JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **150**, 289–306 (1990)

Monotone Semiflows in Scalar Non-Quasi-Monotone Functional Differential Equations

HAL L. SMITH*

Department of Mathematics, Arizona State University, Tempe, Arizona 85287

AND

HORST R. THIEME†

Sonderforschungsbereich 123, Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg, BR Deutschland

Submitted by Stavros N. Busenberg

Received December 18, 1987

By introducing a stronger than pointwise ordering, conditions are found under which scalar functional differential equations generate monotone semiflows even if they are not quasi-monotone. Typically the maximum delay must be the smaller the more quasi-monotonicity is violated. The theory of monotone semiflows is used to show that most solutions converge to equilibrium and that stability of equilibria is essentially the same as for ordinary differential equations. © 1990 Academic Press, Inc

0. INTRODUCTION

In a recent paper [10] one of us applied the ideas of monotone dynamical systems to the functional differential equation

$$x'(t) = f(x_t). \tag{0.1}$$

In (0.1), f is a continuous function from the space $C = C([-\tau, 0], \mathbb{R})$ (with norm $\|\phi\| = \sup\{|\phi(s)|: -\tau \leqslant s \leqslant 0\}$) into the reals and $x_t \in C$ is defined in the usual way by $x_t(s) = x(t+s)$, $-\tau \leqslant s \leqslant 0$. The positive number τ denotes the maximum delay. Here we consider (0.1) to be a scalar equation

* Supported by NSF Grant DMS 8521605. This work began when this author visited Sonderforschungsbereich 123, University of Heidelberg.

† Supported by a Heisenberg scholarship from the Deutsche Forschungsgemeinschaft; visiting Harvey Mudd College, U.S.A., when this work was completed. New address: Department of Mathematics, Arizona State University, Tempe, Arizona 85287.

0022-247X/90 $3.00

for simplicity; systems were considered in [10]. We assume throughout this paper that f is Lipschitz continuous on compact subsets of C. This implies that solutions of (0.1) are uniquely determined by the initial condition $x(s)=\phi(s)$, $-\tau \leqslant s \leqslant 0$, $\phi \in C$, and we write $x(t)=x(t, \phi)$ or $x_t=x_t(\phi)$ for this solution, emphasizing the dependence on the initial data.,

It was shown in [10] that if f satisfies the quasi-monotone condition

(QM) Whenever ϕ_1, $\phi_2 \in C$ satisfy $\phi_1 \leqslant \phi_2$ and $\phi_1(0)=\phi_2(0)$, then $f(\phi_1) \leqslant f(\phi_2)$,

then (0.1) generates a monotone dynamical system on C,

$$\phi \mapsto x_t(\phi), \qquad t \geqslant 0,$$

in the sense that if $\phi_1 \leqslant \phi_2$ then $x_t(\phi_1) \leqslant x_t(\phi_2)$ for $t \geqslant 0$. Inequalities between functions are to be understood to hold in the pointwise sense. If f satisfies an additional condition ((R) in [10]) then a strongly monotone semiflow is generated by (0.1). The results of M. W. Hirsch [4] for strongly monotone dynamical systems can then be applied to (0.1). Roughly speaking, these results imply that most solutions of (0.1) converge to equilibrium. In addition, the theory of positive semigroups (see, e.g., [6]) implies that the stability of steady states of (0.1) is determined by a real characteristic root.

The quasi-monotone condition (QM) is quite special. Consider the delay equation:

$$x'(t)=f(x(t), x(t-\tau)). \tag{0.2}$$

Then (QM) holds provided f satisfies

$$\frac{\partial f}{\partial y}(x, y) \geqslant 0 \tag{0.3}$$

and the strong monotonicity requirement is that strict inequality hold. If strict inequality holds, then the behavior of solutions of (0.2) is essentially the same as for the ordinary differential equation

$$x'(t)=f(x(t), x(t)).$$

In this paper we extend the applicability of monotone methods for (0.1) by determining a new sense in which (0.1) can define a monotone dynamical system. More precisely, we define a new partial ordering which is preserved by the semiflow associated with (0.1) under appropriate conditions. This new partial ordering does not make C a strongly ordered space but it does strongly order the dense subspace consisting of the functions satisfying a Lipschitz condition. We exploit this fact to show that, under appropriate conditions, the semiflow generated by (0.1) is strongly order preserving on C with respect to the new partial order and hence most orbits converge to an equilibrium. Crucial to our proof of this result are results of the authors in [14]. Moreover, the stability of an equilibrium is essentially the same as for the ordinary differential equation

$$x'(t) = \hat{f}(x(t)), \qquad \hat{f}(x) = f(\hat{x})$$

with $\hat{x}$ denoting the function in C which is identically equal to $x \in \mathbb{R}$.

The special case that an invariant attracting region contains only one equilibrium has been considered by one of us some years ago [11] using essentially the same technique but without referring to the alternative ordering introduced in this paper which makes our results in [14] applicable. This ordering has been used before for just the opposite purpose by Hadeler and Tomiuk [2]. Whereas we use it in order to exclude the existence of (attracting) periodic orbits, they use it to prove the existence of periodic solutions.

For the special case (0.2), our results apply when either of the following hold:

$$\begin{aligned}
&\text{(a)}\quad L_2 \geqslant 0, \quad \text{or}\\
&\text{(b)}\quad L_2 < 0 \quad \text{but} \quad L_1 + L_2 > 0, \quad \text{or}\\
&\text{(c)}\quad L_2 < 0, \quad L_1 + L_2 = 0, \quad \text{and} \quad \tau\,|L_2| < 1, \quad \text{or}\\
&\text{(d)}\quad L_2 < 0, \quad L_1 + L_2 < 0, \quad \tau\,|L_2| < 1, \quad \text{and} \quad \tau L_1 - \ln(\tau\,|L_2|) > 1,
\end{aligned} \tag{0.4}$$

where

$$L_1 = \inf \frac{\partial f}{\partial x}$$

$$L_2 = \inf \frac{\partial f}{\partial y}$$

are assumed to be finite. First note from (a) that (0.3) is sufficient and no strong monotonicity is required. Further, (0.3) can be relaxed at the expense of restrictions on the size of the delay τ and the partial derivative of f with respect to x. From a different point of view, if f in (0.1) satisfies a Lipschitz condition

$$|f(\phi) - f(\psi)| \leqslant L\,\|\phi - \psi\|$$

then our results apply provided

$$\tau L e < 1,$$

where e is the base of the exponential function. In other words, (0.1) defines a monotone dynamical system, in a certain sense, provided the delay is sufficiently small (and f satisfies the Lipschitz condition). This is an intuitive result since scalar ordinary differential equations generate monotone dynamical systems on $\mathbb{R}$.

Recent remarkable work of J. Mallet–Paret and G. Sell [8] on (0.2) is relevant to our study. They show that if the strict inequality holds in (0.3) or if the reverse strict inequality holds then the Poincaré–Bendixson alternative holds for limit sets of bounded orbits: a limit set is one of (a) an equilibrium (b) a nontrivial periodic orbit or (c) a structure consisting of a nonempty set E of equilibria and a set of orbits whose positive and negative limit sets are elements of E.

1. Monotonicity

Consider the scalar delay equation

$$x'(t) = f(x_t), \tag{1.1}$$

where $f\colon C \to \mathbb{R}$ is continuous. Recall that to ensure that solutions of initial value problems associated with (1.1) are unique, we assume that f satisfies a Lipschitz condition on each compact subset of C (see [3]). We introduce the following hypothesis concerning f:

(M) There exists $\mu \geqslant 0$ such that whenever $\psi_1, \psi_2 \in C$ and satisfy $\psi_1 \leqslant \psi_2$ and $(\psi_2(s) - \psi_1(s))e^{\mu s}$ is nondecreasing on $[-\tau, 0]$ then

$$\mu(\psi_2(0) - \psi_1(0)) + f(\psi_2) - f(\psi_1) \geqslant 0.$$

Then we have

Proposition 1.1. *Suppose* (M) *holds. If $\phi_1 \leqslant \phi_2$ and $(\phi_2(s) - \phi_1(s))e^{\mu s}$ is nondecreasing, then*

$$x(t, \phi_1) \leqslant x(t, \phi_2)$$

and

$$(x(t, \phi_2) - x(t, \phi_1))e^{\mu t}$$

is nondecreasing for all $t \geqslant 0$ for which both solutions are defined.

Proof. Fix $\varepsilon > 0$ and let $f_\varepsilon\colon C \to \mathbb{R}$ be defined by $f_\varepsilon(\phi) = f(\phi) + \varepsilon\phi(0)$. Given that f satisfies (M), we will establish the result for the solutions of (1.1) with f_ε replacing f and then apply a limit argument. Write $x(t, \phi_i, \varepsilon)$,

$i=1,2$, for the solutions of (1.1) with f_ε replacing f and with initial data $x_0=\phi_i$. Let $t_1\geqslant 0$ be the maximum number such that $x(t,\phi_1,\varepsilon)\leqslant x(t,\phi_2,\varepsilon)$ and $e^{\mu t}(x(t,\phi_2,\varepsilon)-x(t,\phi_1,\varepsilon))$ is monotone nondecreasing in $t\in[0,t_1)$. If t_1 is not the right endpoint of the intersection of the maximal intervals of existence, we may assume that $x(t_1,\phi_2,\varepsilon)>x(t_1,\phi_1,\varepsilon)$. Otherwise we have $\phi_2=\phi_1$ and equality holds beyond t_1. Now, by (M),

$$\begin{aligned}\left.\frac{d^+}{dt}\right|_{t=t_1} & e^{\mu t}[x(t,\phi_2,\varepsilon)-x(t_1,\phi_1,\varepsilon)]\\ &\geqslant e^{\mu t}\{(\mu+\varepsilon)[x(t_1,\phi_2,\varepsilon)-x(t_1,\phi_1,\varepsilon)]+f(x_{t_1}(\phi_2,\varepsilon))-f(x_{t_1}(\phi_1,\varepsilon))\}\\ &\geqslant \varepsilon e^{\mu t}[x(t_1,\phi_2,\varepsilon)-x(t_1,\phi_1,\varepsilon)]>0.\end{aligned}$$

Hence $x(t,\phi_1,\varepsilon)\leqslant x(t,\phi_2,\varepsilon)$ and $e^{\mu t}(x(t,\phi_2,\varepsilon)-x(t,\phi_1,\varepsilon))$ is monotone nondecreasing beyond t_1 in contradiction to the maximality of t_1 and so these properties must hold as long as both solutions exist. Letting $\varepsilon\to 0^+$, $f_\varepsilon\to f$ and continuity of solutions with respect to the data implies that $x(t,\phi_i,\varepsilon)\to x(t,\phi_i)$ as $\varepsilon\to 0^+$ uniformly on compact subsets contained in the maximal intervals of existence. It is evident that the conclusion of the proposition now follows.

Several remarks are appropriate at this point.

Remark 1. It is sufficient for (M) to hold only on some positively invariant open subset U of C for the conclusions of Proposition 1.1 to hold in U.

Remark 2. It follows from Proposition 1.1 that either $\phi_1=\phi_2$ so $x(t,\phi_1)=x(t,\phi_2)$ or $\phi_1(0)<\phi_2(0)$ and $x(t,\phi_1)<x(t,\phi_2)$ for $t>0$ belonging to the domain of existence of both solutions.

Remark 3. Proposition 1.1 can be extended to systems. An appropriate (M) is: there exists $\mu_i\geqslant 0$, $1\leqslant i\leqslant n$, such that whenever $\phi,\psi\in C$, $\phi_i\leqslant\psi_i$, and $[\psi_i(s)-\phi_i(s)]\,e^{\mu_i s}$, $1\leqslant i\leqslant n$, is nondecreasing, then

$$\mu_i(\psi_i(0)-\phi_i(0))+f_i(\psi)-f_i(\phi)\geqslant 0,\qquad 1\leqslant i\leqslant n.$$

We now interpret Proposition 1.1 in an appropriate manner. Define

$$\bar{K}_\mu=\{\phi\in C\colon \phi\geqslant 0 \text{ and } \phi(s)e^{\mu s} \text{ is nondecreasing}\}.$$

$\bar{K}_\mu$ is a closed cone in C. As such, it defines a partial order on C, which we write as $\leqslant_\mu$, by $\phi_1\leqslant_\mu\phi_2$ if and only if $\phi_2-\phi_1\in\bar{K}_\mu$, i.e., $\phi_1\leqslant\phi_2$ and $(\phi_2(s)-\phi_1(s))e^{\mu s}$ is nondecreasing on $[-\tau,0]$. We write $\phi_1<_\mu\phi_2$ if $\phi_1\leqslant_\mu\phi_2$ and $\phi_1\neq\phi_2$.

With this notation, we interpret Proposition 1.1 as follows.

COROLLARY 1.2. *Suppose* (M) *holds and let* ϕ_i, $i=1, 2$, *belong to* C. *Then*

(a) $\phi_1 \leqslant_\mu \phi_2$ *implies* $x_t(\phi_1) \leqslant_\mu x_t(\phi_2)$ *for* $t \geqslant 0$.

(b) $\phi_1 <_\mu \phi_2$ *implies* $x_t(\phi_1) <_\mu x_t(\phi_2)$ *for* $t \geqslant 0$.

Corollary 1.2 means that the semiflow generated by (1.1), Φ_t, defined for suitable $\phi \in C$ by $\Phi_t(\phi) = x_t(\phi)$, is a monotone semiflow in the sense of the ordering $\leqslant_\mu$.

Let us note the following sufficient conditions for (M) to hold:

(L^-) There exists $L>0$ such that for all $\psi_1, \psi_2 \in C$, $\psi_1 \leqslant \psi_2$,

$$f(\psi_2) - f(\psi_1) \geqslant -L \, \|\psi_2 - \psi_1\|,$$

where $e\tau L \leqslant 1$.

Thus, if f is Lipschitz continuous on C, or on some positively invariant open subset, and if the delay is sufficiently small then (M) holds. To see this note that if $\psi_1 \leqslant_\mu \psi_2$ then $f(\psi_2) - f(\psi_1) \geqslant -L(\psi_2(0) - \psi_1(0))e^{\mu\tau}$ so (M) holds if $\mu - Le^{\mu\tau} \geqslant 0$.

Consider the special case of (1.1):

$$x'(t) = f(x(t), x(t-\tau)). \tag{1.2}$$

Then (M) holds if

$$\mu(x_2 - x_1) + f(x_2, y_2) - f(x_1, y_1) \geqslant 0$$

whenever

$$x_2 - x_1 \geqslant (y_2 - y_1)e^{-\mu\tau} \geqslant 0.$$

This in turn holds if there exist L_1 and L_2 such that whenever $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ then

$$f(x_2, y_2) - f(x_1, y_1) \geqslant L_1(x_2 - x_1) + L_2(y_2 - y_1)$$

and

$$\mu + L_1 + L_2^- e^{\mu\tau} \geqslant 0, \tag{1.3}$$

where $u^- = \min\{u, 0\}$. It is easy to verify that (1.3) holds if either

$$\begin{array}{llll} \text{(a)} & L_2 \geqslant 0, & \text{or} & \\ \text{(b)} & L_2 < 0 & \text{but} \quad L_1 + L_2 \geqslant 0, & \text{or} \\ \text{(c)} & L_2 < 0, & L_1 + L_2 < 0, \quad \tau\,|L_2| < 1, & \text{and} \quad \tau L_1 - \ln(\tau\,|L_2|) \geqslant 1. \end{array} \tag{1.4}$$

Of course, if $L_2 \geqslant 0$ then (1.2) satisfies the quasi-monotone hypothesis (QM). Observe that if (a) fails (1.4) may still hold if L_1 is large enough that (b) holds. Also, if f is independent of $x(t)$ then (1.4) holds if $L_2 \geqslant 0$ or $L_2 < 0$ and $\tau\,|L_2| < e^{-1}$. In any case, (1.4) holds if τ is sufficiently small.

Finally, we observe that if f satisfies (QM) and (L^-) then

$$f(\psi_2) - f(\psi_1) + L(\psi_2(0) - \psi_1(0)) \geqslant 0,$$

whenever $\psi_1 \leqslant \psi_2$. It follows that (QM) and (L^-) imply that (M) holds for every $\mu \geqslant L$.

The cones $\bar{K}_\mu$, $\mu \geqslant 0$, have empty interior in C and hence C is not strongly ordered by the ordering $\leqslant_\mu$ in the sense of Hirsch. There is a standard procedure for rectifying this deficiency which we now follow (see, e.g., Amann [1]). Let ξ be the element of $\bar{K}_\mu$ given by $\xi(s) = e^{(1-\mu)s}$ on $[-\tau, 0]$. Fix $\mu \geqslant 0$ and define

$$X = \{\phi \in C\text{: there exists } \beta \geqslant 0 \text{ such that } -\beta\xi \leqslant_\mu \phi \leqslant_\mu \beta\xi\}$$
$$|\phi|_\mu = \inf\{\beta \geqslant 0\text{: } -\beta\xi \leqslant_\mu \phi \leqslant_\mu \beta\xi\}, \qquad \phi \in X.$$

Then $|\circ|_\mu$ is a norm on X and it makes X a Banach space. The set

$$K_\mu = \{\phi \in \bar{K}_\mu\text{: there exists } \beta \geqslant 0 \text{ such that } \phi \leqslant_\mu \beta\xi\} = \bar{K}_\mu \cap X$$

is a closed cone in X with nonempty interior and $|\circ|_\mu$ is monotone on X (see [1]).

LEMMA 1.3. *X consists of the elements of C which satisfy a Lipschitz condition on $[-\tau, 0]$. The norm $|\circ|_\mu$ is equivalent to the norm*

$$|\phi|_{\mathrm{Lip}} = \|\phi\| + \mathrm{Lip}(\phi),$$

where

$$\mathrm{Lip}(\phi) = \sup\left\{\left|\frac{\phi(s) - \phi(t)}{s - t}\right| : s \neq t,\ s,\ t \in [-\tau, 0]\right\}.$$

Proof. If $\phi \in X$, then $-\beta\xi \leqslant_\mu \phi \leqslant_\mu \beta\xi$ for $\beta = |\phi|_\mu$. In particular, $-\beta\xi \leqslant \phi \leqslant \beta\xi$ so $\|\phi\| \leqslant \beta e^{\mu\tau}$. In addition, if $t < s$, then $\phi \leqslant_\mu \beta\xi$ implies that

$$\begin{aligned}\phi(s) - \phi(t) &\leqslant \beta(e^s - e^t)e^{-\mu t} - \phi(s)e^{\mu(s-t)} + \phi(t)\\ &\leqslant \beta(1+\mu)\,e^{\mu\tau}(s-t).\end{aligned}$$

Similarly, $-\beta\xi \leqslant_\mu \phi$ implies the same estimate for $\phi(t) - \phi(s)$. Hence, for $\phi \in X$,

$$\mathrm{Lip}(\phi) \leqslant \beta(1+\mu)e^{\mu\tau}$$

and

$$|\phi|_{\mathrm{Lip}} \leqslant (2+\mu)e^{\mu\tau}\,|\phi|_{\mu}.$$

Conversely, if ψ is Lipschitz continuous on $[-\tau, 0]$ with $\mathrm{Lip}(\psi) = L$, then ψ is absolutely continuous with $|\psi'| \leqslant L$ a.e. As

$$\frac{d}{dt}(\beta\xi(t) - \psi(t))e^{\mu t} = \beta e^{t} - \psi'(t)e^{\mu t} - \mu\psi(t)e^{\mu t}$$

we find that the absolutely continuous function $(\beta\xi(t) - \psi(t))e^{\mu t}$ is monotone nondecreasing if $\beta \geqslant (L + \mu\,\|\psi\|)e^{\tau}$. Hence

$$\psi \leqslant_{\mu} (L + (\mu+1)\,\|\psi\|)e^{\tau}\xi.$$

Similarly,

$$\psi \;{}_{\mu}\!\geqslant\; -(L + (\mu+1)\,\|\psi\|)e^{\tau}\xi.$$

It follows that $\psi \in X$ and

$$|\psi|_{\mu} \leqslant (\mu+1)e^{\tau}\,|\psi|_{\mathrm{Lip}}.$$

It is perhaps worth mentioning that the space X equipped with the norm $|\circ|_{\mu}$ and the ordering $\leqslant_{\mu}$ is a Banach lattice. In other words, given two elements ϕ, $\psi \in X$ there exists in X a supremum $\phi \vee \psi$ and the norm is monotonic (see, e.g., Vulikh [12]). This observation has important implications for stability and bifurcation from equilibria. See Section 3.

Hereafter we will identify X with the Banach space C_L of Lipschitz functions on $[-\tau, 0]$ with norm $|\circ|_{\mathrm{Lip}}$. Note the inclusion $C_L \to C$ is a compact mapping and that C_L is dense in C.

It follows from Lemma 1.3 that the cone K_{μ} in X can be characterized as

$$K_{\mu} = \{\phi \in C_L : \phi \geqslant 0 \text{ and } \phi' + \mu\phi \geqslant 0 \text{ a.e. in } [-\tau, 0]\}.$$

We will need to exploit the smoothing property of the semiflow associated to (1.1) in order to obtain the strong order preserving property. Thus we require the following result. The proof, which is straightforward, is omitted.

LEMMA 1.4. *Let $\phi \in C$ and $x(t, \phi)$ be the solution of* (1.1) *on* $[0, t_1]$ *and suppose that $t_1 \geqslant \tau$. Then there exists a neighborhood U of ϕ in C such that if $\psi \in U$ then $x(t, \psi)$ is defined on* $[0, t_1]$ *and*

$$\psi \rightarrowtail x_{t_1}(\psi)$$

is continuous at ϕ as a map from U into C_L.

The interior of K_μ with respect to C_L is easily seen to be given by

$$\operatorname{Int} K_\mu = \{\phi \in C_L : \phi(s) > 0 \text{ and ess} \inf_{[-\tau, 0]} (\phi' + \mu\phi) > 0\}.$$

Given $\phi_i \in C_L$, $i = 1, 2$, we write $\phi_1 \ll_\mu \phi_2$ if and only if $\phi_2 - \phi_1 \in \operatorname{Int} K_\mu$.

The semiflow, Φ, defined by (1.1) is said to be strongly order preserving on C if it is order preserving in the sense of Corollary 1.2(a) and whenever $\phi_i \in C$, $i = 1, 2$, satisfy $\phi_1 <_\mu \phi_2$, there exist open sets U and V, with $\phi_1 \in U$ and $\phi_2 \in V$, and $t_0 \geqslant 0$ such that $\Phi_{t_0}(U) \leqslant_\mu \Phi_{t_0}(V)$, where the inequality between sets is to be interpreted to hold for any pair of elements, one from each set.

The following hypothesis on f is sufficient for (1.1) to be strongly order preserving:

(SM) There exists $\mu \geqslant 0$ such that whenever $\psi_i \in C_L$, $i = 1, 2$, satisfy $\psi_1 <_\mu \psi_2$ then

$$\mu(\psi_2(0) - \psi_1(0)) + f(\psi_2) - f(\psi_1) > 0.$$

Observe that (SM) implies that M holds. It is not difficult to check that (SM) holds for (0.2) if (0.4) holds.

Proposition 1.5. *Let* (SM) *hold and suppose that* $\phi_1, \phi_2 \in C_L$ *satisfy* $\phi_1 <_\mu \phi_2$. *Then*

$$x_t(\phi_1) \ll_\mu x_t(\phi_2)$$

for all $t \geqslant \tau$ *such that both solutions are defined.*

Proof. Fix $t_1 \geqslant \tau$ such that both solutions $x(t, \phi_i)$ are defined on $[0, t_1]$. Since $\phi_1 \neq \phi_2$, $\phi_1(0) < \phi_2(0)$. Now (M) holds so we have $x(t, \phi_1) < x(t, \phi_2)$ on $[0, t_1]$. Further, by Proposition 1.1, $x_t(\phi_1) \leqslant_\mu x_t(\phi_2)$ for $0 \leqslant t \leqslant t_1$. Then

$$\begin{aligned}(d/dt + \mu)&[x(t, \phi_2) - x(t, \phi_1)] \\ &= \mu[x(t, \phi_2) - x(t, \phi_1)] + f(x_t(\phi_2)) - f(x_t(\phi_1)) > 0,\end{aligned}$$

by (SM). The statement now follows from the characterization of $\operatorname{Int} K_\mu$.

Theorem 1.6. *If* (SM) *holds and all solutions of* (1.1) *extend to* $\mathbb{R}^+$ *then* (1.1) *generates a strongly order preserving semiflow on* C.

Proof. Let ϕ_i, $i = 1, 2$, belong to C and satisfy $\phi_1 <_\mu \phi_2$. By Proposition 1.5, $x_\tau(\phi_1) \ll_\mu x_\tau(\phi_2)$. Since the $x_\tau(\phi_i)$ belong to the strongly ordered space C_L (Proposition 1.4), there exist neighborhoods $\hat{U}$ and $\hat{V}$ of $x_\tau(\phi_1)$ and $x_\tau(\phi_2)$, respectively, in C_L, such that $\hat{U} \leqslant_\mu \hat{V}$. By

Proposition 1.4, there exist neighborhoods U and V in C with $\phi_1 \in U$ and $\phi_2 \in V$ such that $x_\tau(\psi) \in \hat{U}$ whenever $\psi \in U$ and $x_\tau(\psi) \in \hat{V}$ whenever $\psi \in V$. This establishes the strong order preserving property.

2. Dynamical Consequences of Monotonicity

The aim of the present section is to apply theory developed in [14] for the strongly order preserving semiflows on ordered metric spaces to the semiflow defined by (1.1). In order that (1.1) generate a semiflow on C with the required compactness properties, we assume:

(T) f maps bounded subsets of C to bounded subsets of $\mathbb{R}$. For each $\phi \in C$, $x(t, \phi)$ is defined for $t \geqslant 0$ and $\{x(t, \phi) : t \geqslant 0\}$ is bounded. For each compact subset A of C there exists a bounded subset B of C such that $\omega(\phi)$, the positive limit set of the orbit through ϕ, satisfies $\omega(\phi) \subseteq B$ for every $\phi \in A$.

To simplify the statement of the main results in this section we assume that f is defined and satisfies all hypotheses on all of the space C. In applications, however, this is seldom the case. For scalar equations it is commonly the case that the relevant domain is $C^+ = \{\phi \in C: 0 \leqslant \phi\}$. All the results of this and earlier sections (i.e., Propositions 1.1 and 1.5 and Theorem 1.6) continue to hold if the hypotheses hold only on C^+ provided that C^+ is positively invariant for (1.1). A necessary and sufficient condition for the latter to hold is that $f(\phi) \geqslant 0$ whenever $\phi \in C^+$ and $\phi(0) = 0$.

A few definitions are required for the statement of our results. They have their origin in the work of Hirsch [4, 5]. The statements are taken from [14]. A point $\phi \in C$ is a stable point if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|x_t(\phi) - x_t(\psi)\| < \varepsilon$ for $t \geqslant 0$ whenever $\psi \in C$ and $\|\phi - \psi\| < \delta$. The point ϕ is an asymptotically stable point if there exists a neighborhood V of ϕ with the property that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\|x_t(\phi) - x_t(\psi)\| < \varepsilon$ if $t \geqslant t_\varepsilon$ and $\psi \in V$. We let S be the subset of stable points of C and A be the set of asymptotically stable points of C. Clearly, A is an open set. Observe that if $\phi \in S$ then points near to ϕ have limit sets near $\omega(\phi)$ and if $\phi \in A$ then points which are sufficiently near to ϕ have the same limit set as ϕ and the approach to $\omega(\phi)$ is uniform for $\psi \in V$.

Let E be the set of equilibrium points of (1.1). Following Hirsch, we denote by Q the set of quasiconvergent points in C, that is, $Q = \{\phi \in C: \omega(\phi) \subseteq E\}$, and let $\mathscr{C}$ denote the subset of convergent points ϕ for which $x_t(\phi) \to e$ as $t \to \infty$ where $e \in E$.

The result below establishes that there exists a dense open set of stable convergent points if (T) and (SM) hold.

THEOREM 2.1. *Let f satisfy* (T) *and* (SM) *on C. Then* Int *S is dense in C and consists of convergent points.*

Proof. We employ Theorem 3.9 of [14]. The space C is a normally ordered metric space with the metric defined by the uniform norm and the ordering $\leqslant_\mu$. Theorem 1.6 and (T) imply that the semiflow defined by (1.1) on C is strongly order preserving. We must verify the compactness assumption (C) of [14]. Each orbit $\{x_t(\phi) : t \geqslant 0\}$ has compact closure in C by (T). Moreover, for each compact subset K of C, $\bigcup \{\omega(\phi) : \phi \in K\}$ has compact closure in C. Indeed, by (T), there is a bounded subset B of C such that $\omega(\phi) \subseteq B$ for all $\phi \in K$. As f is bounded on B, by (T), and $\omega(\phi)$ is invariant for (1.1), we may conclude that $\sup\{f(\phi): \phi \in B\}$ is a common Lipschitz constant for every $\psi \in \bigcup \{\omega(\phi): \phi \in K\}$. It follows that the union of limit sets has compact closure in C. This establishes that (C) holds. By [14, Theorem 3.9], it follows that $A \cup \text{Int}(S \cap \mathscr{C})$ is dense in C. Now $A \subseteq S \subseteq Q$, by [14, Proposition 3.4]. As A is open, $A \subseteq \text{Int } S$, and so Int S is dense in C and contained in Q. Now we claim that $Q \subseteq \mathscr{C}$. Indeed, the nonordering principle [14, Proposition 2.2] implies that no two distinct points of $\omega(\phi)$ are ordered by $\leqslant_\mu$. But if $\phi \in Q$, then $\omega(\phi)$ consists of equilibria and any two equilibria are related by the partial order $\leqslant_\mu$. Thus $\omega(\phi)$ must consist of precisely one equilibrium if $\phi \in Q$. This completes our proof.

As a corollary of the proof, we note that $A \subseteq S \subseteq Q \subseteq \mathscr{C}$.

Our next result shows that under a mild additional assumption, the set of asymptotically stable convergent points is open and dense in C.

THEOREM 2.2. *Let the hypotheses of Theorem* 2.1 *hold. In addition assume that there does not exist a nontrivial subinterval I of* $\mathbb{R}$ *such that* $f(\hat{x}) = 0$ *for all* $x \in I$. *Then A is dense in C and* $A \subseteq \mathscr{C}$.

Proof. We apply [14, Theorem 3.13]. The additional hypothesis implies the nonexistence of any nontrivial totally ordered arc of equilibria in C. This relies on the observation that the subset E of C can be identified with a subset F of $\mathbb{R}$, by the map $\hat{x} \to x$, and that this map is an order isomorphism of $(E, \leqslant_\mu)$ onto $(F, \leqslant)$, where $\leqslant$ is the restriction of the usual order on $\mathbb{R}$. Thus A is dense in C by [14, Theorem 3.13]. But $A \subseteq \mathscr{C}$ by the arguments of the previous theorem.

3. STABILITY OF EQUILIBRIA

In this section we consider the stability of equilibrium states $x_0 \in \mathbb{R}$ of (1.1) provided that (M) or (SM) hold in a neighborhood of $\hat{x}_0$. Recall that $\hat{x}_0$ is the function in C which is identically equal to x_0 and it is an equilibrium of (1.1) if and only if

$$f(\hat{x}_0) = 0. \tag{3.1}$$

Observe that (1.1) has the same equilibria as the scalar ordinary differential equation

$$x'(t) = \hat{f}(x(t)), \qquad \hat{f}(x) = f(\hat{x}), \qquad x \in \mathbb{R}, \tag{3.2}$$

obtained from (1.1) by "ignoring the delays."

If we assume that f is continuously differentiable in a neighborhood of $\hat{x}_0$, then the stability of the steady state x_0 of (1.1) is determined by the roots of the characteristic equation

$$\lambda = df(\hat{x}_0)(e^{\lambda *}). \tag{3.3}$$

We will show that the monotonicity assumptions greatly simplify the study of (3.3). It is sufficient that f satisfies (M) in a neighborhood of $\hat{x}_0$, in fact, the weaker hypothesis (3.4) below is all we require.

PROPOSITION 3.1. *Let $f: C \to \mathbb{R}$ be continuously differentiable in a neighborhood of $\hat{x}_0$, $x_0 \in \mathbb{R}$. If* (M) *holds for some μ in a C neighborhood of $\hat{x}_0$ then*

$$\mu\phi(0) + df(\hat{x}_0)(\phi) \geqslant 0 \tag{3.4}$$

for all $\phi \in C$ such that $\phi \geqslant_\mu 0$.

Instead of assuming that (M) holds in a neighborhood of $\hat{x}_0$ we will assume that (3.4) holds since it is more easily checked (see Proposition 3.5).

Define

$$\nu_0 = \sup\{\operatorname{Re} \lambda : \lambda \text{ is a root of } (3.3)\}.$$

In our first result we show that ν_0 is a root of (3.3) and thus the stability of x_0 can be determined by examining only the real roots of (3.3). In particular, x_0 cannot lose its stability by a Hopf bifurcation as long as (3.4) holds.

PROPOSITION 3.2. *Let* (3.4) *be satisfied for an equilibrium x_0. Then ν_0 is a root of* (3.3) *satisfying*

$$\nu_0 \geqslant -\mu.$$

Proof. Let $M = df(\hat{x}_0)$. Consider the variational equation

$$y'(t) = My_t.$$

Let $T(t): C \to C$ be the solution operator defined by $T(t)\phi = y_t(\phi)$. It is known that $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on C. Moreover, $T(t)(\bar{K}_\mu) \subset \bar{K}_\mu$ for each $t \geq 0$ by Proposition 1.1 so $\{T(t)\}_{t \geq 0}$ is a positive semigroup. Unfortunately, $\bar{K}_\mu$ is not a total cone in C so we cannot apply the Krein–Rutman theorem. However, we may restrict $T(\tau)$ to the space C_L in which the cone K_μ is total. Observe that C_L is positively invariant for $T(t)$ and thus we may define $T_L(t)$ to be the restriction of $T(t)$ to C_L. Although $\{T_L(t)\}_{t \geq 0}$ is not a strongly continuous semigroup on C_L, one can easily verify that $T_L(t)$ is continuous for each fixed $t \geq 0$, $T_L(\tau)$ is compact, and of course $T_L(\tau)$ maps K_μ into itself by Proposition 1.1. By the Krein–Rutman theorem (see, e.g., [1]), there exists $v \in K_\mu$, $v \neq 0$, such that $T_L(\tau)v = \rho v$, where $\rho > 0$ is the spectral radius of $T_L(\tau)$. Now $T(\tau)$ is compact and maps C into C_L so its spectrum is the same as that of $T_L(\tau)$. Consequently, ρ is the spectral radius of $T(\tau)$ as well. It follows from the theory of linear time-independent functional differential equations that $v = e^{\lambda}$ and $\rho = e^{\lambda\tau}$, where λ must be real since $e^{\lambda t}$ is real for all $t > 0$. Moreover, λ is a root of the characteristic equation. If η is any other root of the characteristic equation then $e^{\eta\tau}$ is an eigenvalue of $T(\tau)$ so $|e^{\eta\tau}| \leq \rho = e^{\lambda\tau}$ and thus the real of η does not exceed λ. Set $\lambda = v_0$.

In order to see that $v_0 \geq -\mu$, put $\phi = e^{vs}$, $v \geq -\mu$, in (3.4), noting that $0 \leq_\mu \phi$, to obtain

$$\mu + M(e^{v*}) \geq 0, \qquad v \geq -\mu.$$

In particular, $M(e^{-\mu*}) \geq -\mu$. If equality holds then clearly $v_0 \geq -\mu$. If not, the fact that $|M(e^{v*})| \leq \|M\|$ for $v = 0$ implies $M(e^{v*})$ is bounded for $v \geq -\mu$ so $M(e^{v*}) < v$ for large v. The intermediate value theorem implies that (3.3) has a root larger than $-\mu$ and the proof is complete.

If (1.1) satisfies the quasi-monotone hypothesis (QM) then the stability of a steady state of (1.1) is the same as the stability of that steady state for (3.2) [10]. If f satisfies (SM) locally, we can essentially make the same statement but we have to leave the case

$$\frac{d}{dx}\hat{f}(x_0) = 0$$

out of consideration. However, the following requires only (3.4).

THEOREM 3.3. *Let* (3.4) *be satisfied for an equilibrium* x_0 *of* (1.1) *and* (3.2). *If*

$$\frac{d\hat{f}}{dx}(x_0) > 0$$

then $v_0 > 0$. *That is, if* x_0 *is unstable as a steady state of* (3.2) *then* x_0 *is unstable as steady state of* (1.1).

Proof. $(d\hat{f}/dx)(x_0) = df(\hat{x}_0)(\hat{1}) = df(\hat{x}_0)(e^{0*})$. Thus, if the derivative is positive, then (3.3) has a positive root by the intermediate value theorem.

Next we derive a condition for local asymptotic stability of (3.2) to imply local asymptotic stability for (1.1). To this end we need to sharpen (3.4) a bit such that (SM) holds in a neighborhood of the equilibrium x_0. Actually, it is necessary for f to satisfy the stronger condition, (3.5) below.

THEOREM 3.4. *Let* f *be differentiable at the equilibrium* $\hat{x}_0$ *and let* $0 < \mu$ *be such that*

$$\mu\phi(0) + df(\hat{x}_0)\phi > 0 \tag{3.5}$$

whenever $\phi \in C_L$ *with* $0 <_\mu \phi$. *If* $(d\hat{f}/dx)(x_0) < 0$ *then* $v_0 < 0$. *That is, if* x_0 *is locally exponentially stable as a steady state of* (3.2) *then it is locally exponentially stable as a steady state of* (1.1).

Proof. As $0 > (d\hat{f}/dx)(x_0) = df(\hat{x}_0)(e^{0*})$, we have

$$\lambda > df(\hat{x}_0)(e^{\lambda *}) \qquad \text{for} \quad \lambda = 0.$$

By (3.5), since $e^{-\mu s} {}_\mu\!> 0$,

$$-\mu < df(\hat{x}_0)(e^{-\mu *}).$$

Hence (3.3) has a real root λ_0 with $-\mu < \lambda_0 < 0$ by the intermediate value theorem. Now, (3.5) implies that the solution operator $T_L(t)$ in the proof of Proposition 3.2 is strongly positive for $t \geqslant \tau$, that is, it maps $K_\mu - \{0\}$ into the interior of K_μ. If $\lambda \geqslant -\mu$ is a root of (3.3), then

$$T(t)e^{\lambda *} = e^{\lambda t} e^{\lambda *}.$$

As strongly positive operators have positive eigenvectors for one eigenvalue only (see, e.g., [1]), there exists only one root $\lambda_0 \geqslant -\mu$ of (3.3), i.e., $v_0 = \lambda_0 < 0$. This completes our proof.

In order to apply Theorem 3.3 or 3.4 it is useful to have conditions under which (3.4) or (3.5) are satisfied. To this end we recall the representation

$$df(\hat{x}_0)\phi = \int_{[-\tau,0]} \phi(\sigma)\, m(d\sigma)$$

with a signed Borel measure m on $[-\tau, 0]$ of bounded total variation. Let m_- denote the negative part of m.

PROPOSITION 3.5. (a) (3.4) *holds if*

$$\mu - \int_{[-\tau,0]} e^{-\mu s} m_-(ds) \geqslant 0. \tag{3.6}$$

(b) (3.5) *holds if the inequality in* (3.6) *is strict.*

Proof. We only show (a) as the proof of (b) is similar. Let $\phi \in C$ with $\phi \geqslant_\mu 0$. Then

$$\mu\phi(0) + df(\hat{x}_0)\phi \geqslant \mu\phi(0) - \int_{[-\tau,0]} \phi(s)\, e^{\mu s} e^{-\mu s} m_-(ds)$$

$$\geqslant \phi(0) \left\{ \mu - \int_{[-\tau,0]} e^{-\mu s} m_-(ds) \right\} \geqslant 0,$$

since $\phi(s)e^{\mu s}$ is nondecreasing and $\phi(0) \geqslant 0$.

4. EXAMPLE: THE LASOTA–WAZEWSKA RED BLOOD CELL MODEL

Following [7] (compare [9]) we consider the particular differential delay equation proposed to model the red blood cell system,

$$x'(t) = f(x_t) = \beta g(x(t-\tau)) - \gamma x(t) \tag{4.1}$$

with

$$g(x) = x^{2n} e^{-\sigma x}. \tag{4.2}$$

Here, n is a natural number and β, γ, σ are positive constants. Nonzero steady states, x, of (4.1) must satisfy the equation

$$x^{2n-1} e^{-\sigma x} = \gamma/\beta. \tag{4.3}$$

Noting that the function defined by the righthand side of (4.3) attains its maximum value at $x = (2n-1)/\sigma$, we have three cases:

Case 1. $((2n-1)/\sigma)^{2n-1} e^{1-2n} < \gamma/\beta$.
Then $x=0$ is the only steady state.

Case 2. $((2n-1)/\sigma)^{2n-1} e^{1-2n} = \gamma/\beta$.
Then there is one nonzero steady state $\xi_1 = (2n-1)/\sigma$.

Case 3. $((2n-1)/\sigma)^{2n-1} e^{1-2n} > \gamma/\beta$.
There are exactly two nonzero steady states ξ_1, ξ_2 satisfying

$$0 < \xi_1 < (2n-1)/\sigma < \xi_2.$$

In all three cases we have

$$\frac{d}{dx}\hat{f}(0) < 0. \tag{4.4}$$

In Case 2 we have

$$\frac{d}{dx}\hat{f}(\xi_1) = 0. \tag{4.5}$$

In Case 3 we have

$$\frac{d}{dx}\hat{f}(\xi_1) > 0 > \frac{d}{dx}\hat{f}(\xi_2). \tag{4.6}$$

Hence, for the ordinary differential equation obtained from (4.1) by ignoring the delay

$$x'(t) = \hat{f}(x(t)) = \beta g(x(t)) - \gamma x(t) \tag{4.7}$$

we have the following asymptotic behavior.

PROPOSITION 4.1. *The steady state $x=0$ of* (4.7) *is locally asymptotically stable.*

Case 1. All solutions converge to $x=0$.

Case 2. Solutions x with $x(0) < \xi_1$ converge to $x=0$; solutions with $x(0) \geqslant \xi_1$ converge to ξ_1.

Case 3. Solutions x with $x(0) < \xi_1$ converge to $x=0$; solutions with $x(0) > \xi_1$ converge to ξ_2. The steady state $x=\xi_1$ is unstable and $x=\xi_2$ is locally asymptotically stable.

Returning to (4.1), we note that the assumption (QM) holds for $\phi \in C$ with $\phi(s) \leqslant 2n/\sigma$, and the stronger assumption (R) in [10] holds for ϕ with the strict inequality above. Some further information can be obtained from the theory in [10].

PROPOSITION 4.2. *If Case 3 holds then ξ_1 is unstable. If further, $(2n/\sigma)^{2n-1} e^{-2n} < \gamma/\beta$ then the set $U = \{\phi \in C: \phi(s) < 2n/\sigma, \ -\tau \leqslant s \leqslant 0\}$ is positively invariant and contains both nonzero equilibria. The equilibrium ξ_2 is locally asymptotically stable. Moreover, U contains an open, dense set of convergent points.*

Proposition 4.2 does not give any information concerning orbits beginning outside U. If

$$(2n/\sigma)^{2n-1} e^{-2n} > \gamma/\beta,$$

then $\xi_2 > 2n/\sigma$ will lie outside the region of quasi-monotonicity. Moreover, U in Proposition 4.2 is no longer a positively invariant set. The theory developed in the present paper can add to our knowledge of the behavior of solutions of (4.1).

If we want (SM) to hold everywhere in C_L then by (0.4) we must require the condition

$$\tau\,|L_2| < 1 \qquad \text{and} \qquad -\gamma\tau - \ln(\tau\,|L_2|) > 1,$$

where $L_2 = \beta \inf g'(\mathbb{R})$. A calculation gives

$$|L_2| = \beta\sqrt{2n}\left(\frac{2n+\sqrt{2n}}{\sigma}\right)^{2n-1} e^{-2n-\sqrt{2n}}.$$

THEOREM 4.3. *Let*

$$-\gamma\tau - \ln\left\{\tau\beta\sqrt{2n}\left(\frac{2n+\sqrt{2n}}{\sigma}\right)^{2n-1} e^{-2n-\sqrt{2n}}\right\} > 1.$$

Then there exists a subset of convergent points which is dense and open in C. Moreover, the steady states $0 = \xi_0 \leqslant \xi_1 \leqslant \xi_2$ have the same stability properties as for the ordinary differential Eq. (4.7) with the possible exception of Case 2. Thus, ξ_0 is locally asymptotically stable and, in Case 3, ξ_1 is unstable and ξ_2 is locally asymptotically stable.

Proof. Theorem 2.2 and Theorem 3.4.

REFERENCES

1. H. AMANN, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
2. K. P. HADELER AND J. TOMIUK, Periodic solutions to difference-differential equations, *Arch. Rational Mech. Anal.* **65** (1977), 87–95.
3. J. K. HALE, "Theory of Functional Differential Equations," Springer-Verlag, New York, 1977.

4. M. W. HIRSCH, Stability and convergence in strongly monotone dynamical systems, *J. Reine Angew. Math.* **383** (1988), 1–53.
5. M. W. HIRSCH, The dynamical systems approach to differential equations, *Bull. Amer. Math. Soc.* **11** (1984), 1–64.
6. W. KERSCHER AND R NAGEL, Asymptotic behavior of one-parameter semigroups of positive operators, *Acta Appl. Math.* **2** (1984), 297–309.
7. A. LASOTA AND M. WAZEWSKA, Mathematical models of the red blood cell system, *Mat. Stos.* **6** (1976), (1976), 25–40.
8. J. MALLET-PARET AND G. SELL, Monotonicity and Morse–Smale differential-delay equations, in preparation.
9. M. MARTELLI, K. SCHMITT, AND H. L. SMITH, Periodic solutions of some nonlinear delay-differential equations, *J. Math. Anal. Appl.* **74** (1980), 494–503.
10. H. L. SMITH, Monotone semiflows generated by functional differential equations, *J. Differential Equations* **66** (1987), 420–442.
11. H. R. THIEME, Global asymptotic behaviour in epidemic models, Proceedings "Equadiff 82" (H.W. Knobloch and K. Schmitt, Eds.), pp. 608–615, Lecture Notes in Mathematics, Vol. 1017, Springer-Verlag, New York, 1983.
12. B. Z. VULIKH, "Introduction to the Theory of Partially Ordered Spaces," Wolters, Groningen, 1967.
13. H. MATANO, Strong comparison principle in nonlinear parabolic equations, Proceedings of "Nonlinear Parabolic Equations: Qualitative Properties of Solutions," Research Notes in Mathematics, Pitman, Vol. 149 (1987), pp. 148–155.
14. H. L. SMITH AND H. R. THIEME, Quasiconvergence and stability for strongly order preserving semiflows, *SIAM J. Math. Anal.*, to appear.