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Convexity of a family of meromorphically univalent functions by using two fixed points

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ABSTRACT

In this paper a new class of meromorphic univalent functions in terms of an integral operator

$$F_c(z) = \int_0^1 cv^c f(vz)\mathrm{d}v, \quad (c \geq 1),$$

is defined. We find some properties of this new class by using two fixed points.

1. Introduction

Let Ω_p denote the class of functions of the form

$$f(z) = \frac{A}{z} + \sum_{n=0}^{\infty} a_{p+n}\, z^{p+n}, \quad A > 0, a_{p+n} > 0, p \in \mathbb{N}, \tag{1.1}$$

which are analytic in the punctured disk $\Delta^* = \{z : 0 < |z| < 1\}$. Also for a function $f(z)$ in Ω_p, we define an integral operator $F_c(z)$ as follows

$$F_c(z) = \int_0^1 cv^c f(vz)\mathrm{d}v, \quad (c \geq 1). \tag{1.2}$$

By a simple calculation we obtain that if $f(z) \in \Omega_p$ then

$$F_c(z) = \frac{A}{z} + \sum_{n=0}^{\infty} \frac{c}{p+n+1}\, a_{p+n}\, z^{p+n} \quad (c \geq 1). \tag{1.3}$$

A function $f(z)$ belonging to the class Ω_p is in class $\Omega_p(\alpha, \beta, \gamma, c)$ if it satisfies the condition

$$\left| \frac{z^3 F_c''(z) + z^2 F_c'(z) - A}{\alpha(1+\gamma)A - A + \gamma z^2 F_c'(z)} \right| < \beta, \tag{1.4}$$

for some $0 \leq \alpha < 1$, $0 < \beta \leq 1$ and $0 \leq \gamma \leq 1$. For a given number z_0 $(0 < z_0 < 1)$, let Ω_{p_j} $(j = 0, 1)$ be a subclass of Ω_p satisfying conditions $z_0 f(z_0) = 1$ and $-z_0^2 f'(z_0) = 1$, respectively. Set

$$\Omega^*_{p_j}(\alpha, \beta, \gamma, c, z_0) := \Omega_p(\alpha, \beta, \gamma, c) \cap \Omega_{p_j} \quad (j = 0, 1) \tag{1.5}$$

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and

$$\Omega^*_{p_0}(\alpha, \beta, \gamma, c, Q) := \bigcup_{z_t \in Q} \Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_t), \tag{1.6}$$

where Q is a nonempty subset of a real interval $[0, 1]$. In this article we are mainly interested in determining a necessary and sufficient condition for a meromorphic univalent function to be in $\Omega^*_{p_j}(\alpha, \beta, \gamma, c, z_0)$ for $j = 0, 1$. Finally we show that $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, Q)$ is a convex family if and only if Q is connected. For other subclasses of meromorphic univalent functions, one may refer to [1–4].

2. Main results

The following theorem gives a coefficient estimate for a function to be in $\Omega_p(\alpha, \beta, \gamma, c)$.

Theorem 2.1. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in \Omega_p(\alpha, \beta, \gamma, c)$ if and only if*

$$\sum_{n=0}^{\infty} \frac{c(p+n)}{p+n+1}(p+n+\gamma\beta)a_{p+n} \le \beta A(1+\gamma)(1-\alpha). \tag{2.1}$$

Proof. Let $f(z) \in \Omega_p(\alpha, \beta, \gamma, c)$, then (1.4) holds true. So by replacing (1.3) in (1.4) we have

$$\left| \frac{\sum_{n=0}^{\infty} \frac{c(p+n)^2}{p+n+1} a_{p+n}\, z^{p+n+1}}{A(1+\gamma)(1-\alpha) - \sum_{n=0}^{\infty} \frac{\gamma c(p+n)}{p+n+1} a_{p+n}\, z^{p+n+1}} \right| < \beta.$$

Since $\operatorname{Re}(z) \le |z|$, for all z, it follows that

$$\operatorname{Re}\left\{ \frac{\sum_{n=0}^{\infty} \frac{c(p+n)^2}{p+n+1} a_{p+n}\, z^{p+n+1}}{A(1+\gamma)(1-\alpha) - \sum_{n=0}^{\infty} \frac{\gamma c(p+n)}{p+n+1} a_{p+n}\, z^{p+n+1}} \right\} < \beta.$$

By letting $z \to 1^-$ through real values, we have

$$\sum_{n=0}^{\infty} \frac{c(p+n)}{p+n+1}(p+n+\gamma\beta)a_{p+n} \le \beta A(1+\gamma)(1-\alpha).$$

Conversely, let (2.1) hold, we have to show that

$$L(f) = |z^3 F_c''(z) + z^2 F_c'(z) - A| - \beta|\alpha(1+\gamma)A - A + \gamma z^2 F_c'(z)| < 0.$$

To this end, let $0 < |z| = r < 1$. Then it follows that

$$\begin{aligned} L(f) &= \left| \sum_{n=0}^{\infty} \frac{c(p+n)^2}{p+n+1} a_{p+n}\, z^{p+n+1} \right| - \beta \left| A(1+\gamma)(1-\alpha) - \sum_{n=0}^{\infty} \frac{\gamma c(p+n)}{p+n+1} a_{p+n}\, z^{p+n+1} \right| \\ &\le \sum_{n=0}^{\infty} \frac{c(p+n)^2}{p+n+1} |a_{p+n}| r^{p+n+1} - \beta A(1+\gamma)(1-\alpha) + \sum_{n=0}^{\infty} \frac{\beta\gamma c(p+n)}{p+n+1} |a_{p+n}| r^{p+n+1} \\ &\le \sum_{n=0}^{\infty} \frac{c(p+n)}{p+n+1}(p+n+\gamma\beta)|a_{p+n}| r^{p+n} - \beta A(\gamma+1)(1-\alpha). \end{aligned} \tag{2.2}$$

Since the above inequality holds for r $(0 < r < 1)$, by (2.1) and letting $r \to 1^-$, we obtain $L(f) \le 0$ and this completes the proof of the theorem. □

Theorem 2.2. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in \Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$ if and only if*

$$\sum_{n=0}^{\infty} \left[\frac{c(p+n)(p+n+\gamma\beta)}{(p+n+1)\beta(1+\gamma)(1-\alpha)} + z_0^{p+n+1} \right] a_{p+n} \le 1. \tag{2.3}$$

Proof. By the equality $f(z) \in \Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$, we obtain

$$z_0 f(z_0) = A + \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1} \quad A \geq 0, \quad a_{p+n} \geq 0,$$

which gives

$$A = 1 - \sum_{n=0}^{\infty} a_{p+n} z_0^{p+n+1}. \tag{2.4}$$

Substituting this value of A in Theorem 2.1, we get the desired assertion. □

Theorem 2.3. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in \Omega^*_{p_1}(\alpha, \beta, \gamma, c, z_0)$ if and only if*

$$\sum_{n=0}^{\infty} \left[\frac{c(p+n)(p+n+\gamma\beta)}{(p+n+1)\beta(1+\gamma)(1-\alpha)} - (p+n) z_0^{p+n+1} \right] a_{p+n} \leq 1. \tag{2.5}$$

Proof. By $-z_0^2 f'(z_0) = 1$, it follows that

$$A = 1 + \sum_{n=0}^{\infty} (p+n) a_{p+n} z_0^{p+n+1}, \tag{2.6}$$

replacing A in (2.1) to give the required result. □

Corollary 2.4. *Let $f(z)$ of the form (1.1) be in the class $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$, then*

$$a_{p+n} \leq \frac{(p+n+1)\beta(1+\gamma)(1-\alpha)}{c(p+n)(p+n+\gamma\beta) + (p+n+1)\beta(1+\gamma)(1-\alpha) z_0^{p+n+1}}. \tag{2.7}$$

Corollary 2.5. *Let $f(z)$ of the form (1.1) be in the class $\Omega^*_{p_1}(\alpha, \beta, \gamma, c, z_0)$ then*

$$a_{p+n} \leq \frac{(p+n+1)\beta(1+\gamma)(1-\alpha)}{(n+p)[c(p+n+\gamma\beta) - (p+n+1)\beta(1+\gamma)(1-\alpha) z_0^{p+n+1}]}. \tag{2.8}$$

Now, we will prove some important properties of $\Omega^*_{p_j}(\alpha, \beta, \gamma, c, z_0)$ $(j = 0, 1)$.

Theorem 2.6. *The class $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$ is closed under convex linear combination.*

Proof. Let $f_k(z)$ $(k = 1, 2)$ defined by

$$f_k(z) = \frac{A_k}{z} + \sum_{n=0}^{\infty} a_{p+n,k}\, z^{p+n}, \quad A_k > 0,\, a_{p+n,k} > 0,\, p \in \mathbb{N}, \tag{2.9}$$

be in the class $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$, it is sufficient to show that the function $G(z)$ defined by

$$G(z) := \lambda f_1(z) + (1-\lambda) f_2(z), \quad 0 \leq \lambda \leq 1$$

is also in the class $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$. Since

$$G(z) = \frac{\lambda A_1 + (1-\lambda) A_2}{z} + \sum_{n=0}^{\infty} [\lambda a_{p+n,1} + (1-\lambda) a_{p+n,2}] z^{p+n},$$

with the aid of Theorem 2.2, we have

$$\sum_{n=0}^{\infty} \frac{c(p+n)(p+n+\gamma\beta)}{(p+n+1)} + \beta(1+\gamma)(1-\alpha) z_0^{p+n+1} [a_{p+n,1} + a_{p+n,2}] z^{p+n} \leq \beta(1-\alpha)(1+\gamma)$$

which implies that $G(z) \in \Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_0)$. □

In a similar manner, by using Theorem 2.3, we can prove the following theorem.

Theorem 2.7. *The class $\Omega^*_{p_1}(\alpha,\beta,\gamma,c,z_0)$ is closed under convex linear combination.*

Now we are ready to prove the main result of the paper. Let Q be a nonempty subset of a real interval $[0,1]$. We define a family $\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q)$ by

$$\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q) := \bigcup_{z_t\in Q}\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_t).$$

For instance, if Q has only one element, then $\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q)$ is known to be a convex family by Theorem 2.6. It is interesting to investigate this class for another subset Q. We shall make use of the following.

Lemma 2.8. *If $f(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_0)\cap\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_1)$ where z_0 and z_1 are distinct positive numbers, then $f(z)=\frac{1}{z}$.*

Proof. Let $f(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_0)\cap\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_1)$, then

$$f(z)=\frac{A}{z}+\sum_{n=0}^{\infty}a_{p+n}\,z^{p+n},\quad A>0, a_{p+n}>0, p\in\mathbb{N}$$

where

$$A=1-\sum_{n=0}^{\infty}a_{p+n}z_0^{p+n+1}=1-\sum_{n=0}^{\infty}a_{p+n}z_1^{p+n+1}.$$

Since $a_{p+n}\geq 0$, $z_0\geq 0$ and $z_1\geq 0$, this implies that $a_{p+n}=0$ for each $n\geq 0$ and hence $f(z)=\frac{1}{z}$. $\square$

Theorem 2.9. *If Q is contained in the interval $[0,1]$, then $\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q)$ is a convex family if and only if Q is connected.*

Proof. Suppose Q is connected and $z_0,z_1\in Q$ with $z_0\leq z_1$. To prove $\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q)$ is a convex family it is enough to show, for

$$f(z)=\frac{A}{z}+\sum_{n=0}^{\infty}a_{p+n}\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_0),$$

$$g(z)=\frac{B}{z}+\sum_{n=0}^{\infty}b_{p+n}\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_1),$$

and $0\leq\lambda\leq 1$, there exists a z_2 ($z_0\leq z_2\leq z_1$) such that

$$h(z)=\lambda f(z)+(1-\lambda)g(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_2).$$

Since $f(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_0)$ and $g(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_1)$, we have $A=1-\Sigma_{n=0}^{\infty}a_{p+n}z_0^{p+n+1}$ and $B=1-\Sigma_{n=0}^{\infty}b_{p+n}$ z_1^{p+n+1}. Therefore

$$\begin{aligned}T(z)&=zh(z)=\lambda A+(1-\lambda)B+\lambda\sum_{n=0}^{\infty}a_{p+n}z^{p+n}+(1-\lambda)\sum_{n=0}^{\infty}b_{p+n}z^{p+n}\\&=1+\lambda\sum_{n=0}^{\infty}a_{p+n}(z^{p+n+1}-z_0^{p+n+1})+(1-\lambda)\sum_{n=0}^{\infty}b_{p+n}(z^{p+n+1}-z_1^{p+n+1}).\end{aligned}\tag{2.10}$$

When z is real, $T(z)$ is real. Also $T(z_0)\leq 1$ and $T(z_1)\geq 1$, so there exists $z_2\in[z_0,z_1]$ such that $T(z_2)=0$. This implies that $h(z)\in\Omega_{p_0}$. Now, in view of (2.10) we have

$$\sum_{n=0}^{\infty}\frac{c(p+n)(p+n+\gamma\beta)}{(p+n+1)}-\beta(1+\gamma)(1-\alpha)z_2^{p+n+1}[a_{p+n}+(1-\lambda)b_{p+n,2}]\leq\beta(1-\alpha)(1+\gamma).$$

Hence we have $h(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_2)$, by Theorem 2.2. Since z_0,z_1 and z_2 are arbitrary, then the family $\Omega^*_{p_0}(\alpha,\beta,\gamma,c,Q)$ is convex. Conversely, if Q is not connected, then there exist z_0,z_1 and z_2 such that $z_0,z_1\in Q$ and $z_2\in[0,1]-Q$, and $z_0<z_2<z_1$. Suppose that both $f(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_0)$ and $g(z)\in\Omega^*_{p_0}(\alpha,\beta,\gamma,c,z_1)$ are not equal to $\frac{1}{z}$. Then for fixed z_2 and $0\leq\lambda\leq 1$ from (2.10), it follows that

$$\begin{aligned}T(z_2)&=z_2h(z_2)\\&=1+\lambda\sum_{n=0}^{\infty}a_{p+n}(z_2^{p+n+1}-z_0^{p+n+1})+(1-\lambda)\sum_{n=0}^{\infty}b_{p+n}(z_2^{p+n+1}-z_1^{p+n+1}).\end{aligned}$$

Since $T(z_2)$ in $\lambda = 0$ is less than 1 and $T(z_2)$ in $\lambda = 1$ is greater than 1, there exists $0 < \eta < 1$ such that $T(z_2) = 1$ or $z_2 h(z_2) = 1$, where $h(z) = \eta f(z) + (1 - \eta)g(z)$. Therefore $h(z) \in \Omega^*_{p_0}(\alpha, \beta, \gamma, c, z_2)$. From Lemma 2.8, $h(z)$ is not in $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, Q)$. Hence $\Omega^*_{p_0}(\alpha, \beta, \gamma, c, Q)$ is not convex. This completes the proof of the theorem. □

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