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A Note on the Unified Forms of Triple I Method

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Abstract—Wang and Fu [1] provided the unified forms of the triple I method for fuzzy modus ponens (FMP for short) and fuzzy modus tollens (FMT for short) and found an interesting duality between them. In this paper, we first answer why there exists the duality, and then establish another unified form of the triple I method for FMT under a weaker condition.

Keywords—Fuzzy reasoning, Triple I method, Unified form, Regular operator, Duality.

1. INTRODUCTION

It is well known that the most fundamental forms of fuzzy reasoning are fuzzy modus ponens (briefly, FMP) and fuzzy modus tollens (briefly, FMT), which can be represented as follows:

FMP: for given $A \to B$ (rule) and A^* (input), calculate B^* (output), (1)

FMT: for given $A \to B$ (rule) and B^* (input), calculate A^* (output), (2)

where $A, A^* \in F(U)$ (the set of all fuzzy subsets of universe U) and $B, B^* \in F(V)$ (the set of all fuzzy subsets of universe V). At present, for the above two problems, the method adopted extensively is Zadeh's [2] CRI (Compositional Rule of Inference) method proposed in 1973. To improve the method of CRI, Wang [3] proposed the triple I (the abbreviation of triple implications) method for solving the above FMP and FMT problems which has been discussed in detail

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later in [4–13]. Its basic idea can be summarized as follows: for $\alpha \in [0,1]$, and A, B and A^* (or B^*), our purpose is to seek the optimal fuzzy subset B^* (or A^*) such that, for all $u \in U$ and $v \in V$,

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha. \tag{3}$$

Inequality (3) indicates that the support degree [3] of $A \to B$ to $A^* \to B^*$ should be required to be greater than or equal to α. Fuzzy set B^* (or A^*) is called the α-triple I solution of FMP (or FMT).

Recently, Wang and Fu [1] provided the unified forms of the triple I method for FMP and FMT where diverse implication operators can be employed under the same way and found an interesting duality between them. In the present paper, first, we seek the reason why there exists the duality between the triple I methods for FMP and FMT, and then, by weakening the condition of implication operators, we establish another unified form of the triple I method for FMT and Wang and Fu's unified form of method for FMT can be as its particular case.

The rest of this paper is organized as follows. Section 2 is the preliminaries. In Section 3, we seek the reason why there exists duality between the two unified forms of the triple I method. In Section 4, we try to weaken the condition that the implication satisfies and establish another unified form of the triple I method for FMT. The final section is the conclusion.

2. PRELIMINARIES

In this section, we recall several definitions of fuzzy logical operators on $[0,1]$, along with some properties that we will use in this paper.

DEFINITION 2.1. *A triangular norm (briefly t-norm) on* $[0,1]$ *is any commutative, associative and nondecreasing in each place* $[0,1]^2 \to [0,1]$ *mapping* T *satisfying* $T(1,x) = x$ *for all* $x \in [0,1]$. *A t-norm* T *is said to be left-continuous if it is left-continuous as a two-place mapping.*

DEFINITION 2.2. *The dual of a t-norm* T *on* $[0,1]$ *w.r.t. negation* $'$ *defined by* $x' = 1 - x$ *for all* $x \in [0,1]$ *is defined as* $S(x,y) = T'(x',y')$ *for all* $x, y \in [0,1]$.

Above S defined by T and negation $'$ is a triangular conorm (briefly, t-conorm) on $[0,1]$.

DEFINITION 2.3. *(See [1].) Let* T *be a t-norm on* $[0,1]$ *and* I *be a* $[0,1]^2 \to [0,1]$ *mapping,* (T, I) *is said to be a residual pair, or* T *and* I *are residual to each other, if the following residuation condition holds, for all* $x, y, z \in [0,1]$:

$$T(x,y) \leq z, \qquad \text{if and only if } y \leq I(x,z).$$

DEFINITION 2.4. *An implication on* $[0,1]$ *is any* $[0,1]^2 \to [0,1]$ *mapping* I *satisfying* $I(0,0) = I(0,1) = I(1,1) = 1$ *and* $I(1,0) = 0$. *The residual implication generated by a t-norm* T *is defined as* $I_T(x,y) = \sup\{\gamma \in [0,1] \mid T(x,\gamma) \leq y\}$ *for all* $x, y \in [0,1]$. *We also write* $I(x,y)$ *as* $x \to y$ *for* $x, y \in [0,1]$ *in the following.*

Since the left-continuity of a t-norm on $[0,1]$ is equivalent to the residuation condition, (T, I_T) always forms a residual pair for a left-continuous t-norm T on $[0,1]$. The residual implication I generated by a left-continuous t-norm on $[0,1]$ is called a regular implication [1]. We now list some properties that we will use in this paper, for all $x, y, z \in [0,1]$.

(P1) $x \leq I(y,z) \Leftrightarrow y \leq I(x,z)$.
(P2) $I(T(x,y),z) = I(x, I(y,z))$.

DEFINITION 2.5. *An implication* I *on* $[0,1]$ *is said to be contrapositive w.r.t. negation* $'$ *if* $I(x,y) = I(y',x')$ *holds for all* $x, y \in [0,1]$. *An implication* I *on* $[0,1]$ *is said to be normal if it is regular and contrapositive w.r.t. negation* $'$.

3. DUALITY BETWEEN TRIPLE I METHODS FOR FMP AND FMT

In this section, we seek the reason why there exists duality between the α-triple I methods for FMP and FMT proposed by Wang and Fu in [1].

Wang and Fu [1] established the following unified form of the α-triple I method for FMP (1).

THEOREM 3.1. *(See [1].) Suppose that the implication I in FMP (1) is regular, then the α-triple I solution of FMP (1) can be expressed as follows:*

$$B^*(v) = \sup_{u\in U} T(\alpha, T(A^*(u), I(A(u), B(v)))), \qquad v \in V, \tag{4}$$

where T is the t-norm residual to implication I.

From the above Theorem 3.1, we have the following.

COROLLARY 3.1. *Suppose that the implication I in FMT (2) is normal, then the α-triple I solution of FMT (2) can be expressed as follows:*

$$A^*(u) = \inf_{v\in V} S(\alpha', S(B^*(v), I'(A(u), B(v)))), \qquad u \in U, \tag{5}$$

where T is the t-norm residual to implication I and S is the dual of T w.r.t. negation $'$.

PROOF. Since implication I is normal, it is contrapositive w.r.t. negation $'$. So, the FMT problem (2) is actually equivalent to the FMP problem (1) at this time and inequality (3) becomes as follows, for all $u \in U$ and $v \in V$:

$$(A(u) \to B(v)) \to \left((B^*(v))' \to (A^*(u))'\right) \geq \alpha. \tag{6}$$

From (4) and the involutivity of $'$, we obtain the α-triple I solution of FMT (2) as follows, for all $u \in U$:

$$\begin{aligned} A^*(u) &= \left(\sup_{v\in V} T\left(\alpha, T\left((B^*(v))', I(A(u), B(v))\right)\right)\right)' \\ &= \inf_{v\in V} S(\alpha', S(B^*(v), I'(A(u), B(v)))). \end{aligned}$$ ∎

REMARK 3.1. The above Corollary 3.1 is exactly Wang and Fu's Theorem 4 [1], where the proof was much more complicated than the one given above.

By comparing the α-triple I solution (4) of FMP and the α-triple I solution (5) of FMT, Wang and Fu found an interesting duality between them, i.e., between the systems $(B^*(\text{FMP}); \sup_{u\in U}, A^*, T, \alpha, I)$ and $(A^*(\text{FMT}); \inf_{v\in V}, B^*, S, \alpha', I')$. Precisely speaking, if one replaces symbols in the expression of an α-triple I conclusion of FMP arranged as in the former system by corresponding symbols in the latter system, then one obtains the α-triple I conclusion of FMT and vice versa. From the proof of Corollary 3.1, we know that the duality of the solution (5) to (4) is actually due to the contrapositivity of implication I w.r.t. negation $'$.

4. UNIFIED FORM OF TRIPLE I METHOD FOR FMT

For FMT (2), Wang and Fu [1] established the unified form (5) of the α-triple I solution, where the implication I is required to be normal. In this section, we establish another unified form of the α-triple I solution of FMT by weakening the condition of implication I from normal to regular, and the unified form (5) can be as its particular case.

THEOREM 4.1. *Suppose that implication I is regular, then the α-triple I solution of FMT (2) can be expressed as follows:*

$$A^*(u) = \inf_{v\in V} I(\alpha, I(I(A(u), B(v)), B^*(v))), \qquad u \in U. \tag{7}$$

PROOF. From (7), we know $A^*(u) \leq I(\alpha, I(I(A(u), B(v)), B^*(v)))$ for all $u \in U$ and $v \in V$. Let T be the t-norm residual to I, then from the residuation condition, we further have

$$\begin{aligned} T(\alpha, A^*(u)) &\leq I(I(A(u), B(v)), B^*(v)) \\ &\Longrightarrow T(I(A(u), B(v)), T(\alpha, A^*(u))) \leq B^*(v) \\ &\Longrightarrow T(T(I(A(u), B(v)), A^*(u)), \alpha) \leq B^*(v) \\ &\Longrightarrow \alpha \leq I(T(I(A(u), B(v)), A^*(u)), B^*(v)). \end{aligned}$$

From (P2), we get $\alpha \leq I(I(A(u), B(v)), I(A^*(u), B^*(v)))$. Hence, the A^* determined by (7) satisfies (3).

We now prove that A^* determined by (7) is the largest fuzzy subset of U satisfying (3).

Assume $E \in F(U)$ and $(A(u) \to B(v)) \to (E(u) \to B^*(v)) \geq \alpha$ for all $u \in U$ and $v \in V$, i.e., $I(I(A(u), B(v)), I(E(u), B^*(v))) \geq \alpha$ for all $u \in U$ and $v \in V$. From the residuation condition and properties (P1) and (P2), it follows that for any $u \in U$,

$$\begin{aligned} T(\alpha, I(A(u), B(v))) &\leq I(E(u), B^*(v)), && \text{for all } v \in V, \\ &\Longrightarrow E(u) \leq I(T(\alpha, I(A(u), B(v))), B^*(v)), && \text{for all } v \in V, \\ &\Longrightarrow E(u) \leq I(\alpha, I(I(A(u), B(v)), B^*(v)), && \text{for all } v \in V, \\ &\Longrightarrow E(u) \leq \inf_{v\in V} I(\alpha, I(I(A(u), B(v)), B^*(v)) = A^*(u). \end{aligned}$$

Summarizing the above and according to the principle of the α-triple I for FMT in [1], we know that A^* determined by (7) is the α-triple I solution of FMT. ∎

COROLLARY 4.1. *Assume that implication I is normal, then formula (7) equals (5).*

PROOF. Since implication I is normal, it is regular and contrapositive w.r.t. negation $'$. Assume that T is the t-norm residual to I and S is the dual of T w.r.t. $'$, then, from Theorem 1 in [14], we know that $I(x, y) = T'(x, y')$, i.e., $I(x, y) = S(x', y)$ for all $x, y \in [0, 1]$. So, formula (5) can be easily followed from (7) and $I(x, y) = S(x', y)$. ∎

The above Corollary 4.1 illustrates that Wang and Fu's unified form (5) is the particular case of unified form (7) when implication I is normal.

5. CONCLUSION

In this paper, we have answered why there exists the duality between the triple I methods for FMP and FMT proposed by Wang and Fu [1]. And then, another unified form of the triple I method for FMT has been established where the implication is required to be regular rather than normal. Wang and Fu's unified form of method for FMT can be as its particular case. As is pointed out in [1], there exist many regular implications since a good many left-continuous t-norms exist.

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