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# A new high accuracy locally one-dimensional scheme for the wave equation[☆]

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ABSTRACT

In this paper, a new locally one-dimensional (LOD) scheme with error of $O(\Delta t^4 + h^4)$ for the two-dimensional wave equation is presented. The new scheme is four layer in time and three layer in space. One main advantage of the new method is that only tridiagonal systems of linear algebraic equations have to be solved at each time step. The stability and dispersion analysis of the new scheme are given. The computations of the initial and boundary conditions for the two intermediate time layers are explicitly constructed, which makes the scheme suitable for performing practical simulation in wave propagation modeling. Furthermore, a comparison of our new scheme and the traditional finite difference scheme is given, which shows the superiority of our new method.



## 1. Introduction

Simulating wave propagation has important applications in geophysics, especially in oil exploration. The simplest, and yet the most important and popular, approach is the finite difference method. One of the first papers on this subject was that of [1] where a finite difference scheme for the acoustic wave equation in homogeneous media was proposed, which is fourth-order accurate in space and second-order accurate in time. Already, a lot of progress has been made in this direction. For instance, some explicit fourth-order finite difference schemes for the acoustic wave equation can be found in [2–5]. To improve the efficiency, one uses the finite difference approach coupled with some splitting methods. Two kinds of splitting methods, the alternating direction implicit (ADI) method and the locally one-dimensional (LOD) method, have been proven to be very efficient in the numerical solution of parabolic and hyperbolic equations (see, e.g., [6–8]). Both methods reduce the multidimensional problem to a sequence of locally one-dimensional problems with tridiagonal systems which can be solved easily. Parallelization of splitting methods has also been investigated [9–12].

The ADI method was first introduced in [13,14] for the heat equation in two space variables. The method was extended to mildly nonlinear problems [15], problems in three space dimensions and elliptic equations [16–18]. Improved forms of the ADI methods are derived in [19]. Moreover, a general formulation of the ADI methods for parabolic and hyperbolic problems is developed in [20]. Furthermore, Kellogg generalized the ADI method for the efficient approximation of operator

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equations [21]. Regarding the analysis of ADI methods, Lees used the energy method to establish the unconditional stability of the ADI methods [22]. Lees also formulated two finite difference methods for the numerical solution of the wave equation [23]. A high order version of the ADI scheme for the wave equation was proposed in [24,18].

One drawback of the ADI method is that it involves a restriction on the computational region, for example, in the case with three space variables, the method is valid under the restriction that the region is a cube [17]. The LOD method was then proposed approximately ten years later; for instance, in [25], Samarskii introduced a LOD finite difference method for solving hyperbolic equations in two space dimensions. The LOD method provides a more competitive option since it can be applied to arbitrary regions, but it was mentioned that "high accuracy difference methods are difficult to use" [7]. We remark that the aforementioned ADI methods such as the method of Lees, and the high accuracy methods of Samarskii [6] and Fairweather and Mitchell [24] cannot be written in LOD form. While high accuracy ADI methods are investigated extensively, to the best of our knowledge there are only a few studies on high accuracy LOD schemes. To name one example, in [26], the fourth-order LOD scheme in time and space based on Richardson extrapolation is discussed. In this paper, we present and analyze a novel high accuracy LOD scheme which has an error of $O(\Delta t^4 + h^4)$ and is easy to implement.

The paper is organized as follows. The derivation of the new LOD scheme is presented in Section 2. The stability and dispersion of the method are analyzed in Sections 3 and 4 respectively. In Section 5, we discuss how the initial and boundary conditions are computed, while in Section 6, numerical results are given as an illustration to confirm our theoretical findings. Finally, a conclusion is given in Section 7.

## 2. The new LOD scheme

In this section, we will describe our new LOD methods. Consider the following wave equation:

$$\frac{\partial^2 u}{\partial^2 t} = v^2\left(\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y}\right) \tag{2.1}$$

with $(x, y, t) \in \overline{\Omega} = \Omega \times [0 \le t \le T]$, where $\Omega = \{(x, y) : 0 < x, y < 1\}$, subject to the initial conditions

$$u(x, y, 0) = f_1(x, y), \qquad \frac{\partial u(x, y, 0)}{\partial t} = f_2(x, y), \quad (x, y) \in \Omega, \tag{2.2}$$

and the boundary conditions

$$u(x, y, t) = g(x, y, t), \qquad (x, y, t) \in \partial\Omega \times [0 \le t \le T], \tag{2.3}$$

where $\partial\Omega$ is the boundary of $\Omega$ and $v$ is the wave velocity. We shall assume that the given initial and boundary conditions are sufficiently smooth for achieving the order of accuracy of the difference scheme under consideration.

The wave equation is typically written as the pair of equations [25]

$$\frac{1}{2}\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial x^2}, \tag{2.4}$$

$$\frac{1}{2}\frac{\partial^2 u}{\partial t^2} = v^2\frac{\partial^2 u}{\partial y^2}. \tag{2.5}$$

In [25], the second-order time derivative is approximated by the standard central difference scheme. In order to obtain our higher order scheme, we use the following expression:

$$\frac{\partial^2 u}{\partial t^2} \approx \frac{au^{n+s} - (a+b)u^n + bu^{n-1+s}}{\Delta t^2}, \quad s \in (0, 1) \tag{2.6}$$

to approximate the derivative $\frac{\partial^2 u}{\partial t^2}$, where $\Delta t$ is the time step, and $s, a, b$ are coefficients that will be determined. Throughout the paper, we use $u^n$ to denote the approximate value of $u(x, y, t)$ at a general grid point $(x_j, y_m, t_n)$, $j = 0, 1, \ldots, N_x$; $m = 0, 1, \ldots, N_y$. Here we have assumed that the velocity $v$ is constant. Using the Taylor expansion, $s$, $a$ and $b$ should satisfy

$$a = \frac{2}{s}, \qquad b = \frac{2}{1-s}. \tag{2.7}$$

Let $h_x$ and $h_y$ be the grid spacings in the $x$ and $y$ directions respectively. Then repeated applications of the finite difference methods to (2.4)–(2.5) lead to the following scheme:

$$\begin{aligned}
\frac{bu^n - (a+b)u^{n-1+s} + au^{n-1}}{\Delta t^2} &= v^2\frac{c_1\delta_x^2 u^{n-1+s} - c_2\delta_x^2 u^n - c_3\delta_x^2 u^{n-1}}{h_x^2},\\
\frac{au^{n+s} - (a+b)u^n + bu^{n-1+s}}{\Delta t^2} &= v^2\frac{c_1'\delta_y^2 u^n - c_2'\delta_y^2 u^{n+s} - c_3'\delta_y^2 u^{n-1+s}}{h_y^2},\\
\frac{bu^{n+1} - (a+b)u^{n+s} + au^n}{\Delta t^2} &= v^2\frac{c_1\delta_x^2 u^{n+s} - c_2\delta_x^2 u^{n+1} - c_3\delta_x^2 u^n}{h_x^2},
\end{aligned} \tag{2.8}$$

which can be written as follows:

$$\begin{aligned}&(b+\tau_x c_2\delta_x^2)u^n+(a+\tau_x c_3\delta_x^2)u^{n-1}=(a+b+\tau_x c_1\delta_x^2)u^{n-1+s},\\&(a+b+\tau_y c_1'\delta_y^2)u^n=(a+\tau_y c_2'\delta_y^2)u^{n+s}+(b+\tau_y c_3'\delta_y^2)u^{n-1+s},\\&(b+\tau_x c_2\delta_x^2)u^{n+1}+(a+\tau_x c_3\delta_x^2)u^n=(a+b+\tau_x c_1\delta_x^2)u^{n+s},\end{aligned}\tag{2.9}$$

with

$$\tau_x=\frac{v^2\Delta t^2}{h_x^2},\qquad \tau_y=\frac{v^2\Delta t^2}{h_y^2},\tag{2.10}$$

where $\delta_x^2$, $\delta_y^2$ are the second-order central difference operators in the $x$, $y$ directions respectively, e.g., $\delta_x^2u_{j,m}^n=u_{j+1,m}^n-2u_{j,m}^n+u_{j-1,m}^n$. The coefficients $c_1$, $c_2$, $c_3$, and $c_1'$, $c_2'$, $c_3'$ will be determined in the following paragraphs.

For ease of notation, let

$$\begin{aligned}&A=b+\tau_x c_2\delta_x^2,\qquad B=a+\tau_x c_3\delta_x^2,\\&C=a+b+\tau_x c_1\delta_x^2,\qquad D=a+b+\tau_y c_1'\delta_y^2,\\&E=a+\tau_y c_2'\delta_y^2,\qquad F=b+\tau_y c_3'\delta_y^2.\end{aligned}\tag{2.11}$$

Then (2.8) can be rewritten as

$$\begin{aligned}&Au^n+Bu^{n-1}=Cu^{n-1+s},\\&Du^n=Eu^{n+s}+Fu^{n-1+s},\\&Au^{n+1}+Bu^n=Cu^{n+s}.\end{aligned}\tag{2.12}$$

Canceling $u^{n+s}$ and $u^{n-1+s}$ in the above expression, we obtain

$$AEu^{n+1}+(AF+BE)u^n+BFu^{n-1}=CDu^n.\tag{2.13}$$

Obviously, $AE=BF$ whenever

$$ac_2=bc_3,\qquad bc_2'=ac_3'.\tag{2.14}$$

Assuming this, (2.13) can be reduced to

$$AE(u^{n+1}-2u^n+u^{n-1})=[CD-(A+B)(E+F)]u^n.\tag{2.15}$$

Let

$$L=u^{n+1}-2u^n+u^{n-1}\approx\Delta t^2\frac{\partial^2u}{\partial t^2}+\frac{\Delta t^4}{12}\frac{\partial^4u}{\partial^4t}+O(\Delta t^6);\tag{2.16}$$

then

$$\begin{aligned}&\delta_x^2L=\Delta t^2h_x^2\frac{\partial^4u}{\partial t^2\partial x^2}+O(\Delta t^2h^4+\Delta t^4h^2),\\&\delta_y^2L=\Delta t^2h_y^2\frac{\partial^4u}{\partial t^2\partial y^2}+O(\Delta t^2h^4+\Delta t^4h^2),\\&\delta_x^2\delta_y^2L=\Delta t^2h_x^2h_y^2\frac{\partial^6u}{\partial t^2\partial x^2\partial y^2}+O(\Delta t^2h^4+\Delta t^4h^2),\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}&\delta_x^2u^n=h_x^2\frac{\partial^2u}{\partial x^2}+\frac{h_x^4}{12}\frac{\partial^4u}{\partial x^4}+O(h^6),\\&\delta_y^2u^n=h_y^2\frac{\partial^2u}{\partial y^2}+\frac{h_y^4}{12}\frac{\partial^4u}{\partial y^4}+O(h^6),\\&\delta_x^2\delta_y^2u^n=h_x^2h_y^2\frac{\partial^4u}{\partial x^2\partial y^2}+\frac{h_x^4h_y^2}{12}\frac{\partial^6u}{\partial x^4\partial y^2}+\frac{h_x^2h_y^4}{12}\frac{\partial^6u}{\partial x^2\partial y^4}+O(h^6),\end{aligned}\tag{2.18}$$

where $h = \max(h_x, h_y)$. Using (2.17) and (2.18) in (2.15), we obtain

$$\begin{aligned}
&ab\frac{\partial^2 u}{\partial t^2} - (a+b)(c_1-c_2-c_3)v^2\frac{\partial^2 u}{\partial x^2} - (a+b)(c_1'-c_2'-c_3')v^2\frac{\partial^2 u}{\partial y^2} + \frac{ab\Delta t^2}{12}\frac{\partial^4 u}{\partial t^4} + ac_2v^2\Delta t^2\frac{\partial^4 u}{\partial t^2\partial x^2} \\
&\quad + bc_2'v^2\Delta t^2\frac{\partial^4 u}{\partial t^2\partial y^2} - \frac{(a+b)(c_1-c_2-c_3)v^2h_x^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{(a+b)(c_1'-c_2'-c_3')v^2h_y^2}{12}\frac{\partial^4 u}{\partial y^4} \\
&\quad - v^4\Delta t^2[c_1c_1' - (c_2+c_3)(c_2'+c_3')]\frac{\partial^4 u}{\partial x^2\partial y^2} + O(\Delta t^4 + h^4) = 0. \qquad (2.19)
\end{aligned}$$

Comparing this equation with (2.1), we have

$$ab = (a+b)(c_1-c_2-c_3), \qquad ab = (a+b)(c_1'-c_2'-c_3'). \qquad (2.20)$$

Inserting (2.7) into (2.20), we get

$$c_1 - c_2 - c_3 = 2, \qquad c_1' - c_2' - c_3' = 2. \qquad (2.21)$$

To keep the truncation errors of our scheme to $O(\Delta t^4 + h^4)$, we require, in (2.19), that

$$\begin{aligned}
&\frac{ab\Delta t^2}{12}\frac{\partial^4 u}{\partial t^4} + ac_2v^2\Delta t^2\frac{\partial^4 u}{\partial t^2\partial x^2} + bc_2'v^2\Delta t^2\frac{\partial^4 u}{\partial t^2\partial y^2} - \frac{(a+b)(c_1-c_2-c_3)v^2h_x^2}{12}\frac{\partial^4 u}{\partial x^4} \\
&\quad - \frac{(a+b)(c_1'-c_2'-c_3')v^2h_y^2}{12}\frac{\partial^4 u}{\partial y^4} - v^4\Delta t^2[c_1c_1' - (c_2+c_3)(c_2'+c_3')]\frac{\partial^4 u}{\partial x^2\partial y^2} = 0. \qquad (2.22)
\end{aligned}$$

Here we have assumed that the velocity $v$ is constant. Using (2.1) and the relationship

$$\frac{\partial^4 u}{\partial t^4} = v^4\left(\frac{\partial^4 u}{\partial x^4} + \frac{2\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right) \qquad (2.23)$$

to replace the time derivatives in the above expression, and letting the coefficients of $\frac{\partial^4 u}{\partial x^4}, \frac{2\partial^4 u}{\partial x^2\partial y^2}, \frac{\partial^4 u}{\partial y^4}$ be zero, we get

$$\begin{aligned}
&\frac{ab}{12}v^2\Delta t^2 + ac_2v^2\Delta t^2 - \frac{(a+b)(c_1-c_2-c_3)}{12}h_x^2 = 0, \\
&\frac{ab}{6} + ac_2 + bc_2' - [c_1c_1' - (c_2+c_3)(c_2'+c_3')] = 0, \qquad (2.24) \\
&\frac{ab}{12}v^2\Delta t^2 + bc_2'v^2\Delta t^2 - \frac{(a+b)(c_1'-c_2'-c_3')}{12}h_y^2 = 0.
\end{aligned}$$

Substituting (2.14) and (2.21) into (2.15), we get $ab = 24$. Therefore $s = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$. For the symmetry of $s$, we choose $s = \frac{1}{2} - \frac{\sqrt{3}}{6}$; then $a = 6 + 2\sqrt{3}$ and $b = 6 - 2\sqrt{3}$. Inserting $a$ and $b$ into (2.25), we obtain

$$c_1 = 1 + \frac{1}{\tau_x}, \qquad c_2 = \frac{2(1-\tau_x)}{a\tau_x}, \qquad c_3 = \frac{2(1-\tau_x)}{b\tau_x}, \qquad (2.25)$$

$$c_1' = 1 + \frac{1}{\tau_y}, \qquad c_2' = \frac{2(1-\tau_y)}{b\tau_y}, \qquad c_3' = \frac{2(1-\tau_y)}{a\tau_y}. \qquad (2.26)$$

Therefore, we have proven the following theorem.

**Theorem 2.1.** *Defining $a = 6 + 2\sqrt{3}$, $b = 6 - 2\sqrt{3}$, $s = \frac{1}{2} - \frac{\sqrt{3}}{6}$ and using the above $c_1, c_2, c_3$ from (2.25) and $c_1', c_2', c_3'$ from (2.26) in (2.8), we see that the new LOD scheme (2.8) has an error of $O(\Delta t^4 + h^4)$.*

## 3. Stability analysis

In this section, we will use the Fourier method to analyze the stability of our new LOD scheme (2.8). Let $u_{j,m}^n = z^n e^{ij\sigma_1} e^{im\sigma_2}$. Then

$$\begin{aligned}
&\delta_x^2 u_{j,m}^n = -4\sin^2\left(\frac{\sigma_1}{2}\right) z^n e^{ij\sigma_1} e^{im\sigma_2}, \\
&\delta_y^2 u_{j,m}^n = -4\sin^2\left(\frac{\sigma_2}{2}\right) z^n e^{ij\sigma_1} e^{im\sigma_2}, \qquad (3.1) \\
&\delta_x^2\delta_y^2 u_{j,m}^n = 16\sin^2\left(\frac{\sigma_1}{2}\right)\sin^2\left(\frac{\sigma_2}{2}\right) z^n e^{ij\sigma_1} e^{im\sigma_2},
\end{aligned}$$

and

$$\begin{aligned}
L_1 u_{j,m}^n &= \left[b - 4\tau_x c_2 \sin^2\left(\frac{\sigma_1}{2}\right)\right] u_{j,m}^n := \widetilde{L}_1 u_{j,m},\\
L_2 u_{j,m}^n &= \left[a - 4\tau_x c_3 \sin^2\left(\frac{\sigma_1}{2}\right)\right] u_{j,m}^n := \widetilde{L}_2 u_{j,m},\\
L_3 u_{j,m}^n &= \left[a + b - 4\tau_x c_1 \sin^2\left(\frac{\sigma_1}{2}\right)\right] u_{j,m}^n := \widetilde{L}_3 u_{j,m},\\
L_4 u_{j,m}^n &= \left[a + b - 4\tau_y c_1' \sin^2\left(\frac{\sigma_2}{2}\right)\right] u_{j,m}^n := \widetilde{L}_4 u_{j,m},\\
L_5 u_{j,m}^n &= \left[a - 4\tau_y c_2' \sin^2\left(\frac{\sigma_2}{2}\right)\right] u_{j,m}^n := \widetilde{L}_5 u_{j,m},\\
L_6 u_{j,m}^n &= \left[b - 4\tau_y c_3' \sin^2\left(\frac{\sigma_2}{2}\right)\right] u_{j,m}^n := \widetilde{L}_6 u_{j,m}.
\end{aligned} \tag{3.2}$$

Substituting the above two sets of formulas into (2.13) yields

$$\rho^2 - \left(2 + \frac{\widetilde{L}_3\widetilde{L}_4 - (\widetilde{L}_1 + \widetilde{L}_2)(\widetilde{L}_5 + \widetilde{L}_6)}{\widetilde{L}_1\widetilde{L}_5}\right)\rho + 1 = 0, \tag{3.3}$$

where $\rho = z^{n+1}/z^n$ is the amplification factor. The necessary and sufficient condition for stability is $|\rho| \le 1$, which is equivalent to

$$-4 \le \frac{\widetilde{L}_3\widetilde{L}_4 - (\widetilde{L}_1 + \widetilde{L}_2)(\widetilde{L}_5 + \widetilde{L}_6)}{\widetilde{L}_1\widetilde{L}_5} \le 0. \tag{3.4}$$

Notice that $b\widetilde{L}_1 = a\widetilde{L}_2$, $b\widetilde{L}_5 = a\widetilde{L}_6$. Therefore (3.4) is reduced to

$$2 \le \frac{\widetilde{L}_3\widetilde{L}_4}{\widetilde{L}_1\widetilde{L}_5} \le 6. \tag{3.5}$$

Since

$$\frac{\widetilde{L}_3}{\widetilde{L}_1} = (3 - \sqrt{3})\frac{3 - (\tau_x + 1)\sin^2\left(\frac{\sigma_1}{2}\right)}{3 - (1 - \tau_x)\sin^2\left(\frac{\sigma_1}{2}\right)}, \tag{3.6}$$

and

$$\frac{2 - \tau_x}{2 + \tau_x} \le \frac{3 - (\tau_x + 1)\sin^2\left(\frac{\sigma_1}{2}\right)}{3 - (1 - \tau_x)\sin^2\left(\frac{\sigma_1}{2}\right)} \le 1, \tag{3.7}$$

we have

$$(3 - \sqrt{3})\frac{2 - \tau_x}{2 + \tau_x} \le \frac{\widetilde{L}_3}{\widetilde{L}_1} \le 3 - \sqrt{3}. \tag{3.8}$$

Analogously, we have

$$(3 + \sqrt{3})\frac{2 - \tau_y}{2 + \tau_y} \le \frac{\widetilde{L}_4}{\widetilde{L}_5} \le 3 + \sqrt{3}. \tag{3.9}$$

Therefore, the necessary and sufficient condition for stability is

$$\gamma_1 \epsilon \left[\frac{1}{3}, 1\right], \qquad \gamma_2 \epsilon \left[\frac{1}{3}, 1\right], \qquad \gamma_1\gamma_2 \ge \frac{1}{3}, \tag{3.10}$$

where

$$\gamma_1 = \frac{2 - \tau_x}{2 + \tau_x}, \qquad \gamma_2 = \frac{2 - \tau_y}{2 + \tau_y}. \tag{3.11}$$

Thus we have proven the following theorem.

**Theorem 3.1.** *The new LOD scheme (2.8) is stable if and only if (3.10) with (3.11) is satisfied.*

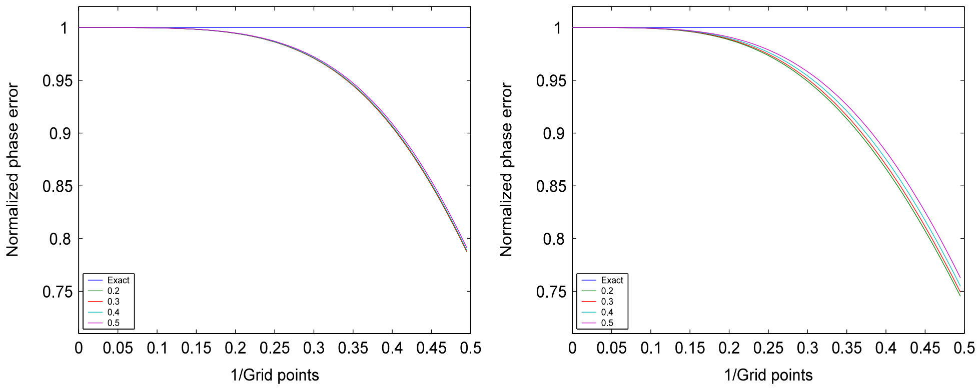


**Fig. 4.1.** The normalized phase error for different CFL numbers at a propagation angle $\theta = 0°$. Left: new LOD scheme. Right: typical fourth-order accuracy scheme in space and time.

## 4. Plane wave analysis

In this section, we will derive the numerical dispersion relation for the scheme (2.8). We first assume that

$$u = e^{\omega t - \boldsymbol{k}\cdot\boldsymbol{x}}, \tag{4.1}$$

where $\omega$ is the pulsation, $\boldsymbol{x} = (x, y)$, $\boldsymbol{k} = (k_1, k_2)$ is the wave propagation angle where $k_1 = |\boldsymbol{k}|\cos(\theta)$ and $k_2 = |\boldsymbol{k}|\sin(\theta)$. For simplicity, we set $h_x = h_y = h$. Inserting (4.1) into (2.8) or its equivalent (2.15) gives the following dispersion relation:

$$\sin^2\frac{\omega\Delta t}{2} = \Big\{\big[a + b - 4\tau_x(c_2 + c_3)S_1\big]\big[a + b - 4\tau_y(c_2' + c_3')S_2\big] - \big[a + b - 4\tau_x c_1 S_1\big]\big[a + b - 4\tau_y c_1' S_2\big]\Big\}\Big\{\big[4(b - 4\tau_x c_2 S_1)(a - 4\tau_y c_2' S_2)\big]\Big\}^{-1}, \tag{4.2}$$

where $S_1 = \sin^2\frac{k_1 h}{2}$ and $S_2 = \sin^2\frac{k_2 h}{2}$. It is interesting to compare (4.2) with the dispersion relation associated with a typical fourth-order scheme, e.g., the one in [5]. For simplicity we consider the case with homogeneous media here. It is straightforward that a difference scheme with accuracy of $O(\Delta t^4 + h^4)$ for (2.1) is

$$u_{j,m}^{n+1} = 2u_{j,m}^n - u_{j,m}^{n-1} + \frac{v^4\Delta t^4}{12h^4}(\delta_x^{4,2} + 2\delta_x^{2,2}\delta_y^{2,2} + \delta_y^{4,2})u_{j,m}^n + \frac{v^2\Delta t^2}{12h^2}(\delta_x^{2,4} + \delta_y^{2,4})u_{j,m}^n, \tag{4.3}$$

where $\delta_x^{4,2}$, $\delta_x^{2,4}$ and $\delta_x^{2,2}$ are difference operators defined by

$$\begin{aligned}
&\delta_x^{4,2}u_{j,m}^n = u_{j+2,m}^n - 4u_{j+1,m}^n + 6u_{j,m}^n - 4u_{j-1,m}^n + u_{j-2,m}^n,\\
&\delta_x^{2,4}u_{j,m}^n = -u_{j+2,m}^n + 16u_{j+1,m}^n - 30u_{j,m}^n + 16u_{j-1,m}^n - u_{j-2,m}^n,\\
&\delta_x^{2,2}u_{j,m}^n (:= \delta_x^2 u_{j,m}^n) = u_{j+1,m}^n - 2u_{j,m}^n + u_{j-1,m}^n.
\end{aligned} \tag{4.4}$$

The difference operators $\delta_y^{4,2}$, $\delta_y^{2,4}$ and $\delta_y^{2,2}$ are defined in a similar way. Substituting (4.1) into (4.3) leads the following dispersion relation:

$$\sin^2\frac{\omega\Delta t}{2} = \frac{v^4\Delta t^4}{48h^4}(T_x^{4,2} - 2T_x^{2,2}T_y^{2,2} + T_y^{4,2}) + \frac{v^2\Delta t^2}{48h^2}(T_x^{2,4} + T_y^{2,4}), \tag{4.5}$$

where

$$\begin{aligned}
&T_x^{4,2} = -16\sin^2\left(\frac{k_1 h}{2}\right) + 4\sin^2(k_1 h), \qquad T_x^{2,2} = 4\sin^2\left(\frac{k_1 h}{2}\right),\\
&T_x^{2,4} = 64\sin^2\left(\frac{k_1 h}{2}\right) - 4\sin^2(k_1 h),
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
&T_y^{4,2} = -16\sin^2\left(\frac{k_2 h}{2}\right) + 4\sin^2(k_2 h), \qquad T_y^{2,2} = 4\sin^2\left(\frac{k_2 h}{2}\right),\\
&T_y^{2,4} = 64\sin^2\left(\frac{k_2 h}{2}\right) - 4\sin^2(k_2 h).
\end{aligned} \tag{4.7}$$

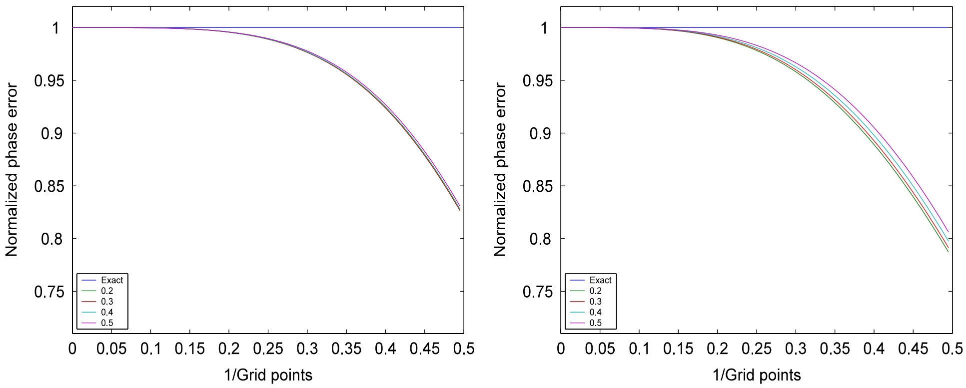


**Fig. 4.2.** The normalized phase error for different CFL numbers at a propagation angle $\theta = 15^\circ$. Left: new LOD scheme. Right: typical fourth-order accuracy scheme in space and time.

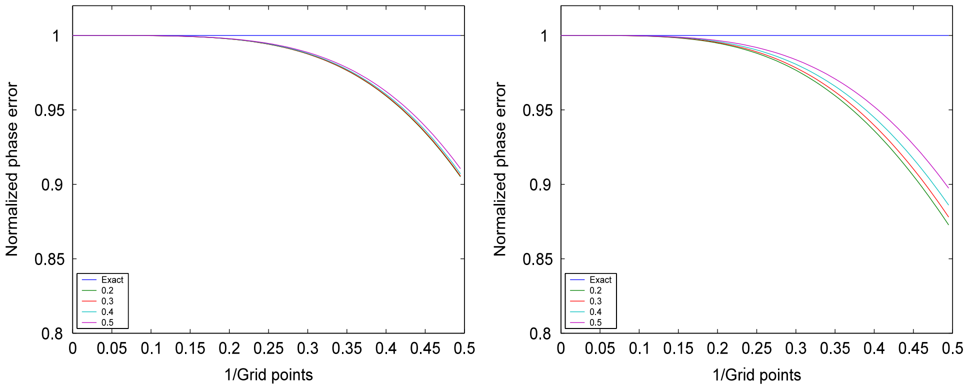


**Fig. 4.3.** The normalized phase error for different CFL numbers at a propagation angle $\theta = 30^\circ$. Left: new LOD scheme. Right: typical fourth-order accuracy scheme in space and time.

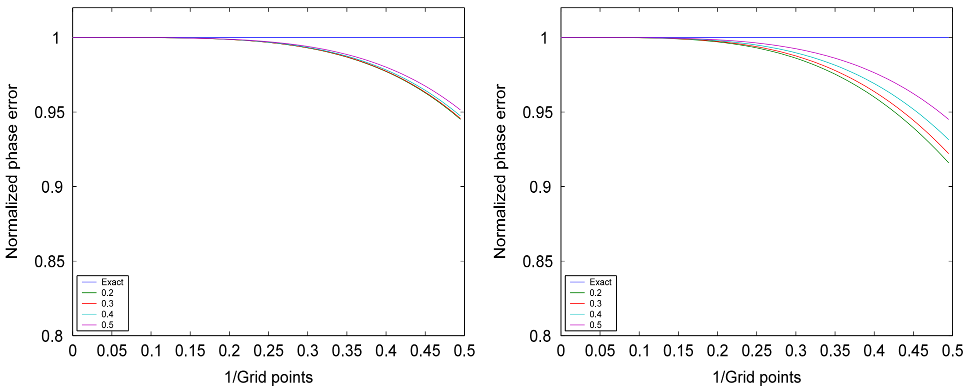


**Fig. 4.4.** The normalized phase error for different CFL numbers at a propagation angle $\theta = 45^\circ$. Left: new LOD scheme. Right: typical fourth-order accuracy scheme in space and time.

The dispersion relation of the exact wave equation (2.1) is

$$\omega^2 = v^2 |\boldsymbol{k}|^2. \tag{4.8}$$

For comparison, in Figs. 4.1–4.4, we have shown the dispersion curves associated with (4.2), (4.5) and (4.8). These curves give the variations of the nondimensional phase as a function of the inverse of the number of grid points per wavelength for various values of the CFL number $\alpha = v\Delta t/h$. Four different propagation angles $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ$ are displayed corresponding to Figs. 4.1–4.4. In Figs. 4.1–4.4, the left figure represents the phase error for the new LOD scheme and the right one shows that for the fourth-order scheme. There are five curves displayed in each figure. The horizontal line represents the dispersion relation of the exact wave equation as it has no phase error. The other four curves correspond to four different CFL numbers, i.e., 0.2, 0.3, 0.4, 0.5. Comparing the left figure and the right figure in Figs. 4.1–4.4 respectively shows that the phase error of the new LOD scheme is smaller than that of the typical fourth-order schemes at different propagation angles.

## 5. Initial and boundary conditions

In this section, we present how the initial and boundary conditions are computed for our new scheme (2.8).

The initial condition $u(x, y, 0)$ is given in (2.2). We consider how to obtain the initial condition $u^{-1}$ first. Suppose $f_1$ and $f_2$ in (2.2) are smooth enough. From

$$\begin{aligned}\frac{u^1-u^{-1}}{\Delta t} &= 2\left[\left(\frac{\partial u}{\partial t}\right)^0 + \frac{\Delta t^2}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)^0\right] + O(\Delta t^4)\\ &= 2\left[f_2 + \frac{v^2\Delta t^2}{6}\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\right] + O(\Delta t^4),\end{aligned} \tag{5.1}$$

we have the following approximation of $u^{-1}$ with fourth-order accuracy in time:

$$u^{-1} = u^1 - 2\Delta t\left[f_2 + \frac{v^2\Delta^2 t}{6}\left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\right)\right], \tag{5.2}$$

since

$$\frac{u^1-2u^0+u^{-1}}{\Delta t^2} = \left(\frac{\partial^2 u}{\partial t^2}\right)^0 + \frac{\Delta t^2}{12}\left(\frac{\partial^4 u}{\partial t^4}\right)^0 + O(\Delta t^4)$$

$$= v^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)^0 + \frac{v^2\Delta t^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4}\right)^0 + O(\Delta t^4) \tag{5.3}$$

$$= v^2\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2}\right) + \frac{v^2\Delta t^2}{12}\left(\frac{\partial^4 f_1}{\partial x^4} + 2\frac{\partial^4 f_1}{\partial x^2\partial y^2} + \frac{\partial^4 f_1}{\partial y^4}\right) + O(\Delta t^4). \tag{5.4}$$

Using (5.2) and (5.4) to cancel $u^1$, we then obtain

$$u^1 = u^0 + \Delta t\left(f_2 + \frac{v^2\Delta t^2}{6}\Delta f_2\right) + \frac{v^2\Delta t^2}{2}\left(\Delta f_1 + \frac{v^2\Delta t^2}{12}\varDelta^2 f_1\right), \tag{5.5}$$

where $\Delta f_1$ and $\Delta f_2$ represent the Laplacians of $f_1$ and $f_2$ respectively, and $\varDelta^2$ is the biharmonic operator. The approximation of $u^1$ with (5.5) reaches fourth-order accuracy in space and time. From the first equation in (2.9) we have

$$(b+\tau_x c_2\delta_x^2)u^1 + (a+\tau_x c_3\delta_x^2)u^0 = (a+b+\tau_x c_1\delta_x^2)u^s, \tag{5.6}$$

and then we have

$$u^s = \frac{bu^1+au^0}{a+b} + \frac{\tau_x c_2\delta_x^2 u^1 + \tau_x c_3\delta_x^2 u^0}{a+b} + \frac{\tau_x c_1\delta_x^2 u^s}{a+b}. \tag{5.7}$$

As

$$\delta_x^2 u^s = \frac{bu^1+au^0}{a+b} + O(h^4), \tag{5.8}$$

thus the following approximation of $u^s$:

$$u^s_{j,m} = \frac{bu^1+au^0}{a+b} + \frac{\tau_x c_2\delta_x^2 u^1_{j,m} + \tau_x c_3\delta_x^2 u^0_{j,m}}{a+b} + \frac{\tau_x c_1(b\delta_x^2 u^1_{j,m} + a\delta_x^2 u^0_{j,m})}{(a+b)^2},$$
$$j = 1, 2, N_x-2, N_x-1;\ m = 0, 1, \ldots, N_y \tag{5.9}$$

reaches fourth-order accuracy in time and space. Using (5.9), we can obtain the initial values of $u^s_{j,m}$ ($j = 1, 2, N_x-2, N_x-1$; $m = 0, 1, \ldots, N_y$), then using these $u^s_{j,m}$, we can obtain all initial values of $u^s_{j,m}$ ($j = 0, 1, \ldots, N_x$; $m = 0, 1, \ldots, N_y$) based on (5.6).

For the boundary conditions we only need to calculate the values of $u^{n+s}$ at the four angle points. On the basis of a similar procedure, we can obtain the four values of $u^{n+s}$ at the four angle points:

$$u^{n+s}_{j,m} = \frac{bu^{n+1}_{j,m}+au^0_{j,m}}{a+b} + \frac{\tau_x c_2\delta_x^2 u^{n+1} + \tau_x c_3\delta_x^2 u^n_{j,m}}{a+b} + \frac{\tau_x c_1(b\delta_x^2 u^{n+1} + a\delta_x^2 u^n_{j,m})}{(a+b)^2},\quad j = 0, N_x;\ m = 0, N_y. \tag{5.10}$$

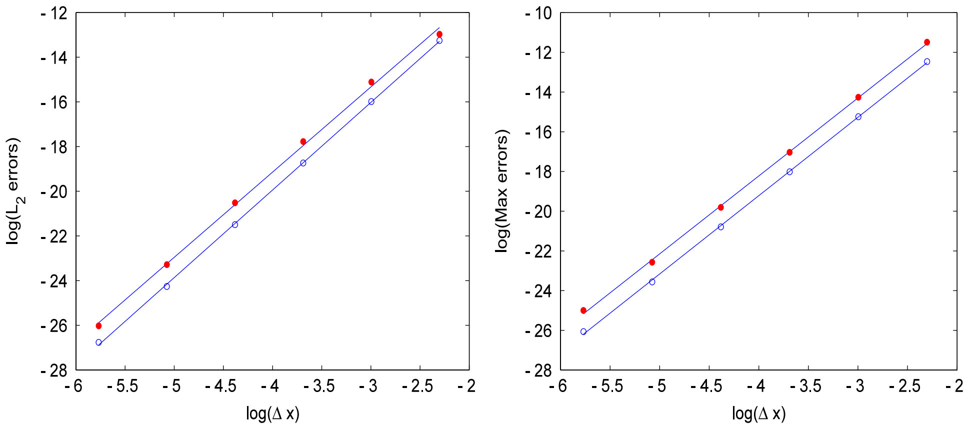


**Fig. 6.1.** A log–log plot for the $L_2$ norm errors (left) and maximum norm errors (right) for our new LOD scheme and the typical difference scheme. Blue curve: the new LOD scheme. The order of convergence is 3.92 measured in $L_2$ norm (left) and 3.95 in maximum norm (right). Red curve: typical difference scheme. The order of convergence is 3.81 measured in $L_2$ norm (left) and 3.93 in maximum norm (right). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Table 6.1**
Numerical error of $\|\cdot\|_2$ and maximum norms for LOD errors, and the comparison with the typical fourth-order schemes when the time sample is fixed.

| $N_x = N_y$ | New LOD scheme | | Typical 4th-order scheme | |
|---|---|---|---|---|
| | $\|\cdot\|_2$ norm | Max norm | $\|\cdot\|_2$ norm | Max norm |
| 10 | 1.750667e−06 | 3.851468e−06 | 2.329224e−06 | 1.021110e−05 |
| 20 | 1.142944e−07 | 2.400185e−07 | 2.739788e−07 | 6.380173e−07 |
| 40 | 7.312223e−09 | 1.499019e−08 | 1.905867e−08 | 3.994203e−08 |
| 80 | 4.624411e−10 | 9.364429e−10 | 1.225941e−09 | 2.497254e−09 |
| 160 | 2.906791e−11 | 5.850472e−11 | 7.749125e−11 | 1.572023e−10 |
| 320 | 2.382459e−12 | 4.789535e−12 | 4.990827e−12 | 1.386324e−11 |

**Table 6.2**
Numerical error of $\|\cdot\|_2$ and maximum norms for LOD errors, and the comparison with the typical fourth-order schemes when the spatial sample is fixed.

| $\Delta t$ | New LOD scheme | | Typical 4th-order scheme | |
|---|---|---|---|---|
| | $\|\cdot\|_2$ norm | Max norm | $\|\cdot\|_2$ norm | Max norm |
| 0.0008 | 3.777039e−12 | 7.568214e−12 | 8.580509e−12 | 1.771971e−11 |
| 0.0001 | 1.254529e−11 | 2.511082e−11 | 3.243765e−11 | 6.530024e−11 |
| 0.0016 | 1.907065e−10 | 3.813522e−10 | 5.187181e−10 | 1.050492e−09 |
| 0.0020 | 6.907978e−10 | 1.381601e−09 | 1.883205e−09 | 3.851588e−09 |
| 0.0025 | 2.432300e−09 | 4.864608e−09 | 6.599718e−09 | 1.370526e−08 |
| 0.0040 | 2.785763e−08 | 5.571525e−08 | 7.006676e−08 | 1.616834e−07 |

## 6. Numerical computations

In this section we will present numerical examples to illustrate our theoretical results obtained in the early sections. For simplicity we set $N_x = N_y := N$. The exact solution of the wave equation (2.1) is chosen as

$$u(x, y, t) = \sin(\pi x)\sin(\pi y)\cos(\sqrt{2}\pi t). \tag{6.1}$$

In Table 6.1, we fix the time step $\Delta t = 0.0002$ and consider six different spatial grids. To measure the approximation error, we use the $L_2$ norm

$$\|\cdot\|_2 := \frac{1}{N}\left\{\sum_{j=1}^{N_x}\sum_{m=1}^{N_y}[u_{j,m}^n - u_{\text{exact}}(x_j, x_m, t^n)]^2\right\}^{\frac{1}{2}}, \tag{6.2}$$

and the maximum norm, where $u_{\text{exact}}(x_j, x_m, t^n)$ is the exact solution. Table 6.1 shows the results after 100 time step extrapolations. We can see that the $L_2$ and the maximum norm errors of the new LOD scheme are smaller than that of the fourth-order scheme. In Fig. 6.1, we have shown log–log plots for the $L_2$ norm errors (left) and maximum norm errors (right) for our new LOD scheme and the typical difference scheme. The blue circles represent the errors for the LOD scheme with various mesh sizes and the blue line represents the least-square fitted line, obtained by using the blue circles. The red dots represent the errors for the typical difference scheme with various mesh sizes and the red line represents the least-square fitted line, obtained by using the red dots. In the left figure of Fig. 6.1, the slope of the blue line is 3.92 which suggests

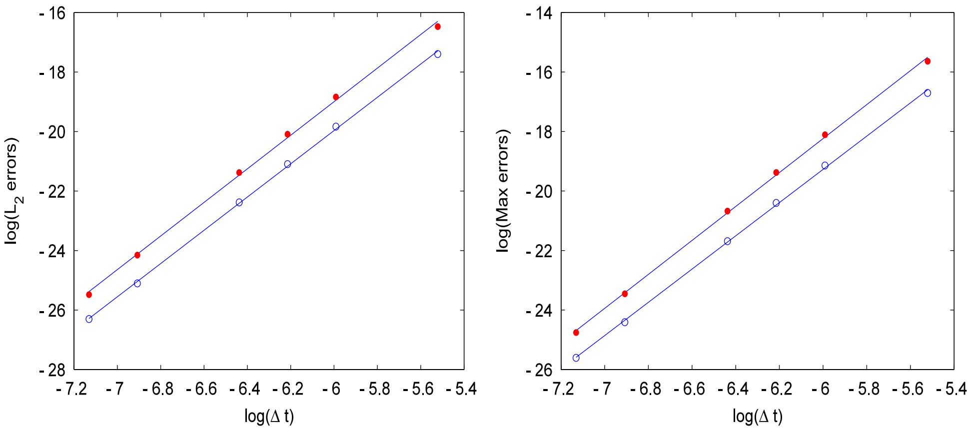


**Fig. 6.2.** A log–log plot for the $L_2$ norm errors (left) and maximum norm errors (right) for our new LOD scheme and the typical difference scheme. Blue curve: the LOD scheme. The order of convergence is 5.59 measured both in $L_2$ norm (left) and in maximum norm (right). Red curve: typical difference scheme. The order of convergence is 5.64 measured in $L_2$ norm (left) and 5.71 in maximum norm (right). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

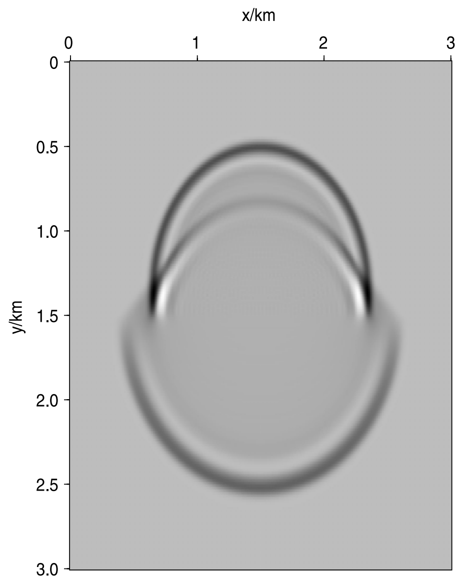


**Fig. 6.3.** A snapshot of the wavefield propagation at time 0.9 s in a two-layer medium. The velocity is 1500 m/s in the upper layer and 2200 m/s in the bottom layer.

that the order of convergence of the new LOD scheme is 3.92 while the slope of the red line is 3.81 which suggests that the order of convergence of the typical difference scheme is 3.81. In the right figure of Fig. 6.1, the slope of the blue line is 3.95 while the slope of the red line is 3.93. From these results, we see that the observed convergence order of our LOD scheme with respect to space may reach very close to 4.

In Table 6.2, we keep the CFL number fixed at 0.2 and consider different time steps where only time error is dominant. The results are those after 100 time step extrapolations. From Table 6.2, we also see that both the $L_2$ norm and the maximum norm errors of the new LOD scheme are smaller than those of the typical fourth-order scheme. The results in Tables 6.1 and 6.2 are consistent with the dispersion analysis results in Section 4. The order of convergence with respect to time can be obtained similarly. In Fig. 6.2, two log–log plots for the $L_2$ norm errors (left) and the maximum norm errors (right) for our new LOD scheme and the typical difference scheme are shown. In the left figure of Fig. 6.2, the slope of the blue line is 5.59 while the slope of the red line is 5.64. In the right figure of Fig. 6.2, the slope of the blue line is 5.64 while the slope of the red line is 5.71. From these results, we see that the observed convergence order for both schemes with respect to time can exceed 4 when the CFL number is fixed.

The second numerical test is for an inhomogeneous model with two layers, and the velocities in the two layers are respectively $c_1 = 1500$ m/s and $c_2 = 2200$ m/s. The initial conditions are $f_1 = f_2 = 0$. A point source is set on the right hand side of (2.1) to induce waves. We use the Ricker wavelet as its shape is similar to that found in a real geophysical oil exploration. The point source is depicted by

$$f(x,z,t) = \delta(x - x_0, z - z_0)\sin(60t)e^{-150t^2}, \tag{6.3}$$

where $\delta$ is the Dirac function, and $(x_0, z_0) = (1500\text{ m}, 900\text{ m})$ is the position of the source. In the numerical calculations, we set $h_x = h_y = 15$ m, $\Delta t = 0.0014$ s. Fig. 6.3 shows a snapshot of the wavefield at time $t = 0.9$ s. The direct wave propagating in two layers and the reflected wave caused by the interface are clearly shown in Fig. 6.3. The wavefronts of direct waves in each layer behave as part of circle while the circle in the second layer is larger than the one in the first layer as the velocity of the second layer is larger. The amplitude of the reflected wave in the first layer is relatively weak.

## 7. Conclusions

We proposed and analyzed a new locally one-dimensional (LOD) scheme with error of $O(\Delta t^4 + h^4)$ for the two-dimensional wave equation. More precisely, the order of convergence, stability and dispersion are rigorously analyzed. The new scheme involves only three layers in space while the traditional finite difference scheme with the same accuracy needs four layers in space. Numerical results prove that the new scheme has even better accuracy than the typical finite difference scheme of the same order. The initial and boundary conditions for the two intermediate time layers are explicitly constructed, which makes the scheme suitable for performing practical simulation in wave simulation modeling.

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