previous proof. Define a function g on V by

$$g(v) = \begin{cases} 1 & \text{for } \lceil (m+n+1)/2 \rceil \text{ of the } v \text{ in } V, \\ -1 & \text{otherwise.} \end{cases}$$

This is a majority dominating function on V and $g(V)$ is the desired value.

Now let g be a majority dominating function on V such that $g(V) = \gamma_{\text{maj}}(G)$. Since $m > n$, by the pigeonhole principle there is some v in V_m such that $g(N[v]) \geqslant 1$. But $g(V) = g(N[v]) \geqslant 1$. Note that when $m + n$ is even, this implies that $g(N[v]) \geqslant 2$. The result now follows. □

Next, we consider the complete multipartite graph with m parts, each of order n.

Theorem 4. *For integers $m \geqslant 2$ and $n \geqslant 3$,*

$$\gamma_{\mathrm{maj}}(K_{n^{(m)}}) = \begin{cases} 2-n & \text{for } m \text{ even, } n \text{ even,} \\ 3-n & \text{for } m \text{ even, } n \text{ odd,} \\ 4-n & \text{for } m \text{ odd.} \end{cases}$$

Proof. We let $G = K_{n^{(m)}}$, V be the vertex set of G, and $V_1, \dots, V_m$ be the n-element partite classes of V. First we consider the case where m is even. We define a function g on V by

$$g(v) = \begin{cases} 1 & \text{for all } v \text{ lying in } V_1 \cup \cdots \cup V_{m/2-1}, \\ 1 & \text{for } \lceil n/2 \rceil + 1 \text{ of } v \text{ in } V_{m/2}, \\ -1 & \text{otherwise.} \end{cases}$$

Clearly, g is a majority dominating function, since $g(v) \geqslant 1$ for all $v \in V_i$ when $i > m/2$. Further $\gamma_{\mathrm{maj}}(G) \leqslant g(V)$, which is the proposed bound.

Next let g be a majority dominating function on V such that $g(V) = \gamma_{\mathrm{maj}}(G)$. There must be some v in V_i such that $g(N[v]) \geqslant 1$. Using the same calculation as usual,

$$\gamma_{\mathrm{maj}}(G) = g(V) = g(N[v]) + g(V_i \backslash \{v\}) \geqslant 1 - (n-1) = 2 - n.$$

Note that we did not use the parity of m in this argument, so this inequality holds for all cases. If n is odd, then $g(V) \geqslant 3 - n$ by the usual parity argument. The result now follows for m even.

Now we consider the case where m is odd. We construct a function g on V defined by

$$g(v) = \begin{cases} 1 & \text{for all } v \text{ lying in } V_1 \cup \cdots \cup V_{(m-1)/2}, \\ 1 & \text{for one } v \text{ in each of } V_{(m+1)/2} \text{ and } V_{(m+1)/2} + 1, \\ -1 & \text{otherwise.} \end{cases}$$

Clearly this is a majority dominating function, since $g(v) \geqslant 1$ for all $v \in V_i$ when $i \geqslant (m+1)/2$. Thus $\gamma_{\mathrm{maj}}(G) \leqslant g(V) = 4 - n$.

Finally, we must show that $\gamma_{\mathrm{maj}}(G) \geqslant 4 - n$ when m is odd. We already know that $\gamma_{\mathrm{maj}}(G) \geqslant 2 - n$. Choose a function g on V such that g is a majority dominating function and $g(V) = \gamma_{\mathrm{maj}}(G)$. Let α_i denote the number of vertices v in V_i such that $g(v) = 1$. If necessary, relabel the partite classes so that $\alpha_1 \geqslant \cdots \geqslant \alpha_m \geqslant 0$.

Suppose α_i is 0 or n for every i. Then $g(V) = kn$ for some integer k. Observe that $k \geqslant 1$; otherwise, g is not a majority dominating function. So in this case, $g(V) = kn \geqslant 4 - n$, and the result follows. If this supposition is false, then there is at least one j such that $0 < \alpha_j < n$. Choose i to be the greatest such j. Observe that $i \geqslant (m+1)/2$; otherwise, $g(N[v]) \leqslant 0$ for all v in V, contradicting that g is a majority dominating function. Notice also that there must be a vertex v in V_j, where $j \leqslant i$ such that $g(N[v]) \geqslant 1$. In this case, we get

$$g(V) = g(N[v]) + g(V_j \backslash \{v\}) \geqslant 1 + 2\alpha_j - n - 1 \geqslant 2\alpha_i - n.$$

We also have

$$g(V) = g(V \backslash V_i) + g(V_i) = g(V \backslash V_i) + 2\alpha_i - n.$$

So if $\alpha_i > 1$ or if $g(V \backslash V_i) > 0$, the desired inequality $g(V) \geqslant 4 - n$ immediately follows. So we will consider the case when $\alpha_i = 1$ and $g(V \backslash V_i) = 0$. In this case, $g(N[v]) \geqslant 1$ if and only if $v \in V_j$ with $j > i$, or $v \in V_j$ with $j \leqslant i$, $\alpha_j = 1$, and $g(v) = 1$. We can calculate the maximum possible number s of vertices with positive weight, in terms of i, to be $s = (m - i)n + (2i - m + 1)/2$. We can then use the fact that $i \geqslant (m+1)/2$ to get $s \leqslant (mn - n + 2)/2 < mn/2$ since $n \geqslant 3$, contradicting the fact that g is a majority dominating function. Note that this estimation relies on the fact that $n \geqslant 3$. So finally, the inequality $g(V) \geqslant 4 - n$ follows, concluding the proof. □

The first half of the proof above also works for m even and $n = 2$. We handle the remaining problem of m odd and $n = 2$ now.

Theorem 5. $\gamma_{\mathrm{maj}}(K_{2^{(m)}}) = 0$.

Proof. Suppose g is a majority dominating function on $G = K_{2^{(m)}}$ such that $g(V) = \gamma_{\mathrm{maj}}(G)$. Consider a vertex v in V such that $g(N[v]) \geqslant 1$. Denote the other element of the partite class of which v is a member by u. Then

$$g(V) = g(N[v]) + g(u) \geqslant 1 - 1 = 0.$$

Thus, $\gamma_{\mathrm{maj}}(G) \geqslant 0$.

Further, define a function g by

$$g(v) = \begin{cases} 1 & \text{for exactly one vertex in each partite class,} \\ -1 & \text{otherwise.} \end{cases}$$

Such a function is a majority dominating function on V, so $\gamma_{\mathrm{maj}}(G) \leqslant g(V) = 0$. The result follows. □

3. Disjoint unions of complete graphs

Broere et al. prove the following Proposition.

Proposition 1 (Broere et al. [1]). *Given two integers $n \geqslant m \geqslant 1$,*

$$\gamma_{\mathrm{maj}}(K_m \cup K_n) = \begin{cases} 1 - m & \text{for } n \text{ odd}, \\ 2 - m & \text{for } n \text{ even}. \end{cases}$$

As in the previous section, we use a different counting technique to generalize this.

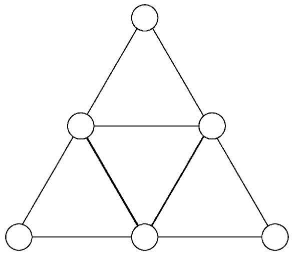

Fig. 1. The Hajös graph.

Theorem 6. *Suppose that* $n > m \geqslant 1$. *If* G *is a graph of order* m, *and* $H = K_n \cup G$ *then*

$$\gamma_{\text{maj}}(H) = \begin{cases} 1 - m & \text{for } n \text{ odd}, \\ 2 - m & \text{for } n \text{ even}. \end{cases}$$

The same result is true when $n = m \geqslant 1$ *if and only if the signed domination of* G *satisfies* $\gamma_s(G) \geqslant 1$.

Proof. Suppose first that $n > m \geqslant 1$. Let V_n be the set of vertices of K_n and V_m be the set of vertices of G. Then $V = V_n \cup V_m$. We will assume that g is a majority dominating function with $g(V) = \gamma_{\text{maj}}(H)$. There must be a vertex $v \in V_n$ such that $g(N[v]) \geqslant 1$. But then $\gamma_{\text{maj}}(K_n \cup G) = g(V) = g(N[v]) + g(V_m) \geqslant 1 - m$. Then by a parity argument, $\gamma_{\text{maj}}(H)$ is at least the desired amount.

Next we will define a majority dominating function on H as follows:

$$g(v) = \begin{cases} 1 & \text{for } \lceil (n+1)/2 \rceil \text{ vertices } v \text{ in } V_n, \\ -1 & \text{otherwise.} \end{cases}$$

Clearly, g is a majority dominating function on H and $g(V)$ is the desired amount, showing that $\gamma_{\text{maj}}(H)$ is at most the desired amount, completing the first part of the proof.

Finally, suppose that $n = m$. Then a function g is a majority dominating function on H if and only if it is either a signed dominating function on G or a signed dominating function on K_n. This proves the second statement of the theorem. There are graphs that have $\gamma_s(G) < 1$, for example the Hajös graph H as given in Fig. 1.

In this case, $\gamma_s(H) = 0$, and so $\gamma_{\text{maj}}(K_6 \cup H) = -6$, not -4, as the theorem would predict. Thus, the theorem does not hold in these cases. $\square$

Proposition 1 now follows as a corollary. Some additional corollaries are stated below.

Corollary 2. *For integers $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m \geqslant 1$ and $m \geqslant 2$* such that $n_1 \geqslant n_2 + \cdots + n_m$,

$$\gamma_{\text{maj}}\left(\bigcup_{i=1}^{m} K_{n_i}\right) = \begin{cases} 1-(n_2+\cdots+n_m) & \textit{for } n_1 \textit{ odd}, \\ 2-(n_2+\cdots+n_m) & \textit{for } n_1 \textit{ even}. \end{cases}$$

Proof. This follows from Theorem 6 if we let $G = \bigcup_{i=2}^{m} K_{n_i}$. Notice that the theorem follows for $n_1 = n_2 + \cdots + n_m$ since $\gamma_s(\bigcup_{i=2}^{m} K_{n_i}) \geqslant 1$. $\square$

Now we consider the complement of $K_{1^{(m)},n}$, namely $\bar{K}_m \cup K_n$. As above, we have two different results depending on the relation between n and m. The first is a corollary of Theorem 6 and the second is an independent result.

Corollary 3. *For integers $n > m \geqslant 1$,*

$$\gamma_{\text{maj}}(\bar{K}_m \cup K_n) = \begin{cases} 1-m & \textit{for } n \textit{ odd}, \\ 2-m & \textit{for } n \textit{ even}. \end{cases}$$

Proof. This follows directly from Theorem 6 if we let $G = \bar{K}_m$. $\square$

Theorem 7. *For integers $m > n \geqslant 2$,*

$$\gamma_{\text{maj}}(\bar{K}_m \cup K_n) = \begin{cases} 1-n & \textit{for } m \textit{ and } n \textit{ odd}, \\ 2-n & \textit{for } m \textit{ even}, \\ 3-n & \textit{for } n \textit{ even and } m \textit{ odd}. \end{cases}$$

Proof. Let G, V, V_n and V_m be defined as in the previous proof. Define a function g on V by

$$g(v) = \begin{cases} 1 & \text{for } \lceil (n+1)/2 \rceil \text{ of the vertices } v \text{ in } V_n, \\ 1 & \text{for } \lceil (m-n)/2 \rceil \text{ of the vertices } v \text{ in } V_m, \\ -1 & \text{otherwise}. \end{cases}$$

It can be verified that g is a majority dominating function with

$$\gamma_{\text{maj}}(G) \leqslant g(V) = \begin{cases} 1-n & \text{for } m \text{ and } n \text{ odd}, \\ 2-n & \text{for } m \text{ even}, \\ 3-n & \text{for } n \text{even and } m \text{ odd}. \end{cases}$$

Now suppose that we have a majority dominating function g such that $g(V) = \gamma_{\text{maj}}(G)$. If there is an element v in V_n with the property that $g(N[v]) \geqslant 1$, then there must also be at least $\lceil (m+n)/2 \rceil - n$ vertices v in V_m with $g(v) = 1$. Then we get that

$$\gamma_{\text{maj}}(G) = g(V) \geqslant \begin{cases} 1-m+2\lceil (m-n)/2 \rceil & \text{for } n \text{ odd}, \\ 2-m+2\lceil (m-n)/2 \rceil & \text{for } n \text{ even}. \end{cases}$$

It is easy to check that this is the desired value, using the parity of m.

Finally, if there is no v in V_n such that $g(N[v]) \geqslant 1$, then there must be $\lceil (m+n)/2 \rceil$ vertices v in V_m with $g(v) = 1$. We get

$$g(V) = g(V_m) + g(V_n) \geqslant \left\lceil \frac{m+n}{2} \right\rceil - \left(m - \left\lceil \frac{m+n}{2} \right\rceil \right) - n \geqslant 2 \left\lceil \frac{m+n}{2} \right\rceil - m - n,$$

which is at least the desired value. The result now follows. □

Next we consider the complement of $K_{n^{(m)}}$, that is, $\bigcup_{i=1}^m K_n$. First, we need the following Lemma.

Lemma 1 (Broere et al. [1]). *A majority dominating function g on a graph G is minimal only if for every vertex $v \in V$ with $g(v) = 1$, there exists a vertex $u \in N[v]$ with $g(N[u]) \in \{1, 2\}$.*

Theorem 8. *For integers $m > 2$ and $n \geqslant 2$,*

$$\gamma_{\mathrm{maj}} \left(\bigcup_{i=1}^m K_n \right) = \begin{cases} \lceil m/2 \rceil - n \lfloor m/2 \rfloor & \text{for } n \text{ odd}, \\ 2 \lceil m/2 \rceil - n \lfloor m/2 \rfloor & \text{for } n \text{ even}. \end{cases}$$

Proof. Let G denote the graph in question, V the vertex set, and $V_1, \ldots, V_m$ the vertex sets of the complete subgraphs. First consider the function g on V, defined by

$$g(v) = \begin{cases} 1 & \text{for } \lceil (n+1)/2 \rceil \text{ of } v \text{ in } V_i \text{ for } i \leqslant \lceil \frac{m}{2} \rceil, \\ -1 & \text{otherwise.} \end{cases}$$

This is clearly a majority dominating function and shows that $\gamma_{\mathrm{maj}}(G)$ is at most the proposed value.

Suppose g is a majority dominating function on V and $g(V) = \gamma_{\mathrm{maj}}(G)$. Notice that $g(V_i) = g(N[v])$ for all v in V_i. Since $g(V) = \gamma_{\mathrm{maj}}(G)$, g is minimal, and so if $g(v) = 1$, then there is some u in $N[v]$ such that $g(N[u])$ is 1 or 2 (Lemma 1). Notice that $g(N[u]) = g(N[v])$ in this case. So, $g(V_i)$ will be either $-n$ or 1 or 2. Since g is a majority dominating function, at least $\lceil mn/2 \rceil$ of the vertices must have positive weight, implying that $\lceil m/2 \rceil$ of the complete subgraphs must have positive weight, and the remaining $\lfloor m/2 \rfloor$ of the complete subgraphs have negative weight. Thus $\gamma_{\mathrm{maj}}(G)$ is at least the desired amount. The result now follows. □

4. Complexity results

We will show that the following decision problem is NP-complete.

Majority domination of disjoint union of complete graphs (MUK)

Instance: A finite graph G that is the disjoint union of complete graphs of sizes $n_1, \ldots, n_m$ and an integer t.

Question: Is there a majority dominating function of weight less than or equal to t for G?

We do this by presenting a polynomial transformation from the well-known NP-complete decision problem Partition, stated in [5].

Partition

Instance: A finite set A and a 'size' $s(a) \in Z^+$ for each a in A.

Question: Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A \backslash A'} s(a)?$$

Theorem 9. *The decision problem MUK is* NP-*complete.*

Proof. Obviously, MUK is in NP.

We will show that a special case of the MUK problem is equivalent to *Partition*. Let A be a finite set with a 'size' $s(a)$ for each a in A. Define a constant $M = 2|A|$ and let $\tilde{s}(a) = Ms(a)$. Let $t = |A| - \frac{1}{2}\sum_{a \in A} \tilde{s}(a)$. Notice that $t > 0$ since $\tilde{s}(a) \geqslant 2$. Now define a graph $G = \bigcup_{a \in A} K_{\tilde{s}(a)}$. Denote the vertices of K_{n_a} by V_a, for all $a \in A$. We now show that there is a partition $A = A' \cup B$ satisfying the Partition conditions if and only if the graph G and the positive integer t satisfy the conditions of MUK.

Suppose there is a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A \backslash A'} s(a).$$

Assume, without loss of generality, that $|A'| \geqslant |A \backslash A'|$. We can substitute $\tilde{s}(a)$ for $s(a)$, since multiplying by a constant does not change this equality. Now define a majority dominating function g on G by

$$g(v) = \begin{cases} 1 & \text{for } \tilde{s}(a)/2 + 1 \text{ vertices in } V_a, \text{ for } a \in A \backslash A', \\ -1 & \text{otherwise.} \end{cases}$$

It is clear that g is a majority dominating function. Furthermore, we can compute

$$g(V) = 2|A \backslash A'| - \sum_{a \in A'} \tilde{s}(a) \leqslant |A| - \frac{\sum_{a \in A} \tilde{s}(a)}{2} = t,$$

satisfying the condition of MUK.

Now suppose there is a majority dominating function g on G such that $g(V) \leqslant t$. Without loss of generality, we may assume that $g(V_a) = 2$ or $-\tilde{s}(a)$ for each $a \in A$. Let B be the set of all elements $a \in A$ with $g(V_a) = 2$ and $A' = A \backslash B$. We note that, by the definition of a majority dominating function,

$$\sum_{a \in A'} \tilde{s}(a) \leqslant \sum_{a \in B} \tilde{s}(a),$$

so

$$\sum_{a \in A'} s(a) \leqslant \sum_{a \in B} s(a).$$

By $g(V) \leqslant t$, we have

$$2|A'| - \sum_{a \in A'} \tilde{s}(a) \leqslant t = |A| - \frac{1}{2}\sum_{a \in A} \tilde{s}(a)$$

or

$$1 - \frac{2|A'|}{|A|} + \sum_{a \in A'} s(a) \geqslant \sum_{a \in B} s(a).$$

As $1 - (2|A'|)/|A| < 1$, we have

$$\sum_{a \in A'} s(a) \geqslant \sum_{a \in B} s(a).$$

Thus

$$\sum_{a \in A'} s(a) = \sum_{a \in B} s(a).$$

Consequently, the condition for *Partition* is satisfied. Thus *Partition* is polynomially reducible to a special case of MUK, and so MUK is NP-complete. □

5. Regular graphs

In general, given a k-regular graph G, we do not know $\gamma_{\text{maj}}(G)$. However, we can construct k-regular graphs with arbitrarily negative majority domination numbers.

Theorem 10. *Given two integers $k \geqslant 2$ and $n > 0$, there exists a k-regular graph G such that $\gamma_{\text{maj}}(G) \leqslant -n$.*

Proof. Let $G = \bigcup_{i=1}^{2n} K_{k+1}$. Then, using Theorem 8, we calculate

$$\gamma_{\text{maj}}\left(\bigcup_{i=1}^{2n} K_{k+1}\right) = \begin{cases} \lceil 2n/2 \rceil - (k+1)\lfloor 2n/2 \rfloor \leqslant n - 3n \leqslant -n & \text{for } k \text{ even}, \\ 2\lceil 2n/2 \rceil - (k+1)\lfloor 2n/2 \rfloor \leqslant 2n - 4n \leqslant -n & \text{for } k \text{ odd}. \end{cases} \qquad \square$$

We can also give a lower bound on the majority domination number of a regular graph. This is a corollary of a result in [10], but the proof presented here is shorter.

Theorem 11. *Given G, a k-regular graph on n vertices, $\gamma_{\text{maj}}(G) \geqslant -(n/2) \cdot (k/k+1)$.*

Proof. Let G be a k-regular graph on n vertices, with vertex set V. Suppose g is a majority dominating function on V such that $g(V) = \gamma_{\text{maj}}(G)$. Let V^+ denote the elements v of V such that $g(N[v]) \geqslant 1$ and let $V^- = V \backslash V^+$. Consider the sum

$$\sum_{v \in V^+} g(N[v]) + \sum_{v \in V^-} g(N[v]) = \sum_{v \in V} g(N[v]) = \sum_{v \in V} \sum_{u \in N[v]} g(u) = (k+1)\gamma_{\mathrm{maj}}(G).$$

Notice that $g(N[v]) \geqslant 1$ when v is in V^+ and $g(N[v]) \geqslant -(k+1)$ when v is in V^-. So the first sum is at least $|V^+| - |V^-|(k+1)$. Notice that $|V^+| \geqslant n/2$ and $-|V^-| \geqslant -n/2$. Hence, the sum is at least

$$\frac{n}{2} - \frac{n}{2}(k+1) = -\frac{n}{2}k.$$

Combining this sum with the fact that the sum is exactly $(k+1)\gamma_{\mathrm{maj}}(G)$, the result immediately follows. □

6. Open problems

There are many problems relating to majority domination that remain open. Some of these problems are included below.

1. Given two connected graphs G and H, with vertex sets V_G and V_H, respectively, and $|V_G| \geqslant |V_H|$, then $\gamma_{\mathrm{maj}}(G \cup H) \leqslant \gamma_s(G) - |V_H|$. In fact, this is an equality in the case of two complete graphs (Proposition 1). When is there a better bound?
2. What is the relationship between the majority domination number of a graph and that of its complement?
3. Is the decision problem associated with determining the majority domination number of $K_{n_1,\ldots,n_m}$ also NP-complete?

7. Uncited references

[3,6–9]

Acknowledgements

This research was done under the supervision of Joseph Gallian at the Summer Research Program at the University of Minnesota, Duluth sponsored by the National Science Foundation (DMS-9629078) and the National Security Agency (MDA-904-96-I-0044). In addition, the author would like to thank the referees for their comments and suggestions.

References

[1] I. Broere, J.H. Hattingh, M.A. Henning, A.A. McRae, Majority domination in graphs, Discrete Math. 138 (1995) 125–135.

[2] E.J. Cockayne, C. Mynhardt, On a generalization of signed domination functions of graphs, Ars Combin. 43 (1996) 235–245.

[3] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, A. McRae, Minus domination in regular graphs, Discrete Math. 149 (1996) 311–312.
[4] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, P.J. Slater, Signed domination in graphs, Graph theory, combinatorics, and algorithms, Vols. 1 and 2, Wiley, New York, 1995, 311–321.
[5] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-completeness, Freeman, New York, 1979.
[6] J.H. Hattingh, J.H., Majority domination and its generalizations, in: Haynes, Hedetniemi, Slater (Eds.), Domination in Graphs: Advanced Topics, Marcel Dekker, Inc., 1997 (Chapter 4).
[7] M.A. Henning, Domination in regular graphs, Ars Combin. 43 (1996) 263–271.
[8] M.A. Henning, H.R. Hind, Strict majority functions on graphs, J. Graph Theory 28 (1998) 49–56.
[9] H.G. Yeh, G.J. Chang, Algorithmic aspect of majority domination, Taiwanese J. Math. 1 (1997) 343–350.
[10] B. Zelinka, Some remarks on domination in cubic graphs, Discrete Math. 158 (1996) 249–255.