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Some Mathieu-type series associated with the Fox–Wright function

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ABSTRACT

Closed-form integral expressions are derived here for a family of convergent Mathieu-type series and its alternating variant when their terms contain the Fox–Wright ${}_p\Psi_q$ function. Some two-sided exponential bounding inequalities are then obtained for a class of Fox–Wright ${}_p\Psi_q$ functions, thereby generalizing certain results of Luke. Finally, by means of the integral expressions obtained here, a number of two-sided exponential bounding inequalities are given for the aforementioned series.

1. Introduction and preliminaries

In a series of recent papers, the authors as well as Tomovski have studied certain families of Mathieu-type **a**-series and alternating Mathieu-type **a**-series, whose terms contain the familiar Gauss hypergeometric function ${}_2F_1$, the generalized hypergeometric function ${}_pF_q$, the Fox–Wright ${}_1\Psi_2$ function, Meijer's G function and Fox's H function (see, for example, [1–7]). The results derived in these earlier works are concerned mainly with closed-form integral expressions and two-sided bounding inequalities for these series. Here, in our present investigation, we generalize these results to Mathieu-type series whose terms contain the Fox–Wright ${}_p\Psi_q$ function and its alternating variants improving substantially the earlier results. The closed-form integral expressions in conjunction with the two-sided exponential inequalities obtained here are shown to lead to a number of two-sided bounding inequalities for the aforementioned series.

Here, *and in what follows*, we use ${}_p\Psi_q$ to denote the Fox–Wright generalization of the familiar hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters (see [8,9]), defined by (cf., e.g., [7, p. 4, Eq. (2.4)])

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\middle| z\right] = {}_p\Psi_q\left[\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\middle| z\right] := \sum_{m=0}^{\infty}\frac{\prod_{\ell=1}^{p}\Gamma(a_\ell+\alpha_\ell m)}{\prod_{\ell=1}^{q}\Gamma(b_\ell+\beta_\ell m)}\frac{z^m}{m!} $$

$$ \left(\alpha_\ell\in\mathbb{R}_+\ (\ell=1,\ldots,p);\ \beta_j\in\mathbb{R}_+\ (j=1,\ldots,q);\ 1+\sum_{\ell=1}^{q}\beta_\ell-\sum_{j=1}^{p}\alpha_j>0\right), \tag{1} $$

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for suitably bounded values of $|z|$. The generalized hypergeometric function ${}_pF_q$ is defined by

$$ {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\bigg| z\right] = {}_pF_q\left[\begin{matrix} a_p \\ b_q \end{matrix}\bigg| z\right] := \sum_{m=0}^{\infty} \frac{\prod_{\ell=1}^{p}(a_\ell)_m}{\prod_{\ell=1}^{q}(b_\ell)_m} \frac{z^m}{m!} \tag{2} $$

where, as usual, we make use of the following notation:

$$ (\tau)_0 := 1 \quad \text{and} \quad (\tau)_m := \tau(\tau+1)\cdots(\tau+m-1) = \frac{\Gamma(\tau+m)}{\Gamma(\tau)} \quad (m \in \mathbb{N}) $$

to denote the *shifted factorial* or the Pochhammer symbol. Obviously, we find from the definitions (1) and (2) that

$$ {}_p\Psi_q\left[\begin{matrix} (a_p,1) \\ (b_q,1) \end{matrix}\bigg| z\right] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)} \cdot {}_pF_q\left[\begin{matrix} a_p \\ b_q \end{matrix}\bigg| z\right] \quad (a_j > 0;\ b_k \notin \mathbb{Z}_0^-). \tag{3} $$

Throughout this paper, we adopt the following convention for the real sequence $\mathbf{c}$:

$$ \mathbf{c} : 0 < c_1 < c_2 < \cdots < c_n \uparrow \infty. \tag{4} $$

It is useful here to consider the function $c : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$ c(x)|_{x\in\mathbb{N}} = \mathbf{c}. $$

We also denote by $c^{-1}(x)$ the inverse function of $c(x)$.

We denote the integer part of a real number a by $[a]$ and we write

$$ \{a\} = a - [a] $$

for the *fractional* part of the real number a.

In this paper, we investigate the Mathieu-type series $\mathfrak{B}$ and its alternating variant $\widetilde{\mathfrak{B}}$, which are defined by

$$ \mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\mathbf{c};r\right) := \sum_{j=1}^{\infty} \frac{c_j^{-\lambda}}{(c_j+r)^\mu} \cdot {}_{p+1}\Psi_q\left[\begin{matrix} (a_{p+1},\alpha_{p+1}) \\ (b_q,\beta_q) \end{matrix}\bigg| \frac{r}{c_j}\right] \tag{5} $$

and

$$ \widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\mathbf{c};r\right) := \sum_{j=1}^{\infty} \frac{(-1)^{j-1}c_j^{-\lambda}}{(c_j+r)^\mu} \cdot {}_{p+1}\Psi_q\left[\begin{matrix} (a_{p+1},\alpha_{p+1}) \\ (b_q,\beta_q) \end{matrix}\bigg| \frac{r}{c_j}\right], \tag{6} $$

where it is tacitly assumed that all of the required constraints for the parameters involved are satisfied for the series in (5) and (6) to be convergent.

We remark in passing that, by means of different approaches, similar series have been presented in integral forms by (for example) Pogány [3, Theorem 2], and by Pogány and Tomovski [5, Theorem 1] who made use of the generalized hypergeometric function ${}_pF_q$ and Meijer's G function.

2. Integral representations of $\mathfrak{B}_{\lambda,\mu}({}_{p+1}\Psi_q;\mathbf{c};r)$ and $\widetilde{\mathfrak{B}}_{\lambda,\mu}({}_{p+1}\Psi_q;\mathbf{c};r)$

In the course of our investigation, one of the main tools is the following result providing the Laplace transform of $x^{\lambda-1}{}_p\Psi_q(\cdot|\omega x)$:

$$ \int_0^\infty \mathrm{e}^{-\rho x}\, x^{\lambda-1}{}_p\Psi_q\left[\begin{matrix} (a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix}\bigg| \omega x\right]\mathrm{d}x = \rho^{-\lambda}{}_{p+1}\Psi_q\left[\begin{matrix} (\lambda,1),(a_p,\alpha_p) \\ (b_q,\beta_q) \end{matrix}\bigg| \frac{\omega}{\rho}\right] $$
$$ \left(\alpha_\ell\ (\ell=1,\ldots,p);\ \beta_j > 0\ (j=1,\ldots,q);\ \Re(\rho) > 0;\ \Re(\lambda) > 0\right), \tag{7} $$

which can easily be derived by using the definition (1) and an elementary Gamma-function property given by

$$ \rho^{-\lambda}\Gamma(\lambda) = \int_0^\infty \mathrm{e}^{-\rho t} t^{\lambda-1}\mathrm{d}t \quad (\Re(\rho) > 0; \Re(\lambda) > 0). \tag{8} $$

We now consider the series (5) for $\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\mathbf{c};r\right)$ and set

$$ \rho = c_j, \qquad \omega = r, \qquad a_{p+1} = \lambda \quad \text{and} \quad \alpha_{p+1} = 1 $$

in (7). Then, using (8) with $\rho = c_j + r$, we find that

$$\begin{aligned}\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\,\mathbf{c};\,r\right) &= \frac{1}{\Gamma(\mu)}\sum_{j=1}^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-c_j s-(c_j+r)t} s^{\lambda-1} t^{\mu-1}\,{}_p\Psi_q\left[\left.\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right| rs\right]\mathrm{d}s\,\mathrm{d}t\\ &= \frac{1}{\Gamma(\mu)}\int_0^{\infty}\int_0^{\infty}\left(\sum_{j=1}^{\infty}\mathrm{e}^{-c_j(s+t)}\right)\mathrm{e}^{-rt}s^{\lambda-1}t^{\mu-1}\,{}_p\Psi_q\left[\left.\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right| rs\right]\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

The *inner-most* Dirichlet series:

$$\mathcal{D}_{\mathbf{c}}(y) = \sum_{j=1}^{\infty}\mathrm{e}^{-c_j y} \quad (y := s+t > 0)$$

possesses a Laplace integral form (see [1,2]). Since

$$\left[c^{-1}(x)\right] \equiv 0 \quad (x \in [0, c_1)),$$

we thus obtain

$$\mathcal{D}_{\mathbf{c}}(y) = y\int_0^{\infty}\mathrm{e}^{-yx}\left(\sum_{j:c_j\leqq x}1\right)\mathrm{d}x = y\int_{c_1}^{\infty}\mathrm{e}^{-yx}\left[c^{-1}(x)\right]\mathrm{d}x.$$

Therefore, we conclude that

$$\begin{aligned}\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\,\mathbf{c};\,r\right) &= \int_{c_1}^{\infty}\frac{[c^{-1}(x)]}{\Gamma(\mu)}\left(\int_0^{\infty}\mathrm{e}^{-(r+x)t}t^{\mu-1}\mathrm{d}t\right)\cdot\left(\int_0^{\infty}\mathrm{e}^{-xs}s^{\lambda}\,{}_p\Psi_q\left[\left.\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right| rs\right]\mathrm{d}s\right)\mathrm{d}x\\ &\quad+\int_{c_1}^{\infty}\frac{[c^{-1}(x)]}{\Gamma(\mu)}\left(\int_0^{\infty}\mathrm{e}^{-(r+x)t}t^{\mu}\mathrm{d}t\right)\cdot\left(\int_0^{\infty}\mathrm{e}^{-xs}s^{\lambda-1}\,{}_p\Psi_q\left[\left.\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right| rs\right]\mathrm{d}s\right)\mathrm{d}x\\ &= \int_{c_1}^{\infty}\frac{[c^{-1}(x)]}{x^{\lambda+1}(r+x)^{\mu}}\,{}_{p+1}\Psi_q\left[\left.\begin{matrix}(\lambda+1,1),(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right|\frac{r}{x}\right]\mathrm{d}x \qquad (9)\\ &\quad+\mu\int_{c_1}^{\infty}\frac{[c^{-1}(x)]}{x^{\lambda}(r+x)^{\mu+1}}\,{}_{p+1}\Psi_q\left[\left.\begin{matrix}(\lambda,1),(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right|\frac{r}{x}\right]\mathrm{d}x. \qquad (10)\end{aligned}$$

Finally, by introducing the following integral:

$$\mathfrak{I}_{\mathbf{c}}^{\Psi}(u,v) := \int_{c_1}^{\infty}\frac{\left[c^{-1}(x)\right]}{x^u(r+x)^v}\cdot{}_{p+1}\Psi_q\left[\left.\begin{matrix}(u,1),(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right|\frac{r}{x}\right]\mathrm{d}x, \qquad (11)$$

we can easily express $\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\,\mathbf{c};\,r\right)$ as a linear combination of two $\mathfrak{I}_{\mathbf{c}}^{\Psi}(\cdot,\cdot)$ functions.

Theorem 1. *Let*

$$\lambda,\mu,r > 0 \quad \textit{and} \quad (a_{p+1},\alpha_{p+1}) = (\lambda, 1).$$

Then the Mathieu-type series $\mathfrak{B}$ *defined by* (5) *possesses the following closed-form integral representation:*

$$\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\,\mathbf{c};\,r\right) = \mathfrak{I}_{\mathbf{c}}^{\Psi}(\lambda+1,\mu) + \mu\,\mathfrak{I}_{\mathbf{c}}^{\Psi}(\lambda,\mu+1). \qquad (12)$$

We next sum the following *alternating* Dirichlet series $\widetilde{\mathcal{D}}_{\mathbf{c}}(y)$ that appears in the derivation of the integral expression for $\widetilde{\mathfrak{B}}$ (see, for details, [3,5], and [6, Section 4]):

$$\widetilde{\mathcal{D}}_{\mathbf{c}}(y) = \sum_{j=1}^{\infty}(-1)^{j-1}\mathrm{e}^{-c_j y} = y\int_{c_1}^{\infty}\mathrm{e}^{-yx}\sin^2\left(\frac{\pi}{2}\left[c^{-1}(x)\right]\right)\mathrm{d}x. \qquad (13)$$

Obviously, by repeating the same calculations as above with (13), we can deduce the following result for the *alternating* Mathieu-type series.

Theorem 2. *Let*

$$\lambda,\mu,r > 0 \quad \textit{and} \quad (a_{p+1},\alpha_{p+1}) = (\lambda, 1).$$

Then the Mathieu-type series $\widetilde{\mathfrak{B}}$ defined by (6) possesses the following closed-form integral representation:

$$\widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q;\mathbf{c};r\right)=\widetilde{\mathfrak{I}}^{\Psi}_{\mathbf{c}}(\lambda+1,\mu)+\mu\widetilde{\mathfrak{I}}^{\Psi}_{\mathbf{c}}(\lambda,\mu+1) \tag{14}$$

where

$$\widetilde{\mathfrak{I}}^{\Psi}_{\mathbf{c}}(u,v):=\int_{c_1}^{\infty}\frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)}{x^u(r+x)^v}\cdot{}_{p+1}\Psi_q\left[\left.\begin{matrix}(u,1),(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right|\frac{r}{x}\right]\mathrm{d}x. \tag{15}$$

3. Two-sided exponential inequalities for the Fox–Wright function ${}_p\Psi_q$

Exponential inequalities for the confluent hypergeometric function ${}_pF_p$ were investigated by Luke [10, Theorem 16, Eq. (5.5–7)]. In fact, by assuming that

$$b_j\geqq a_j>0\quad(j=1,\ldots,p),$$

he showed that (see [10, Theorem 16, Eq. 5.5–7)])

$$\mathrm{e}^{\theta|x|}\leqq{}_pF_p\left[\left.\begin{matrix}a_p\\b_p\end{matrix}\right|x\right]\leqq 1-\theta\left(1-\mathrm{e}^{|x|}\right)\quad\left(\theta:=\prod_{j=1}^{p}\left(\frac{a_j}{b_j}\right);x\in\mathbb{R}\right). \tag{16}$$

Recently, Pogány and Tomovski [5] extended Luke's inequality to a class of generalized hypergeometric functions ${}_pF_q$ $(p\leqq q)$. Because ${}_pF_q\left[a_p;b_q|x\right]$ is symmetric with respect to the permutations of the parameters a and b, we can assume that

$$a_1\leqq a_2\leqq\cdots\leqq a_p\quad\text{and}\quad b_1\leqq b_2\leqq\cdots\leqq b_q.$$

So, whenever there is the p-tuple:

$$1\leqq i_1<\cdots<i_p\leqq q$$

such that

$$b_{i_j}\geqq a_j>0\quad(j=1,\ldots,p)$$

we say that the considered generalized hypergeometric function ${}_pF_q\left[a_p;b_q|x\right]$ belongs to the function class ${}_p\mathbb{F}_q$. The case $p=q$ would obviously correspond to Luke's condition:

$$a_j\leqq b_j\quad(j=1,\ldots,p)$$

for the confluent case ${}_p\mathbb{F}_p$. We recall the related result in [5, Theorem 3] as follows.

Lemma 1. *Let*

$$a_j>0\quad(j=1,\ldots,p)\quad\text{and}\quad p\leqq q.$$

Then, for $g\in{}_p\mathbb{F}_q$, the following two-sided bounding inequality holds true:

$$\mathrm{e}^{\theta_0|x|}\leqq g(x)\leqq 1-\theta_0+\theta_0\mathrm{e}^{|x|}\quad\left(\theta_0:=\frac{\prod\limits_{j=1}^{p}a_j}{\prod\limits_{j=1}^{q}b_j};x\in\mathbb{R}\right). \tag{17}$$

The motivation for this type of results is derived from the following observation. In the case of the generalized hypergeometric series:

$${}_pF_q\left[\left.\begin{matrix}a_p\\b_q\end{matrix}\right|x\right]=1+\sum_{m=1}^{\infty}\mathfrak{f}_m\frac{x^m}{m!}\quad\left(\mathfrak{f}_m:=\frac{\prod\limits_{\ell=1}^{p}(a_\ell)_m}{\prod\limits_{\ell=1}^{q}(b_\ell)_m};m\in\mathbb{N}\right)$$

the following two-sided inequalities hold true [5, Eq. (35)]:

$$\theta_0^m\leqq\mathfrak{f}_m\leqq\mathfrak{f}_1\equiv\theta_0. \tag{18}$$

By applying (18) to the above-generalized hypergeometric series, we easily get the two-sided inequality (17) asserted by Lemma 1.

We are now interested in extending (17) to a certain subclass of the Fox–Wright generalized hypergeometric ${}_p\Psi_q$ functions. This will be done by Theorem 3, which contains a key result for determining some remarkably general two-sided exponential inequalities leading, in turn, to a number of two-sided exponential inequalities for hypergeometric-type functions as its corollaries.

Theorem 3. *Let the real-valued function $f \geqq 0$ be twice continuously differentiable at the origin. Suppose also that*

$$f(0) = 1 \quad \text{and} \quad f'(0) > f''(0) > [f'(0)]^2. \tag{19}$$

Then

$$e^{f'(0)|x|} \leqq f(x) \leqq 1 - \left(1 - e^{|x|}\right) f'(0), \tag{20}$$

where the equality holds true in (20) when $x = 0$.

Proof. We assume that $x > 0$ and, for some *positive* real numbers α and β, we consider the following auxiliary functions:

$$F_R^+(x) := f(x) - 1 + \beta\left(1 - e^x\right) \quad \text{and} \quad F_L^+(x) := f(x) - e^{\alpha x}.$$

Since

$$\left(F_R^+(0)\right)' = \left(f'(x) - \beta e^x\right)|_{x=0} = 0 \quad \left(\beta = f'(0)\right)$$

and

$$\left(F_R^+(0)\right)'' = f''(0) - f'(0) < 0,$$

the origin $x = 0$ is the abscissa of the local maximum of $F_R^+(x)$. Therefore, we have

$$F_R^+(x) \leqq F_R^+(0) = 0,$$

which readily implies that

$$f(x) \leqq 1 - \left(1 - e^x\right) f'(0).$$

Similar arguments would lead us to the left-hand estimate in (20). For *nonpositive* values of x, we write

$$F_R(x) := f(-x) - 1 + \beta\left(1 - e^{-x}\right) \quad \text{and} \quad F_L(x) := f(-x) - e^{-\alpha x} \quad (x \geqq 0).$$

By repeating the above procedure, we get

$$f(-x) \leqq 1 - f'(0) + f'(0)e^{-x}.$$

Then, keeping (19) in mind, we deduce that the condition $\alpha = f'(0)$ is necessary to ensure a local minimum of $F_L(x)$ at the origin. We thus conclude that

$$F_L(x) \geqq f(-x) - e^{-f'(0)x} \geqq 0.$$

This completes the proof of the two-sided inequality (20) asserted by Theorem 3. □

Rewriting the Fox–Wright function as follows:

$${}_p\Psi_q\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| x\right] = \sum_{m=0}^{\infty} \Psi_m \frac{x^m}{m!} \qquad \left(\Psi_m := \frac{\prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j m\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j m\right)};\ m \in \mathbb{N}_0\right)$$

and noting that

$$\Psi_0 := \frac{\prod_{j=1}^{p} \Gamma\left(a_j\right)}{\prod_{j=1}^{q} \Gamma\left(b_j\right)} \neq 1,$$

we will henceforth concentrate upon the so-called *normalized Fox–Wright generalized hypergeometric function* defined by

$${}_p\Psi_q^\star\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| x\right] := \Psi_0^{-1}\, {}_p\Psi_q\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| x\right].$$

Clearly, we have

$${}_p\Psi_q^\star[\cdot|0] = 1.$$

If ${}_p\Psi_q^\star[\cdot|x]$ satisfies Eq. (19), that is, if

$$\frac{\partial}{\partial x}\, {}_p\Psi_q^\star\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| 0\right] > \frac{\partial^2}{\partial x^2}\, {}_p\Psi_q^\star\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| 0\right] > \left(\frac{\partial}{\partial x}\, {}_p\Psi_q^\star\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| 0\right]\right)^2, \tag{21}$$

then, by Theorem 3, we can deduce an exponential inequality like Eq. (20) for ${}_p\Psi_q^\star[\cdot|x]$. Of course, the same follows for ${}_p\Psi_q[\cdot|x]$, but under the following condition:

$$\frac{\Psi_1}{\Psi_0} < \frac{\Psi_2}{\Psi_1} < 1 \Longleftrightarrow \left(\Psi_2 < \Psi_1 \quad \text{and} \quad \Psi_1^2 < \Psi_2\Psi_0\right), \tag{22}$$

which is equivalent to (21). The building blocks of Ψ_m $(m = 0, 1, 2)$ are Gamma functions, and the ones involved in (22) are closely related to the well-known *Gautschi quotient* [11]:

$$Q(\lambda, \mu) := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} =: (\lambda)_\mu \quad (\min\{\lambda, \mu\} > 0) \tag{23}$$

and to the so-called *Gurland's ratio* [11]:

$$T(\lambda, \mu) := \frac{\Gamma(\lambda)\Gamma(\mu)}{\left[\Gamma\left(\frac{\lambda+\mu}{2}\right)\right]^2} \quad (\min\{\lambda, \mu\} > 0). \tag{24}$$

Indeed, for

$$a_\ell, b_j, \alpha_\ell, \beta_j > 0 \quad (\ell = 1, \ldots, p; j = 1, \ldots, q),$$

the constraint $\Psi_2 < \Psi_1$ is well defined and it is equivalent to the following inequality:

$$\prod_{j=1}^{p} \frac{\Gamma(a_j + 2\alpha_j)}{\Gamma(a_j + \alpha_j)} \leqq \prod_{j=1}^{q} \frac{\Gamma(b_j + 2\beta_j)}{\Gamma(b_j + \beta_j)}$$

which, in the notation for the Gautschi quotient given by (23), assumes the form:

$$\prod_{j=1}^{p} Q(a_j + \alpha_j, \alpha_j) \leqq \prod_{j=1}^{q} Q(b_j + \beta_j, \beta_j). \tag{25}$$

The second constraint for the parameters involved in (22) is

$$\Psi_1^2 < \Psi_2\Psi_0,$$

which can be rewritten as follows:

$$\prod_{j=1}^{q} T(b_j, b_j + 2\beta_j) \leqq \prod_{j=1}^{p} T(a_j, a_j + 2\alpha_j) \tag{26}$$

by using the notation for the Gurland quotient in (24).

Let us now denote the parameter space of the exponential inequalities for the Fox–Wright function ${}_p\Psi_q$ by

$$\begin{aligned} {}_p\mathbb{D}_q(Q, T) := \Bigg\{ &(\mathbf{a}_p, \mathbf{b}_q, \boldsymbol{\alpha}_p, \boldsymbol{\beta}_q) \in \mathbb{R}_+^{2(p+q)} \colon \prod_{j=1}^{p} Q(a_j + \alpha_j, \alpha_j) \leqq \prod_{j=1}^{q} Q(b_j + \beta_j, \beta_j) \\ &\text{and} \quad \prod_{j=1}^{q} T(b_j, b_j + 2\beta_j) \leqq \prod_{j=1}^{p} T(a_j, a_j + 2\alpha_j) \Bigg\}, \end{aligned} \tag{27}$$

where

$$\mathbf{t}_k := (t_1, \ldots, t_k).$$

We remark that

$${}_p\mathbb{D}_q(Q, T) \neq \emptyset.$$

Indeed, by specifying the parameters as follows:

$$\alpha_\ell = \beta_j = 1 \quad (\ell = 1, \ldots, p; j = 1, \ldots, q)$$

and keeping in mind that all

$$a_\ell, b_j > 0 \quad (\ell = 1, \ldots, p; j = 1, \ldots, q),$$

we arrive at

$$\Psi_2 < \Psi_1 \Longleftrightarrow \prod_{j=1}^{p} (1 + a_j) \leqq \prod_{j=1}^{q} (1 + b_j) \tag{28}$$

and

$$\Psi_1^2 < \Psi_2\Psi_0 \Longleftrightarrow \prod_{j=1}^{p} a_j \leqq \prod_{j=1}^{q} b_j. \tag{29}$$

The relations (28) and (29) clearly show the claim that ${}_p\mathbb{D}_q(Q, T)$ is not empty.

Theorem 4. *Let* $(\mathbf{a}_p, \mathbf{b}_q, \boldsymbol{\alpha}_p, \boldsymbol{\beta}_q) \in {}_p\mathbb{D}_q(Q, T)$. *Then*

$$\Psi_0 \mathrm{e}^{\Psi_1\Psi_0^{-1}|x|} \leqq {}_p\Psi_q\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| z\right] \leqq \Psi_0 - \left(1-\mathrm{e}^{|x|}\right)\Psi_1 \quad (x \in \mathbb{R}), \tag{30}$$

where the equality holds true when $x = 0$ *and*

$$\Psi_0 = \frac{\prod_{j=1}^{p}\Gamma\left(a_j\right)}{\prod_{j=1}^{q}\Gamma\left(b_j\right)} \quad \text{and} \quad \Psi_1 = \frac{\prod_{j=1}^{p}\Gamma\left(a_j+\alpha_j\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_j\right)}.$$

Proof. Since, by hypothesis,

$$(a_j, b_k, \alpha_j, \beta_k) \in {}_p\mathbb{D}_q(Q, T),$$

the condition (21) is fulfilled. Furthermore, because the Fox–Wright function ${}_p\Psi_q^{\star}[\cdot|x]$ is normalized and it is continuously differentiable twice at the origin, all prerequisites of Theorem 3 are satisfied. Consequently, by means of Theorem 3, it follows that

$$\mathrm{e}^{\Psi_1\Psi_0^{-1}|x|} \leqq {}_p\Psi_q^{\star}\left[\left.\begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right| x\right] \leqq 1 - \Psi_1\Psi_0^{-1}\left(1-\mathrm{e}^{|x|}\right),$$

which proves Theorem 4. □

At this point we are faced with the following important dilemma:

(i) To accept the general result (30) which is hardly applicable because of the description constraints of the set ${}_p\mathbb{D}_q(Q, T)$ involving the T and Q expressions; or

(ii) To find some simpler domain reducing the description constraints of ${}_p\mathbb{D}_q(Q, T)$ by some already known estimates for Gurland's ratio T and Gautschi's ratio Q.

In order to realize the option (ii), we will need a few auxiliary results.

Lemma 2. *The following two-sided inequality holds true for the Gurland quotient:*

$$\left(1+\frac{\alpha}{a}\right)^{2\alpha}\left(1+\frac{1}{a}\right)^{-\alpha^2} \leqq T(a, a+2\alpha) \leqq \left(1+\frac{\alpha}{a}\right)^{\alpha} \quad (\alpha \in [0, 1];\ a > 0) \tag{31}$$

with equality on both sides for $\alpha \in \{0, 1\}$.

Proof. We recall the following two-sided inequality reported by Merkle [11, Corollary 1, Eq. (22–23)]:

$$\frac{(a+\alpha)^{2\alpha}}{a^{\alpha(2-\alpha)}(a+1)^{\alpha^2}} \leqq T(a, a+2\alpha) \leqq \frac{(a+\alpha)^{1+\alpha}}{a(a+1)^{\alpha}} \quad (\alpha \in [0, 1];\ a > 0).$$

Obvious transformations and the use of the classical Bernoulli inequality would give us (31) as asserted by Lemma 2. □

Consider $\Psi_1^2 < \Psi_0\Psi_2$, that is, Eq. (26) in the form:

$$\mathcal{T} := \prod_{j=1}^{q} T(b_j, b_j+2\beta_j) \leqq \prod_{j=1}^{p} T(a_j, a_j+2\alpha_j) =: \mathcal{P}.$$

We now majorize $\mathcal{T}$ and minorize $\mathcal{P}$ by means of (31). Since all of the α and β parameters are positive, for the inequality $\mathcal{T} \leqq \mathcal{P}$ to hold true, it is sufficient to show that

$$\prod_{j=1}^{q}\left(1+\frac{\beta_j}{b_j}\right)^{\beta_j} \leqq \prod_{j=1}^{p}\left(1+\frac{\alpha_j}{a_j}\right)^{2\alpha_j}\left(1+\frac{1}{a_j}\right)^{-\alpha_j^2} \quad (\alpha_j, \beta_k \in [0, 1];\ j = 1, \ldots, p;\ k = 1, \ldots, q).$$

We next consider (25) in the form:

$$Q_1 := \prod_{j=1}^{p} Q\,(a_j, +\alpha_j, \alpha_j) \leqq \prod_{j=1}^{q} Q\,(b_j + \beta_j, \beta_j) =: Q_2. \tag{32}$$

In order to transform the relation (25) in the above manner, we need the following lemma.

Lemma 3. *The following inequality holds true for the Gautschi quotient:*

$$Q\,(a, \alpha) = \frac{\Gamma(a+\alpha)}{\Gamma(a)} \geqq \left(a - \frac{1-\alpha}{2}\right)^{\alpha} \quad \left(a > \frac{1-\alpha}{2}; \alpha \in [0, 1]\right). \tag{33}$$

Moreover, for all $a, \alpha > 0$,

$$Q\,(a, \alpha) = \frac{\Gamma(a+\alpha)}{\Gamma(a)} \leqq a^{\alpha} \left(1 + \frac{\alpha}{a}\right)^{\alpha - \frac{1}{2}}. \tag{34}$$

Proof. The first inequality (33) is due to Lazarević and Lupaş [12, p. 248, Corollary]; the second inequality (34) can easily be deduced from a known result [13, Remark 1, Eq. (5)]. □

Majorizing Q_1 and simultaneously minorizing Q_2 by means of the estimates (33) and (34), we get

$$Q_1 \leqq \prod_{j=1}^{p} a_j^{\alpha_j} \left(1 + \frac{\alpha_j}{a_j}\right)^{\alpha_j - 1/2} =: \widehat{Q}_1$$

and

$$Q_2 \geqq \prod_{j=1}^{q} \left(b_j - \frac{1-\beta_j}{2}\right)^{\beta_j} =: \widehat{Q}_2.$$

Obviously, the inequality $\widehat{Q}_1 \leqq \widehat{Q}_2$ suffices for the inequality $\Psi_2 < \Psi_1$ to hold true.

Finally, if we define the new domain:

$$\begin{aligned} {}_p\mathbb{D}'_q := \Bigg\{ (\mathbf{a}_p, \mathbf{b}_q, \boldsymbol{\alpha}_p, \boldsymbol{\beta}_q) \colon & \prod_{j=1}^{q} \left(1 + \frac{\beta_j}{b_j}\right)^{\beta_j} \leqq \prod_{j=1}^{p} \left(1 + \frac{\alpha_j}{a_j}\right)^{2\alpha_j} \left(1 + \frac{1}{a_j}\right)^{-\alpha_j^2} \\ \text{and} \quad & \prod_{j=1}^{p} a_j^{\alpha_j} \left(1 + \frac{\alpha_j}{a_j}\right)^{\alpha_j - \frac{1}{2}} \leqq \prod_{j=1}^{q} \left(b_j - \frac{1-\beta_j}{2}\right)^{\beta_j} \Bigg\}, \end{aligned} \tag{35}$$

then we can deduce the following result.

Corollary. *Let*

$$(\mathbf{a}_p, \mathbf{b}_q, \boldsymbol{\alpha}_p, \boldsymbol{\beta}_q) \in {}_p\mathbb{D}'_q$$
$$\left(a_j > \frac{1-\alpha_j}{2}; b_k > \frac{1-\beta_j}{2};\ \alpha_j, \beta_k \in [0, 1]; j = 1, \ldots, p; k = 1, \ldots, q\right).$$

Then the following two-sided exponential inequality holds true:

$$\Psi_0 \mathrm{e}^{\Psi_1 \Psi_0^{-1} |x|} \leqq {}_p\Psi_q \left[\begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \middle| z \right] \leqq \Psi_0 - \Psi_1 \left(1 - \mathrm{e}^{|x|}\right) \quad (x \in \mathbb{R}) \tag{36}$$

with the equalities when $x = 0$.

Remark 1. The previous exposition does not cover the case when $\alpha \geqq 1$. In this respect, we notice that the relationship:

$$\Gamma(a+\alpha) = (a+\alpha-1)_{[\alpha]} \Gamma\,(a + \{\alpha\}) \quad (\alpha \geqq 1)$$

reduces the case of a general α to the case of Lemmas 2 and 3, since (by definition) $\{\alpha\} \in [0, 1)$. However, we can list another set of suitable estimates for Gurland's ratio, derived by Kečkić and Vasić [14], in which (by extending the range of α) we lose the unit interval from the domain of a. In our setting, the result of Kečkić and Vasić [14] reads as follows:

$$\frac{a^a\,(a+2\alpha)^{a+2\alpha}}{(a+\alpha)^{a+\alpha}} \leqq T(a, a+2\alpha) \leqq \frac{(a-1)^{a-1}\,(a+2\alpha-1)^{a+2\alpha-1}}{(a+\alpha-1)^{2(a+\alpha-1)}}, \tag{37}$$

provided that

$$\min\{a, a+2\alpha\} > 1. \quad \square$$

4. Two-sided exponential inequalities for $\mathfrak{B}_{\lambda,\mu}({}_{p+1}\Psi_q; \mathbf{c}; r)$ and $\widetilde{\mathfrak{B}}_{\lambda,\mu}({}_{p+1}\Psi_q; \mathbf{c}; r)$

It is clear by Theorems 1 and 2 that, in order to obtain exponential inequalities for the Mathieu-type series (5) and its *alternating* variant (6), we are firstly confronted by the problem of estimating $\mathfrak{J}_{\mathbf{c}}^{\Psi}(u, v)$ and $\widetilde{\mathfrak{J}}_{\mathbf{c}}^{\Psi}(u, v)$, but now for ${}_{p+1}\Psi_q$. Indeed, we have already proved that

$$\mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) = \mathfrak{J}_{\mathbf{c}}^{\Psi}(\lambda+1, \mu) + \mu\, \mathfrak{J}_{\mathbf{c}}^{\Psi}(\lambda, \mu+1)$$

and

$$\widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) = \widetilde{\mathfrak{J}}_{\mathbf{c}}^{\Psi}(\lambda+1, \mu) + \mu\widetilde{\mathfrak{J}}_{\mathbf{c}}^{\Psi}(\lambda, \mu+1).$$

Therefore, by means of Theorem 4, and keeping (11) and (15) in mind, we have the following result.

Theorem 5. *Let*

$$(\mathbf{a}_{p+1}, \mathbf{b}_q, \boldsymbol{\alpha}_{p+1}, \boldsymbol{\beta}_q) \in {}_{p+1}\mathbb{D}_q(Q, T) \quad (a_{p+1} = u;\ \alpha_{p+1} = 1).$$

Then

$$L_1 \leqq \mathfrak{J}_{\mathbf{c}}^{\Psi}(u, v) \leqq R_1 \quad \textit{and} \quad \widetilde{L}_1 \leqq \widetilde{\mathfrak{J}}_{\mathbf{c}}^{\Psi}(u, v) \leqq \widetilde{R}_1, \tag{38}$$

where

$$L_1 := \Psi_0 \int_{c_1}^{\infty} \frac{[c^{-1}(x)]}{x^u (x+r)^v}\, e^{\frac{r\Psi_1}{\Psi_0 x}} dx, \tag{39}$$

$$R_1 := (\Psi_0 - \Psi_1) \int_{c_1}^{\infty} \frac{[c^{-1}(x)]}{x^u (x+r)^v} dx + \frac{\Psi_1}{r^{u+v-1}} \int_0^{\frac{r}{c_1}} \frac{[c^{-1}(\frac{r}{x})]}{(1+x)^v}\, x^{u+v-2} e^x dx, \tag{40}$$

$$\widetilde{L}_1 := \Psi_0 \int_{c_1}^{\infty} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)}{x^u (x+r)^v}\, e^{\frac{r\Psi_1}{\Psi_0 x}} dx, \tag{41}$$

$$\widetilde{R}_1 := (\Psi_0 - \Psi_1) \int_{c_1}^{\infty} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)}{x^u (x+r)^v} dx + \frac{\Psi_1}{r^{u+v-1}} \int_0^{\frac{r}{c_1}} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(\frac{r}{x})]\right)}{(1+x)^v}\, x^{u+v-2} e^x dx, \tag{42}$$

and

$$\Psi_0 := \Psi_0(u) = \frac{\Gamma(u) \prod_{j=1}^{p} \Gamma\left(a_j\right)}{\prod_{j=1}^{q} \Gamma\left(b_j\right)} \quad \textit{and} \quad \Psi_1 := \Psi_1(u) = \frac{\Gamma(u+1) \prod_{j=1}^{p} \Gamma\left(a_j + \alpha_j\right)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j\right)}. \tag{43}$$

Because (38) does not completely describe the behaviour of the lower and upper bounding functions, we have to specify the asymptotics of $c(x)$ for large x, bearing the convergence of the underlying integrals in mind. Therefore, by assuming that

$$c(x) = \mathcal{O}\left(x^{1/(u+v-1-\epsilon)}\right) \quad (x \to \infty; \epsilon > 0) \tag{44}$$

we easily see that all of the constituting integrals for $L_1, \widetilde{L}_1, R_1, \widetilde{R}_1$ converge for this decay rate. We, therefore, prescribe in what follows that $c(x)$ has the asymptotics indicated in (44).

Theorem 6. *Let*

$$(\mathbf{a}_{p+1}, \mathbf{b}_q, \boldsymbol{\alpha}_{p+1}, \boldsymbol{\beta}_q) \in {}_{p+1}\mathbb{D}_q(Q, T) \quad \left(a_{p+1} = u; \alpha_{p+1} = 1\right)$$

and

$$c(x) \leqq K_c\, x^{1/(u+v-1-\epsilon)} \quad (x \to \infty;\ \epsilon > 0; K_c \in \mathbb{R}_+). \tag{45}$$

Then

$$L_2 \leqq \mathfrak{J}_{\mathbf{c}}^{\Psi}(u, v) \leqq R_2, \tag{46}$$

where

$$L_2 := \frac{\Psi_0}{c_1^{u+v-1}(u+v-1)} \left(1 + \frac{u+v-1}{u+v} \frac{r}{c_1}\right)^{-v} + \frac{r\Psi_1}{c_1^{u+v}(u+v)} \left(1 + \frac{u+v}{u+v+1} \frac{r}{c_1}\right)^{-v} \tag{47}$$

and

$$R_2 := \frac{\Psi_1}{c_1^{\epsilon}(\epsilon+1)\,K_c^{u+v-1-\epsilon}}\left(\frac{1}{\epsilon}+\frac{c_1^v}{(c_1+r)^v}+\frac{e^{r/c_1}-1}{\epsilon+2}\left(1+\frac{(\epsilon+1)c_1^v}{(c_1+r)^v}\right)\right) + \frac{\Psi_0-\Psi_1}{r^{\epsilon}K_c^{u+v-1-\epsilon}}B(\epsilon, v-\epsilon) \quad (\epsilon - v \notin \mathbb{N}_0;\, r\in[0,c_1]). \tag{48}$$

Moreover, without any growth assumption upon $c(x)$,

$$\widetilde{\mathfrak{I}}_{\mathbf{c}}^{\Psi}(u,v) \leqq \frac{\Psi_1}{c_1^{u+v-1}(u+v)}\left(\frac{1}{u+v-1}+\frac{c_1^v}{(c_1-r)^v}+\frac{e^{r/c_1}-1}{u+v-1}\left(1+\frac{(u+v)c_1^v}{(c_1-r)^v}\right)\right) + \frac{\Psi_0-\Psi_1}{r^{u+v-1}}B(1-u, u+v-1) \quad (u<1;\, u+v-1>0;\, r\in[0,c_1]). \tag{49}$$

Here $B(\alpha,\beta)$ given by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\mathrm{d}t \quad (\min\{\Re(\alpha),\Re(\beta)\}>0) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\beta,\alpha)$$

is the classical Beta function.

Proof. By noting that

$$[c^{-1}(x)] \geqq 1 \quad \text{when } x \geqq c_1$$

(in view of the monotonous character of the sequence $\mathbf{c}$) and that

$$e^x \geqq 1+x \quad (x\in\mathbb{R}),$$

we now make use of the following known formula [15, p. 313, Eq. 3.194(1)]:

$$\int_0^A \frac{x^{\mu-1}\mathrm{d}x}{(1+x)^{\nu}} = \frac{A^{\mu}}{\mu}\,{}_2F_1\left[\begin{matrix}\nu,\mu\\ \mu+1\end{matrix}\middle| -A\right] \quad (|\arg(1+A)|<\pi;\, \Re(\mu)>0) \tag{50}$$

and Luke's inequality [10, Theorem 13, Eq. (4.20)]:

$$\frac{1}{(1+\theta x)^{\sigma}} \leqq {}_{p+1}F_p\left[\begin{matrix}\sigma, a_p\\ b_p\end{matrix}\middle| -x\right] \leqq 1-\theta+\frac{\theta}{(1+x)^{\sigma}} \quad \left(\theta=\prod_{j=1}^{p}\frac{a_j}{b_j};\, x>0;\, \sigma>0;\, b_j\geqq a_j\ (j=1,\ldots,p)\right). \tag{51}$$

Putting $p=1$ in (51), we thus get

$$\begin{aligned}\mathfrak{I}_{\mathbf{c}}^{\Psi}(u,v) &\geqq \frac{\Psi_0^{u+v}}{r^{u+v-1}\Psi_1^{u-1}}\int_0^{\frac{r\Psi_1}{c_1\Psi_0}}\frac{x^{u+v-2}\,(1+x)}{(\Psi_1+\Psi_0x)^v}\mathrm{d}x\\ &= \frac{\Psi_0}{c_1^{u+v-1}(u+v-1)}\,{}_2F_1\left[\begin{matrix}v, u+v-1\\ u+v\end{matrix}\middle| -\frac{r}{c_1}\right] + \frac{r\Psi_1}{c_1^{u+v}(u+v)}\,{}_2F_1\left[\begin{matrix}v, u+v\\ u+v+1\end{matrix}\middle| -\frac{r}{c_1}\right]\\ &\geqq \frac{\Psi_0}{c_1^{u+v-1}(u+v-1)}\left(1+\frac{u+v-1}{u+v}\frac{r}{c_1}\right)^{-v} + \frac{r\,\Psi_1}{c_1^{u+v}(u+v)}\left(1+\frac{u+v}{u+v+1}\frac{r}{c_1}\right)^{-v}.\end{aligned}$$

This proves the lower bound asserted by Theorem 6.

Denoting by R_{11} and R_{12} the first and the second integrals on the right-hand side of (40), respectively, straightforward calculation would yield

$$R_{11} = \int_0^{\infty}\frac{[c^{-1}(x)]}{x^u(x+r)^v}\mathrm{d}x \leqq \frac{1}{K_c^{\star}}\int_0^{\infty}\frac{x^{v-1-\epsilon}\mathrm{d}x}{(x+r)^v} = \frac{B(\epsilon, v-\epsilon)}{K_c^{\star}r^{\epsilon}} \quad (K_c^{\star} := K_c^{u+v-1-\epsilon}). \tag{52}$$

Since R_{12} contains e^x, we take its arc one estimates by the secant on $[0, \frac{r}{c_1}]$, that is,

$$e^x \leqq 1+\frac{c_1}{r}\left(e^{r/c_1}-1\right)x \quad \left(x\in\left[0,\frac{r}{c_1}\right]\right). \tag{53}$$

We thus conclude that

$$\begin{aligned} R_{12} &= \int_0^{r/c_1} \frac{[c^{-1}(r/x)]}{(1+x)^v} x^{u+v-2} \mathrm{e}^x \mathrm{d}x \leqq \left(\frac{r}{K_c}\right)^{u+v-1-\epsilon} \int_0^{r/c_1} \frac{x^{\epsilon-1}\,\mathrm{e}^x}{(1+x)^v} \mathrm{d}x \\ &= \left(\frac{r}{K_c}\right)^{u+v-1-\epsilon} \int_0^{r/c_1} \frac{x^{\epsilon-1}}{(1+x)^v} \mathrm{d}x + \frac{r^{u+v-2-\epsilon} c_1 (\mathrm{e}^{r/c_1}-1)}{K_c^{u+v-1-\epsilon}} \int_0^{r/c_1} \frac{x^{\epsilon}}{(1+x)^v} \mathrm{d}x. \end{aligned}$$

Applying (50) with

$$\mu = \epsilon, \qquad A = \frac{r}{c_1}, \qquad \text{and} \qquad \nu = v,$$

we express the upper bound for R_{12} in terms of the Gaussian hypergeometric function ${}_2F_1$ as follows:

$$R_{12} \leqq \frac{r^{u+v-1}}{c_1^{\epsilon} K_c^{\star}} \left(\epsilon^{-1}\, {}_2F_1 \left[\begin{matrix} v, \epsilon \\ \epsilon+1 \end{matrix} \middle| -\frac{r}{c_1} \right] + \frac{\mathrm{e}^{r/c_1}-1}{\epsilon+1} {}_2F_1 \left[\begin{matrix} v, \epsilon+1 \\ \epsilon+2 \end{matrix} \middle| -\frac{r}{c_1} \right] \right) \quad (r \in [0, c_1]).$$

Finally, by means of (51), we finish the evaluation procedure resulting in the inequality:

$$R_{12} \leqq \frac{r^{u+v-1}}{c_1^{\epsilon}\,(\epsilon+1)\, K_c^{u+v-1-\epsilon}} \left(\epsilon^{-1} + \frac{c_1^v}{(c_1+r)^v} + \frac{\mathrm{e}^{r/c_1}-1}{\epsilon+2} \left(1 + \frac{(\epsilon+1) c_1^v}{(c_1+r)^v} \right) \right). \tag{54}$$

Inserting (52) and (54) into (40), we complete the proof of the upper bound in (46).

It remains to give a proof of the second main inequality (49). In this case, combining the obvious estimate:

$$0 \leqq \sin^2(\theta) \leqq 1 \quad (\theta \in \mathbb{R})$$

with (53), we obtain

$$\begin{aligned} \widetilde{\mathfrak{I}}_{\mathbf{c}}^{\Psi}(u, v) &\leqq (\Psi_0 - \Psi_1) \int_0^{\infty} \frac{\mathrm{d}x}{x^u (x+r)^v} + \frac{\Psi_1}{r^{u+v-1}} \int_0^{r/c_1} \frac{x^{u+v-2} \mathrm{e}^x}{(1+x)^v} \mathrm{d}x \\ &= \frac{\Psi_0 - \Psi_1}{r^{u+v-1}} B(1-u, u+v-1) + \frac{\Psi_1}{c_1^{u+v-1}\,(u+v-1)} {}_2F_1 \left[\begin{matrix} v, u+v-1 \\ u+v \end{matrix} \middle| -\frac{r}{c_1} \right] \\ &\quad + \frac{\Psi_1 (\mathrm{e}^{r/c_1}-1)}{c_1^{u+v-1}\,(u+v)} {}_2F_1 \left[\begin{matrix} v, u+v \\ u+v+1 \end{matrix} \middle| -\frac{r}{c_1} \right]. \end{aligned}$$

Now, by applying Luke's formula (51) once more, we evaluate the ${}_2F_1[\cdot]$-terms in the last relation. Some obvious calculations lead to the asserted upper bound. □

Remark 2. The above-derived bounds for $\mathfrak{I}_{\mathbf{c}}^{\Psi}(u, v)$ and $\widetilde{\mathfrak{I}}_{\mathbf{c}}^{\Psi}(u, v)$ are neither unique nor sharp. Because

$$\left[c^{-1}(x)\right] \equiv 0 \quad (x \in [0, c_1)),$$

we can enlarge the integration domain of R_{11} from $[c_1, \infty)$ to $\mathbb{R}_+$ without any loss of generality. But, by evaluating that integral, we get the Beta function, while (in the first case) we arrive at the *incomplete Beta function* or, equivalently, a *special* Gaussian ${}_2F_1[\cdot]$ expression. We have just chosen the simpler way here.

Theorem 7. *Let*

$$(\mathbf{a}_{p+1}, \mathbf{b}_q, \boldsymbol{\alpha}_{p+1}, \boldsymbol{\beta}_q) \in {}_{p+1}\mathbb{D}_q(Q, T) \quad (a_{p+1} = \lambda;\ \alpha_{p+1} = 1).$$

Also let

$$\lambda, \mu > 0 \quad \text{and} \quad r \in [0, c_1].$$

Then

$$L_3 \leqq \mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) \leqq R_3 \quad \text{and} \quad \widetilde{L}_3 \leqq \widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) \leqq \widetilde{R}_3, \tag{55}$$

where

$$L_3 := \Psi_0 \int_{c_1}^{\infty} \frac{[c^{-1}(x)]}{x^{\lambda}(x+r)^{\mu}} \left(\frac{\lambda}{x} \mathrm{e}^{\frac{\Psi_1 r}{\lambda \Psi_0 x}} + \frac{\mu}{x+r} \right) \mathrm{e}^{\frac{\Psi_1 r}{\Psi_0 x}} \mathrm{d}x, \tag{56}$$

$$R_3 := \int_{c_1}^{\infty} \frac{[c^{-1}(x)]}{x^{\lambda}(x+r)^{\mu}} \left(\frac{\lambda(\Psi_0 - \Psi_1) - \Psi_1}{x} + \frac{\mu(\Psi_0 - \Psi_1)}{x+r} \right) dx + \frac{\Psi_1}{r^{\lambda+\mu}} \int_0^{r/c_1} \frac{[c^{-1}(r/x)]\, x^{\lambda+\mu-1}}{(1+x)^{\mu}} \left(\lambda + 1 + \frac{\mu}{1+x} \right) e^x dx, \tag{57}$$

$$\widetilde{L}_3 := \Psi_0 \int_{c_1}^{\infty} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)}{x^{\lambda}(x+r)^{\mu}} \left(\frac{\lambda}{x} e^{\frac{\Psi_1 r}{\lambda \Psi_0 x}} + \frac{\mu}{x+r} \right) e^{\frac{\Psi_1 r}{\Psi_0 x}} dx, \tag{58}$$

$$\widetilde{R}_3 := \int_{c_1}^{\infty} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(x)]\right)}{x^{\lambda}(x+r)^{\mu}} \left(\frac{\lambda(\Psi_0 - \Psi_1) - \Psi_1}{x} + \frac{\mu(\Psi_0 - \Psi_1)}{x+r} \right) dx + \frac{\Psi_1}{r^{\lambda+\mu}} \int_0^{r/c_1} \frac{\sin^2\left(\frac{\pi}{2}[c^{-1}(r/x)]\right) x^{\lambda+\mu-1}}{(1+x)^{\mu}} \left(\lambda + 1 + \frac{\mu}{1+x} \right) e^x dx. \tag{59}$$

Here the constants Ψ_0 and Ψ_1 are given by

$$\Psi_0 := \Psi_0(\lambda) = \frac{\Gamma(\lambda) \prod_{j=1}^{p} \Gamma(a_j)}{\prod_{j=1}^{q} \Gamma(b_j)} \quad \text{and} \quad \Psi_1 := \Psi_1(\lambda) = \frac{\Gamma(\lambda+1) \prod_{j=1}^{p} \Gamma(a_j + \alpha_j)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j)}.$$

Proof. The proof of Theorem 7 is essentially a synthesis of previous results, except for the fact that the constants Ψ_k $(k = 0, 1)$ change according to $u \in \{\lambda, \lambda + 1\}$ in (12) and (14). But, by using the familiar relationship:

$$\Gamma(x+1) = x\Gamma(x),$$

we have

$$\Psi_0(\lambda+1) = \lambda \Psi_0(\lambda) \quad \text{and} \quad \Psi_1(\lambda+1) = (\lambda+1)\Psi_1(\lambda).$$

By these observations, we can complete the proof of the above-asserted results immediately. □

At the end of this section, we prescribe a reasonable polynomial growth order behaviour upon $c(x)$. The lower bound of $\widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right)$ will then become zero. However, all three other bounds are of interest, too.

Theorem 8. *Let*

$$(\mathbf{a}_{p+1}, \mathbf{b}_q, \boldsymbol{\alpha}_{p+1}, \boldsymbol{\beta}_q) \in {}_{p+1}\mathbb{D}_q(Q, T) \quad (a_{p+1} = \lambda;\ \alpha_{p+1} = 1).$$

Suppose also that

$$c(x) \leqq K_c x^{1/(\lambda+\mu-\epsilon)} \quad (\epsilon > 0;\ K_c \in \mathbb{R}_+). \tag{60}$$

Then

$$L_4 \leqq \mathfrak{B}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) \leqq R_4, \tag{61}$$

where

$$L_4 := \frac{\Psi_0}{(\lambda+\mu)c_1^{\lambda+\mu}} \left(1 + \frac{(\lambda+\mu)r}{(\lambda+\mu+1)c_1} \right)^{-\mu} \left(\lambda + \frac{\mu(\lambda+\mu+1)c_1}{(\lambda+\mu)(c_1+r)+c_1} \right) \tag{62}$$

and

$$R_4 := \frac{\lambda+\mu}{(\epsilon+1)c_1^{\epsilon} K_c^{\star}} \left(\frac{\Psi_0}{\epsilon} + \frac{\lambda+\mu+1}{\lambda+\mu} \frac{c_1 \Psi_1 (e^{r/c_1} - 1)}{(\epsilon+2)r} \right) + \frac{c_1^{\mu-\epsilon}}{K_c^{\star}(c_1+r)^{\mu}} \left(\lambda + \frac{\mu c_1}{c_1+r} \right) \left(\frac{\Psi_0}{\epsilon+1} + \frac{c_1 \Psi_1 (e^{r/c_1} - 1)}{(\epsilon+2)r} \left(1 + \frac{c_1+r}{(\lambda+\mu)c_1 + \lambda r} \right) \right). \tag{63}$$

Moreover, without prescribing any growth rate for the sequence $\mathbf{c}$*, the following inequality holds true:*

$$0 \leqq \widetilde{\mathfrak{B}}_{\lambda,\mu}\left({}_{p+1}\Psi_q; \mathbf{c}; r\right) \leqq \frac{\Psi_0}{(\lambda+\mu+1)c_1^{\lambda+\mu}} \left(1 + \frac{(2c_1+r)c_1^{\mu}}{(c_1+r)^{\mu+1}} \right). \tag{64}$$

Proof. As in the case of Theorem 7, the proof of the results asserted by Theorem 8 is essentially a synthesis of previous results. So, we give here only its sketch and leave the details involved as an exercise for the interested reader.

By the obvious lower bounds:

$$[c^{-1}(x)] \geqq 1 \quad (x \geqq c_1),$$

we make the following change of variables:

$$rx^{-1} \mapsto x.$$

Finally, by taking into account the obvious inequality:

$$e^{\lambda x} \geqq 1 \quad (\lambda \in \mathbb{R}_+)$$

in the case when

$$x \in \left[0, \frac{r}{c_1}\right] \quad (r \leqq c_1),$$

we get

$$\begin{aligned} L_3 &\geqq \frac{\Psi_0}{r^{\lambda+\mu-1}} \int_0^{r/c_1} \frac{x^{\lambda+\mu-2}}{(1+x)^\mu} \left(\lambda + \frac{\mu x}{1+x}\right) dx \\ &= \frac{\Psi_0 r}{(\lambda+\mu)c_1^{\lambda+\mu}} \left(\lambda\, {}_2F_1\left[\begin{matrix}\mu, \lambda+\mu \\ \lambda+\mu+1\end{matrix}\middle| -\frac{r}{c_1}\right] + \mu\, {}_2F_1\left[\begin{matrix}\mu+1, \lambda+\mu \\ \lambda+\mu+1\end{matrix}\middle| -\frac{r}{c_1}\right]\right) \\ &\geqq \frac{\Psi_0 r}{(\lambda+\mu)c_1^{\lambda+\mu}} \left(1 + \frac{(\lambda+\mu)r}{(\lambda+\mu+1)c_1}\right)^{-\mu} \left(\lambda + \mu\left(1 + \frac{(\lambda+\mu)r}{(\lambda+\mu+1)c_1}\right)^{-1}\right), \end{aligned}$$

which is equivalent to (62). By the way, we have applied here the result (50) and the left-hand inequality in (51).

To achieve the upper bound R_4, we start with estimating the integrand in R_3 by means of the following inequality:

$$\left[c^{-1}(x)\right] \leqq c^{-1}(x) \leqq \left(\frac{x}{K_c}\right)^{\lambda+\mu-\epsilon} \qquad (K_c^{\lambda+\mu-\epsilon} =: K_c^\star).$$

After some routine calculation, we thus deduce that

$$\begin{aligned} R_3 &\leqq \frac{\lambda\Psi_0}{\epsilon c_1^\epsilon K_c^\star}\, {}_2F_1\left[\begin{matrix}\mu, \epsilon \\ \epsilon+1\end{matrix}\middle| -\frac{r}{c_1}\right] + \frac{\mu\Psi_0}{\epsilon\, c_1^\epsilon K_c^\star}\, {}_2F_1\left[\begin{matrix}\mu+1, \epsilon \\ \epsilon+1\end{matrix}\middle| -\frac{r}{c_1}\right] + \frac{(\lambda+1)\Psi_1(e^{r/c_1}-1)}{(\epsilon+1)\, c_1^{\epsilon-1} r K_c^\star}\, {}_2F_1\left[\begin{matrix}\mu, \epsilon+1 \\ \epsilon+2\end{matrix}\middle| -\frac{r}{c_1}\right] \\ &\quad + \frac{\mu\,\Psi_1(e^{r/c_1}-1)}{(\epsilon+1)\, c_1^{\epsilon-1}\, r\, K_c^\star}\, {}_2F_1\left[\begin{matrix}\mu+1, \epsilon+1 \\ \epsilon+2\end{matrix}\middle| -\frac{r}{c_1}\right] =: H_1. \end{aligned}$$

It remains now to make use of Luke's upper bound in (51) to the above hypergeometric terms. This results in the following inequality:

$$\begin{aligned} H_1 &\leqq \frac{(\lambda+\mu)\Psi_0}{\epsilon(\epsilon+1)c_1^\epsilon K_c^\star} + \frac{c_1^\mu \Psi_0}{(\epsilon+1)c_1^\epsilon\, K_c^\star (c_1+r)^\mu} \left(\lambda + \frac{\mu\, c_1}{c_1+r}\right) \\ &\quad + \frac{(\lambda+\mu+1)c_1^{1-\epsilon}\Psi_1(e^{r/c_1}-1)}{(\epsilon+1)(\epsilon+2)\, rK_c^\star} + \frac{c_1^{\mu+1-\epsilon}\,\Psi_1(e^{r/c_1}-1)}{(\epsilon+2)rK_c^\star(c_1+r)^\mu} \left(\lambda + 1 + \frac{\mu c_1}{c_1+r}\right), \end{aligned} \tag{65}$$

which is equivalent to (63).

Finally, by applying the elementary inequality:

$$\sin^2(\theta) \leqq 1 \quad (\theta \in \mathbb{R}_+)$$

to the integrand of R_3, followed by the same tools and estimates as before, this last expression assumes the form:

$$R_3 \leqq \frac{\Psi_0}{c_1^{\lambda+\mu}} \left(\lambda\, {}_2F_1\left[\begin{matrix}\mu, \lambda+\mu \\ \lambda+\mu+1\end{matrix}\middle| -\frac{r}{c_1}\right] + \mu\, {}_2F_1\left[\begin{matrix}\mu+1, \lambda+\mu \\ \lambda+\mu+1\end{matrix}\middle| -\frac{r}{c_1}\right]\right), \tag{66}$$

which leads us to the asserted upper-bound result in (64). □

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