A SECOND-ORDER APPROXIMATION FOR THE VARIANCE OF A RENEWAL REWARD PROCESS *

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Let $\{C(t), t \geq 0\}$ be a renewal reward process. We obtain the approximation $\operatorname{Var} C(t) = ct + d + o(1)$, and explicitly identify c and d.

renewal process	cumulative process
renewal reward process	variance time curve

1. Introduction

Consider a sequence of independent random vectors $\{(X_i, Y_i), i = 0, 1, 2, \ldots\}$, where $(X_i, Y_i), i \geq 1$, are identically distributed. Assume that $\{X_i, i = 0, 1, \ldots\}$ is a renewal sequence. Define $S_j = \Sigma_{i=0}^{j} X_i$ for $j = 0, 1, \ldots$, and $N(t) = \{\min j\colon S_j > t\}$. Consider the process

$$C(t) = \begin{cases} 0, & t < X_0, \\ \sum_{i=0}^{N(t)-1} Y_i, & t \geq X_0, \end{cases} \qquad t \geq 0.$$

The process C is called a renewal reward process, and is a generalization of a renewal process and a special case of a cumulative process.

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Renewal reward processes occur in various stochastic optimization models ([6], [7, pp.51–54]), particularly in Markov and semi-Markov decision processes [7, pp.156–161]. Examples are found in inventory models, queues, counter models, dispatching problems and many others. In these models, Y_i represents the reward or cost associated with a given policy over the renewal interval $(S_{i-1}, S_i]$. The reward or cost is assumed to occur at the end of the interval rather than accumulate gradually.

Many results for renewal processes generalize to renewal reward processes. Some of these are the strong law and elementary renewal theorem [8, pp.27–28], central limit theorem [8, p.30], and Blackwell and key renewal theorems [1].

We derive the approximation $\mathrm{Var}\, C(t) = ct + d + o(1)$, where c and d are explicitly computed (Theorem 1 and Corollary 1). This generalizes the well-known approximation to $\mathrm{Var}\, N(t)$ (see [9, p.28]) which has been useful in inference for renewal processes ([2], [3, p.81]). Smith [8, p.28] has shown under suitable conditions that $\mathrm{Var}\, C(t) = ct + o(t)$, so that the coefficient c is known. Our contribution is showing that under stronger conditions $\mathrm{Var}\, C(t) = ct + d + o(1)$, and we evaluate d explicitly. The sharpened approximation to $\mathrm{E}\{C(t)\}$ (Lemma 1) and $\mathrm{Var}\, C(t)$ (Theorem 1 and Corollary 1) should be useful in sharpening the central limit (normal) approximation to the distribution of $C(t)$.

2. Derivation of results

We will prove a few lemmas needed for our main result (Theorem 1, Corollary 1). We use the notation $C_0(t)$ for a renewal reward process with $X_0 \equiv Y_0 \equiv 0$, and $C(t)$ for a general renewal reward process. We use $N_0(t)$ for an ordinary renewal process ($X_0 \equiv 0$, $Y_i \equiv 1$, $i = 0, 1, \ldots$) and $N(t)$ for a general renewal process ($Y_i \equiv 1$, $i = 0, 1, \ldots$). Define

$$D_0(t) = \mathrm{E}\{C_0(t)\}, \qquad D(t) = \mathrm{E}\{C(t)\},$$

$$M_0(t) = \mathrm{E}\{N_0(t)\}, \qquad M(t) = \mathrm{E}\{N(t)\}.$$

Denote the distribution of X_0 by F_0 and of X_1 (and thus of X_i for $i \geqslant 1$) by F. F is said to be *non-lattice* if there does not exist a $w > 0$ such that $\Sigma_{n=0}^{\infty} F\{nw\} = 1$. F is said to belong to the class $\mathcal{G}$ if some convolution of F has an absolutely continuous component. Define

$$\mu_j = \mathrm{E}\{X^j\}, \qquad \lambda_j = \mathrm{E}\{Y^j\}, \qquad r_{ij} = \mathrm{E}\{X^i Y^j\},$$

whenever these expectations exist. By *existence of an expectation* $\mathrm{E}\{g(X,Y)\}$ we mean that $\mathrm{E}\{|g(X,Y)|\} < \infty$. A function h on $[0,\infty)$ is said to be of *bounded variation* on $[0,\infty)$ if the total variation of h over $[0,\infty)$ is finite. We will use the fact that an integrable function of bounded variation on $[0,\infty)$ is directly Riemann integrable [4, p. 362].

Lemma 1. *If F is non-lattice and μ_2, λ_1 and n_{11} exist, then*

$$D_0(t) = at + b + \mathrm{o}(1),$$

where $a = \lambda_1/\mu_1$ and $b = \frac{1}{2}\mu_1^{-2}\mu_2\lambda_1 - \mu_1^{-1}n_{11}$. If, in addition, $\mathrm{E}\{X_0\}$ and $\mathrm{E}\{Y_0\}$ exist, then

$$D(t) = at + b + \mathrm{E}\{Y_0\} - a\mathrm{E}\{X_0\} + \mathrm{o}(1)\,.$$

Proof. The lemma follows by a standard-type application of the key renewal theorem, similar to [4, p. 366] or [5]. We condition on X_1, obtaining

$$D_0(t) = \int_0^t D_0(t-s)\,\mathrm{d}F(s) + \int_0^t \mathrm{E}\{Y \mid X=s\}\,\mathrm{d}F(s)\,;$$

thus

$$D_0(t) - at = \int_0^t [D_0(t-s) - a(t-s)]\,\mathrm{d}F(s) + a\int_t^\infty (1-F(s))\,\mathrm{d}s - \int_t^\infty \mathrm{E}\{Y \mid X=s\}\,\mathrm{d}F(s)\,.$$

Under the above assumptions,

$$a\int_t^\infty (1-F(s))\,\mathrm{d}s - \int_t^\infty \mathrm{E}\{Y \mid X=s\}\,\mathrm{d}F(s)$$

is directly Riemann integrable. Applying the key renewal theorem,

$$\lim_{t\to\infty}\{D_0(t) - at\} = \mu_1^{-1}\int_{t=0}^\infty \left[a\int_{s=t}^\infty (1-F(s))\,\mathrm{d}s - \int_{s=t}^\infty \mathrm{E}\{Y \mid X=s\}\,\mathrm{d}F(s)\right]\mathrm{d}t$$

$$= \mu_1^{-1}\left(\tfrac{1}{2}\mu_1^{-1}\lambda_1\mu_2 - n_{11}\right).$$

For general C,

$$D(t)-at=\int_0^t [D_0(t-s)-a(t-s)]\,\mathrm{d}F_0(s)-at(1-F_0(t))$$
$$-a\int_0^t s\,\mathrm{d}F_0(s)+\int_0^t \mathrm{E}\{Y_0\mid X_0=s\}\,\mathrm{d}F_0(s)\,.$$

Now $D_0(t)-at\to b$, thus there exists T such that $t>T$ implies $|D_0(t)-at-b|\leqslant 1$. Therefore

$$\sup_t\{|D_0(t)-at|\}\leqslant \sup_{t\leqslant T}\{|D_0(t)|\}+aT+|b|+1\,.$$

But

$$|D_0(t)|=\mathrm{E}\left\{\left|\sum_{i=1}^{N(t)-1} Y_i\right|\right\}\leqslant \mathrm{E}\left\{\sum_{i=1}^{N(t)}|Y_i|\right\}=\mathrm{E}\{N(t)\}\,\mathrm{E}\{|Y|\}$$

by Wald's identity (see [7]). Thus

$$\sup_{t\leqslant T}\{|D_0(t)|\}\leqslant \mathrm{E}\{N(T)\}\,\mathrm{E}\{|Y|\}<\infty,$$

thus

$$\sup_t\{|D_0(t)-at|\}<\infty.$$

Thus by the dominated convergence theorem,

$$\int_0^t [D_0(t-s)-a(t-s)]\,\mathrm{d}F_0(s)\to b\,.$$

Also, since $1-F_0(t)$ is monotone and $\int_0^\infty(1-F_0(t))\,\mathrm{d}t=\mathrm{E}\{X_0\}<\infty$, it follows that $t(1-F_0(t))\to 0$. Thus

$$D(t)-at=b+\mathrm{E}\{Y_0\}-a\mathrm{E}\{X_0\}+\mathrm{o}(1)\,.$$

This concludes the proof. □

Define

$$r(t)=M_0(t)-\mu_1^{-1}t-\tfrac{1}{2}\mu_1^{-2}\mu_2\,,$$

and set $I=\int_0^\infty r(t)\,\mathrm{d}t$ whenever $\int_0^\infty|r(t)|\,\mathrm{d}t<\infty$. Define $V_r(x)$ to be the total variation of r over $[-x,\infty]$.

Lemma 2. *If $F \in \mathcal{G}$ and $\mu_3 < \infty$, then*

$$l = \tfrac{1}{4}\mu_1^{-3}\mu_2^2 - \tfrac{1}{6}\mu_1^{-2}\mu_3 .$$

Proof. Smith [10, p. 2] derived the powerful result that $F \in \mathcal{G}$ and $\mu_3 < \infty$ implies

(i) $\lim_{t\to\infty} \{t\, r(t)\} = 0$,
(ii) $\int_0^\infty |r(t)|\, dt < \infty$,
(iii) $r(t)$ is of bounded variation on $[0, \infty)$.

First note that for $x > 0$,

$$V_r(x) \leqslant V_r(0) + \mu_1^{-1}x.$$

Consider the function

$$g(t) = \int_{x=0}^{t} x\, r(t-x)\, dF(x) .$$

Now

$$\int_0^\infty |g(t)|\, dt \leqslant \mu_1 \int_0^\infty |r(t)|\, dt < \infty$$

(by (ii)), thus g is integrable, and

$$\int_0^\infty g(t)\, dt = l\mu_1 .$$

Moreover, g is of bounded variation on $[0, \infty)$ since

$$\sum_i |g(t_i) - g(t_{i-1})| \leqslant \int_0^\infty x \left(\sum_i |r(t_i - x) - r(t_{i-1} - x)|\right) dF(x)$$

$$\leqslant \int_0^\infty x\, V_r(x)\, dF(x) \leqslant V_r(0)\, \mu_1 + \mu_1^{-1}\mu_2$$

($V_r(0) < \infty$ by (iii)). Therefore g is directly Riemann integrable.

Now

$$M_0(t) = 1 + \int_0^t M_0(t-x)\, dF(x) .$$

Subtract $\mu_1^{-1}t + \frac{1}{2}\mu_1^{-2}\mu_2$ from both sides and then multiply by t to obtain

$$t\,r(t) = \int_0^t (t-x)\,r(t-x)\,\mathrm{d}F(x) + Z(t)\,,$$

where

$$Z(t) = g(t) + \mu_1^{-1}t\int_t^\infty (1-F(x))\,\mathrm{d}x - \tfrac{1}{2}\mu_1^{-2}\mu_2 t(1-F(t))\,.$$

By the key renewal theorem

$$\lim_{t\to\infty}\{t\,r(t)\} = \mu_1^{-1}\int_0^t Z(t)\,\mathrm{d}t = \mu_1^{-1}\,[l\mu_1 + \tfrac{1}{6}\mu_1^{-1}\mu_3 - \tfrac{1}{4}\mu_1^{-2}\mu_2^2\,]\,.$$

But, by (i), $\lim_{t\to\infty}\{t\,r(t)\} = 0$, thus $l = \frac{1}{4}\mu_1^{-3}\mu_2^2 - \frac{1}{6}\mu_1^{-2}\mu_3$. This concludes the proof. □

Define

$$r^*(t) = D_0(t) - at - b,$$

and set $l^* = \int_0^\infty r^*(t)\,\mathrm{d}t$ whenever $\int_0^\infty |r^*(t)|\,\mathrm{d}t < \infty$.

Lemma 3. *If $F \in \mathcal{G}$, μ_3, λ_1 and n_{21} exist, then*

$$l^* = \lambda_1 l + \tfrac{1}{2}\mu_1^{-1}n_{21} - \tfrac{1}{2}\mu_1^{-2}\mu_2 n_{11}\,.$$

Moreover, r^* is directly Riemann integrable and

$$\lim_{t\to\infty}\{t\,r^*(t)\} = 0\,.$$

Proof. Note that

$$\mathrm{E}\{X|Y|\} \leqslant [\mathrm{E}\{X^2|Y|\}\,\mathrm{E}\{|Y|\}]^{\frac{1}{2}}\,,$$

so that n_{11} exists. Also,

$$C_0(t) = \sum_{i=1}^{N_0(t)-1} Y_i = \sum_{i=1}^{N_0(t)} Y_i - Y_{N_0(t)}\,.$$

Since $N_0(t)$ is a stopping time [7],

$$D_0(t) = \lambda_1 M_0(t) - \mathrm{E}\{Y_{N_0(t)}\}\ .$$

Subtracting $at + b$ from both sides we obtain

$$r^*(t) = \lambda_1 r(t) + \mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}.$$

Thus if $\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}$ is integrable, then

$$l^* = \lambda_1 l + \int_0^\infty (\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\})\, \mathrm{d}t\ .$$

But

$$\mu_1^{-1} n_{11} = \int_0^\infty \mathrm{E}\{Y \mid X=x\}\ \mu_1^{-1} x\ \mathrm{d}F(x)\ , \tag{1}$$

and

$$\mathrm{E}\{Y_{N_0(t)}\} = \int_0^\infty \mathrm{E}\{Y \mid X=x\}\ (M(t) - M(t-x))\, \mathrm{d}F(x)\ . \tag{2}$$

(2) follows by letting $h(t) = \mathrm{E}\{Y_{N_0(t)}\}$ and setting up the renewal equation

$$h(t) = \int_0^t h(t-y)\ \mathrm{d}F(y) + \int_t^\infty \mathrm{E}\{Y \mid X=x\}\ \mathrm{d}F(x)\ .$$

The solution to this renewal equation is

$$h(t) = \int_{z=0}^{t} \int_{x=t-z}^{\infty} \mathrm{E}\{Y \mid X=x\}\ \mathrm{d}F(x)\ \mathrm{d}M(z)$$

$$= \int_{x=0}^{\infty} \int_{z=t-x}^{t} \mathrm{d}M(z)\ \mathrm{E}\{Y \mid X=x\}\ \mathrm{d}F(x)$$

$$= \int_{x=0}^{\infty} (M(t) - M(t-x))\ \mathrm{E}\{Y \mid X=x\}\ \mathrm{d}F(x)\ .$$

It follows from (1) and (2) that

$$\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\} = \int_{x=0}^{\infty} \mathrm{E}\{Y \mid X=x\}\ (r(t-x) - r(t))\, \mathrm{d}F(x)\ . \tag{3}$$

Thus

$$\int_0^\infty |\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}|\,\mathrm{d}t \leq$$

$$\leq \int_{x=0}^\infty \left[\int_{t=0}^\infty (|r(t-x)| + |r(t)|)\,\mathrm{d}t\right] \mathrm{E}(|Y|\,|\,X=x)\,\mathrm{d}F(x)$$

$$\leq \int_{x=0}^\infty \left[2\int_0^\infty |r(t)|\,\mathrm{d}t + \tfrac{1}{2}\mu_1^{-1}x^2 + \tfrac{1}{2}\mu_1^{-2}\mu_2 x\right] \mathrm{E}\{|Y|\,|\,X=x\}\,\mathrm{d}F(x)$$

$$= 2\mathrm{E}\{|Y|\}\int_0^\infty |r(t)|\,\mathrm{d}t + \tfrac{1}{2}\mu_1^{-1}\,\mathrm{E}\{X^2|Y|\} + \tfrac{1}{2}\mu_1^{-2}\mu_2\mathrm{E}\{X|Y|\}\,.$$

Thus $\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}$ is integrable, and

$$\int_0^\infty (\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\})\,\mathrm{d}t =$$

$$= \int_{x=0}^\infty \left(\int_{t=-x}^0 r(t)\,\mathrm{d}t\right) \mathrm{E}\{Y\,|\,X=x\}\,\mathrm{d}F(x)$$

$$= \tfrac{1}{2}\mu_1^{-1} n_{21} - \tfrac{1}{2}\mu_1^{-2}\mu_2 n_{11}\,.$$

Since r is directly Riemann integrable and

$$r^* = \lambda_1 r + \mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}\,,$$

to show that r^* is directly Riemann integrable it suffices to show that $\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}$ is of bounded variation on $[0, \infty)$. But by (3),

$$\sum_i |\mathrm{E}\{Y_{N_0(t_i)}\} - \mathrm{E}\{Y_{N_0(t_{i-1})}\}| \leq$$

$$\leq \int_{s=0}^\infty \left[\mathrm{E}\{|Y|\,|\,X=s\}\left[\sum_i |r(t_i-s) - r(t_{i-1}-s)| + \sum_i |r(t_i) - r(t_{i-1})|\right]\right]\mathrm{d}F(s)$$

$$\leq \int_{s=0}^\infty \mathrm{E}\{|Y|\,|\,X=s\}\,[2V_r(0) + \mu_1^{-1}s]\,\mathrm{d}F(s)$$

$$= 2V_r(0)\,\mathrm{E}\{|Y|\} + \mu_1^{-1}\mathrm{E}\{X\,|Y|\}\,.$$

Next we want to show that $tr^*(t) \to 0$. It suffices (since $tr(t) \to 0$) to show that

$$t(\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}) \to 0 .$$

But by (3),

$$t(\mu_1^{-1} n_{11} - \mathrm{E}\{Y_{N_0(t)}\}) = \int_0^\infty \mathrm{E}\{Y \mid X=x\}\, t(r(t-x)-r(t))\, \mathrm{d}F(x).$$

Note that $tr(t) \to 0$; thus for some T and all $t \geqslant T$, $|tr(t)| < 1$, and thus

$$\sup_{t\geqslant 0}\{|tr(t)|\} \leqslant T[M_0(T) + \mu_1^{-1} T + \tfrac{1}{2}\mu_1^{-2}\mu_2] + 1 ;$$

similarly,

$$\sup_{t\geqslant 0}\{|r(t)|\} < \infty .$$

Therefore

$$\begin{aligned}|t(r(t-x)-r(t))| &\leqslant 2 \sup_{t\geqslant -x}\{|tr(t)|\} + x \sup_{t\geqslant -x}\{|r(t)|\}\\ &\leqslant 2 \sup_{t\geqslant 0}\{|tr(t)|\} + x \sup_{t\geqslant 0}\{|r(t)|\}\\ &\quad + 2(\mu_1^{-1}x^2 + \tfrac{1}{2}\mu_1^{-2}\mu_2 x) + (\mu_1^{-1}x^2 + \tfrac{1}{2}\mu_1^{-2}\mu_2 x).\end{aligned}$$

Since n_{11} and n_{21} exist, $t(r(t-x)-r(t))\mathrm{E}\{Y \mid X=x\}$ is dominated by an integrable function, and thus the dominated convergence theorem shows that $t(\mu_1^{-1}n_{11} - \mathrm{E}\{Y_{N_0(t)}\}) \to 0$. This concludes the proof. □

Define

$$\widetilde{D}_0(t) = \mathrm{E}\left\{\sum_{i=1}^{N_0(t)-1} Y_i^2\right\}, \qquad D_0^{(2)}(t) = \int_0^t D_0(t-s)\, \mathrm{d}D_0(s) .$$

Let $F^{(K)}$ be the K^{th} convolution of F, and let $f_K = \mathrm{d}F^{(K)}/\mathrm{d}M_0$. Since $M_0 = \Sigma_{K=0}^{\infty} F^{(K)}$, it follows that $M_0 = 0$ implies that all $F^{(K)} = 0$; thus $F^{(K)} \ll M_0$, and f_K is well defined.

Lemma 4. $\operatorname{Var} C_0(t) = \widetilde{D}_0(t) + 2D_0^{(2)}(t) - (D_0(t))^2$.

Proof. Since

$$\operatorname{Var} C_0(t) + (D_0(t))^2 - \tilde{D}_0(t) = 2\mathrm{E}\left\{\sum_{i<j\leq N_0(t)-1} Y_i Y_j\right\},$$

our task is to show that

$$\mathrm{E}\left\{\sum_{i<j\leq N_0(t)-1} Y_i Y_j\right\} = D_0^{(2)}(t).$$

Now

$$D_0(t) = \mathrm{E}\left\{\sum_{i=1}^{\infty} Y_i I_{S_i<t}\right\} = \sum_{i=1}^{\infty} \int_0^t \mathrm{E}\{Y_i \mid S_i = s\}\, \mathrm{d}F^{(i)}(s)$$

$$= \int_0^t \left[\sum_{i=1}^{\infty} \mathrm{E}\{Y_i \mid S_i = s\}\, f_i(s)\right] \mathrm{d}M_0(s).$$

Therefore

$$\frac{\mathrm{d}D_0(s)}{\mathrm{d}M_0(s)} = \sum_{i=1}^{\infty} \mathrm{E}\{Y_i \mid S_i = s\}\, f_i(s).$$

Next

$$\mathrm{E}\left\{\sum_{i<j\leq N_0(t)-1} Y_i Y_j\right\} = \mathrm{E}\left\{\sum_{i<j} Y_i Y_j I_{S_j<t}\right\}$$

$$= \sum_{i<j} \int_{w=0}^{t} \int_{s=w}^{t} \mathrm{E}\{Y_i \mid S_i = w\}\, \mathrm{E}\{Y_{j-i} \mid S_{j-i} = s-w\}\, \mathrm{d}F^{(i)}(w)\, \mathrm{d}F^{(j-i)}(s-w)$$

$$= \int_{w=0}^{t} \int_{s=w}^{t} \left(\sum_i \mathrm{E}\{Y_i \mid S_i = w\}\, f_i(w)\right)$$

$$\times \left(\sum_{k=1}^{\infty} \mathrm{E}\{Y_k \mid S_k = s-w\}\, f_k(s-w)\right) \mathrm{d}M_0(s-w)\, \mathrm{d}M_0(w)$$

$$= \int_{w=0}^{t} D_0(t-w) \left(\frac{\mathrm{d}D_0(w)}{\mathrm{d}M_0(w)}\right) (\mathrm{d}M_0(w))$$

$$= \int_{w=0}^{t} D_0(t-w)\, \mathrm{d}D_0(w) = D_0^{(2)}(t).$$

Theorem 1. *If $F \in \mathcal{G}$ and μ_3, λ_2 and n_{12} exist, then*

$$\operatorname{Var} C_0(t) = ct + d + o(1),$$

where

$$c = \mu_1^{-3}\mu_2\lambda_1^2 - 2\mu_1^{-2}n_{11}\lambda_1 + \mu_1^{-1}\lambda_2 = \mu_1^{-1}\operatorname{Var}(Y - aX),$$

$$d = \tfrac{5}{4}\mu_1^{-4}\mu_2^2\lambda_1^2 - \tfrac{2}{3}\mu_1^{-3}\mu_3\lambda_1^2 + 2\mu_1^{-2}n_{21}\lambda_1$$
$$- 3\mu_1^{-3}\mu_2\lambda_1 n_{11} + \mu_1^{-2}n_{11}^2 + \tfrac{1}{2}\mu_1^{-2}\mu_2\lambda_2 - \mu_1^{-1}n_{12}.$$

Proof. First note that $\mathrm{E}\{X\,|Y|\} \leqslant [\mathrm{E}\{XY^2\}\ \mathrm{E}\{X\}]^{\frac{1}{2}}$ and $\mathrm{E}\{X^2\,|Y|\} \leqslant [\mathrm{E}\{XY^2\}\ \mathrm{E}\{X^3\}]^{\frac{1}{2}}$, so that n_{11} and n_{21} exist.

It follows from Lemma 1 that

$$\widetilde{D}_0(t) = \mu_1^{-1}\lambda_2 t + \tfrac{1}{2}\mu_1^{-2}\mu_2\lambda_2 - \mu_1^{-1}n_{12} + o(1), \tag{4}$$

$$(D_0(t))^2 = a^2t^2 + 2abt + b^2 + o(1). \tag{5}$$

Now

$$D_0^{(2)}(t) = \int_0^t D_0(t-s)\,\mathrm{d}D_0(s)$$
$$= \int_0^t r^*(t-s)\,\mathrm{d}D_0(s) + \int_0^t (a(t-s)+b)\,\mathrm{d}D_0(s).$$

By Lemma 3, r^* is directly Riemann integrable. It thus follows from a generalization of the key renewal theorem to renewal reward processes [1, p. 101] that

$$\int_0^t r^*(t-s)\,\mathrm{d}D_0(s) = al^* + o(1).$$

Thus

$$D_0^{(2)}(t) = al^* + \int_0^t (a(t-s)+b)\,\mathrm{d}D_0(s) + o(1).$$

Next

$$\int_0^t (a(t-s)+b)\,\mathrm{d}D_0(s) = bD_0(t) + a\int_0^t D_0(s)\,\mathrm{d}s$$
$$= bD_0(t) + a\int_0^t r^*(s)\,\mathrm{d}s + a\int_0^t (as+b)\,\mathrm{d}s$$
$$= \tfrac{1}{2}a^2t^2 + 2abt + al^* + b^2 + o(1).$$

Combining, we have

$$2D_0^{(2)}(t) = a^2t^2 + 4abt + 4al^* + 2b^2 + o(1). \tag{6}$$

It follows from (4), (5), (6) and Lemma 4 that

$$\operatorname{Var} C_0(t) = (2ab + \mu_1^{-1}\lambda_2)t$$

$$+ (4al^* + b^2 + \tfrac{1}{2}\mu_1^{-2}\mu_2\lambda_2 - \mu_1^{-1}n_{12}) + o(1).$$

Substituting for a and b (Lemma 1) and l^* (Lemma 3), we obtain the result. □

Define $\bar{r}(t) = \operatorname{Var} C_0(t) - ct - d$.

Corollary 1. *If $F \in \mathcal{G}$, and μ_3, λ_2, n_{12}, $\mathrm{E}\{X_0^2\}$ and $\mathrm{E}\{Y_0^2\}$ exist, then*

$$\operatorname{Var} C(t) = ct + d - c\,\mathrm{E}\{X_0\} + \operatorname{Var}(Y_0 - aX_0) + o(1),$$

where c and d are given in Theorem 1.

Proof. $C(t) = I_{X_0 \le t}[Y_0 + C_0(t - X_0)]$; thus

$$(C(t))^2 = I_{X_0 \le t}[Y_0^2 + 2Y_0C_0(t - X_0) + C_0^2(t - X_0)].$$

Now

$$\mathrm{E}\{I_{X_0 \le t}Y_0^2\} = \int_0^t \mathrm{E}\{Y_0^2 \mid X_0 = s\}\,\mathrm{d}F_0(s) = \mathrm{E}\{Y_0^2\} + o(1).$$

Next

$$2\mathrm{E}\{I_{X_0 \le t}Y_0C_0(t - X_0)\} =$$

$$= 2\mathrm{E}[I_{X_0 \le t}\,\mathrm{E}(Y_0 \mid X_0)\,[a(t - X_0) + b + r^*(t - X_0)]]$$

$$= 2at\,\mathrm{E}\{Y_0\} - 2a\,\mathrm{E}\{X_0Y_0\} + 2b\,\mathrm{E}\{Y_0\} + o(1)$$

since $\lim_{t\to\infty}\{r^*(t)\} = 0$, $\sup_t\{|r^*(t)|\} < \infty$ and $\mathrm{E}\{|Y_0|\} < \infty$ imply

$$\int_0^t r^*(t - s)\,\mathrm{E}\{Y_0 \mid X_0 = s\}\,\mathrm{d}F_0(s) \to 0.$$

Finally,

$$\mathrm{E}\{I_{X_0<t}\, C_0^2(t-X_0)\} =$$

$$= \mathrm{E}\{I_{X_0<t}[(c(t-X_0)+d+\bar{r}(t-X_0)) + (a(t-X_0)+b+r^*(t-X_0))^2]\}$$

$$= \mathrm{E}\{I_{X_0<t}[a^2t^2 + t(c-2a^2X_0+2ab) + (d-cX_0+a^2X_0^2+b^2-2abX_0)$$

$$+ (\bar{r}(t-X_0) + (r^*(t-X_0))^2 + 2(a(t-X_0)+b)\, r^*(t-X_0)]\}.$$

By Lemma 1 and Theorem 1, $\bar{r}(t)$ and $r^*(t) \to 0$. Moreover, $\sup_t\{|\bar{r}(t)|\} < \infty$ and $\sup_t\{|r^*(t)|\} < \infty$, thus $\int_0^t \bar{r}(t-x)\, \mathrm{d}F_0(x)$, $\int_0^t (r^*(t-x))^2\, \mathrm{d}F_0(x)$ and $\int_0^t r^*(t-x)\, \mathrm{d}F_0(x)$ all converge to 0. Moreover, by Lemma 3, $t\,r^*(t) \to 0$, and it easily follows that $\sup_t\{|t\,r^*(t)| < \infty\}$; thus

$$\mathrm{E}\{I_{X_0<t}(t-X_0)\, r^*(t-X_0)\} = \int_0^t (t-x)\, r^*(t-x)\, \mathrm{d}F_0(x) \to 0\,.$$

Therefore

$$\mathrm{E}\{I_{X_0<t}\, C_0^2(t-X_0)\} = a^2t^2 + t(c - 2a^2\, \mathrm{E}\{X_0\} + 2ab)$$

$$+ (d - c\,\mathrm{E}\{X_0\} + a^2\,\mathrm{E}\{X_0^2\} + b^2 - 2ab\,\mathrm{E}\{X_0\})$$

$$+ \mathrm{o}(1)\,.$$

Thus

$$\mathrm{E}\{C^2(t)\} = a^2t^2 + t\,(c - 2a^2\,\mathrm{E}\{X_0\} + 2ab + 2a\,\mathrm{E}\{Y_0\})$$

$$+ (d - c\,\mathrm{E}\{X_0\} + a^2\,\mathrm{E}\{X_0^2\} - 2a\,\mathrm{E}\{X_0Y_0\}$$

$$+ \mathrm{E}\{Y_0^2\} + b^2 - 2ab\,\mathrm{E}\{X_0\} + 2b\,\mathrm{E}\{Y_0\}) + \mathrm{o}(1)\,.$$

By Lemma 1,

$$(\mathrm{E}\{C(t)\})^2 = a^2t^2 + t(2ab - 2a^2\,\mathrm{E}\{X_0\} + 2a\,\mathrm{E}\{Y_0\})$$

$$+ (a^2(\mathrm{E}\{X_0\})^2 - 2a\,\mathrm{E}\{X_0\}\mathrm{E}\{Y_0\}$$

$$+ (\mathrm{E}\{Y_0\})^2 + b^2 - 2ab\,\mathrm{E}\{X_0\} + 2b\,\mathrm{E}\{Y_0\}) + \mathrm{o}(1)\,.$$

Subtracting $(\mathrm{E}\{C(t)\})^2$ from $\mathrm{E}\{C^2(t)\}$ yields the result. □

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