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Comparison morphisms and the Hochschild cohomology ring of truncated quiver algebras ☆

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ABSTRACT

A main contribution of this paper is the explicit construction of comparison morphisms between the standard bar resolution and Bardzell's minimal resolution for truncated quiver algebras over arbitrary fields (TQA's).
As a direct application we describe explicitly the Yoneda product and derive several results on the structure of the cohomology ring of TQA's over a field of characteristic zero. For instance, we show that the product of odd degree cohomology classes is always zero. We prove that TQA's associated with quivers with no cycles or with neither sinks nor sources have trivial cohomology rings. On the other side we exhibit a fundamental example of a TQA with nontrivial cohomology ring. Finally, for truncated polynomial algebras in one variable, we construct explicit cohomology classes in the bar resolution and give a full description of their cohomology ring.

1. Introduction

To any finite quiver Δ and any field k one associates a k-algebra kΔ, the *path algebra* or *quiver algebra* of Δ, where the set of vertices Δ_0 and the sets of k-paths Δ_k form a k-basis and the product is given by concatenation of paths (see Section 2).

Quiver algebras and their quotients arise in many contexts and have been extensively studied. A result of Gabriel [G] establishes that for every finite dimensional k-algebra A such that $A/r = \mathrm{k} \times \cdots \times \mathrm{k}$, where r is the Jacobson radical of A, there exists a finite quiver Δ, the Gabriel quiver of A,

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and an epimorphism $\varphi : \mathrm{k}\Delta \to A$ such that $(\Delta_N) \subseteq \ker\varphi \subseteq (\Delta_2)$ for some $N \geqslant 2$. Here (Δ_k) is the two-sided ideal generated by Δ_k.

Monomial algebras are those for which $\ker\varphi$ is generated by monomials. In the particular case when $\ker\varphi = (\Delta_N)$, the algebra A is an *N-truncated quiver algebra*, denoted N-TQA, or simply TQA, from now on. This property of A turns out to be intrinsic [Ci2] which makes TQA's a distinguished class.

For these classes of algebras Bardzell [Ba] introduced a minimal resolution that plays a key role in the treatment of homological questions and problems.

A main contribution of this paper is the explicit construction of comparison morphisms between the standard bar resolution and Bardzell's minimal resolution for TQA's. We believe that such morphisms, sought for a long time, should have many applications. Our construction in this case could inspire others to find comparison morphisms for wider classes of algebras, hopefully for all monomial algebras.

As a direct application we describe explicitly the Yoneda product and derive several results on the structure of the cohomology ring of TQA's over a field k of characteristic zero. In the near future we will complete a full description of this ring.

1.1. A brief account of known results

Since the early nineties several authors have investigated different homological questions for a number of classes of monomial algebras A, including TQA's.

In the context of truncated quiver algebras, Cibils [Ci1] proved that their n-th homology group $H_n(A, A)$ vanishes for all $n > 0$ if the quiver has no oriented cycles. A shorter proof of this fact was later given in [Ci2]. On the other hand, Liu and Zhang [LZ] showed that $H_n(A, A) = 0$ for all $n > 0$ if and only if the quiver has no oriented cycles of some specific lengths. More recently, Sköldberg [Sk] gave a complete description of the homology of TQA's. His computations are based on the use of Bardzell's minimal resolution. In the same paper he treats also the case of quadratic monomial algebras with an analogous approach.

Bardzell's resolution was introduced in [Ba] for monomial algebras, and has shown to be an efficient tool for computations in contrast to the usual bar resolution. Recently, Marconnet in [M] constructed a comparison morphism for the first nontrivial degree, in the context of cubic Artin–Schelter regular algebras.

The first cohomology computations for TQA's appeared in [Ci2] where the second cohomology group is described to study formal deformations and to characterize rigidity. In the subsequent paper [Ci3] these results were extended to the class of monomial algebras.

A description of the whole cohomology of TQA's, over fields of characteristic zero, was given by Locateli in [Lo]. Her computations also rely on the use of Bardzell's minimal resolution, and cohomology classes are represented by pairs of parallel paths. The particular case of truncated *cycle algebras*, those associated to an n-cycle quiver, is treated separately. Recently in [XHJ] the case of arbitrary characteristic was solved.

The determination of the structure of the full cohomology ring is still a difficult problem that has been addressed in a number of cases.

For instance, for a radical square zero algebra, a description of the Yoneda product on Hochschild cohomology is given in [Ci4] and it is shown that this algebra is finitely generated only for the case when the underlying quiver is a cycle or it has no oriented cycles.

For truncated cycle algebras, the complete structure of the cohomology ring was determined in [BLM] and independently in [EH], showing in particular that the Yoneda product is nontrivial and the cohomology ring is finitely generated (see also [Ho] and [FS]). Cycle algebras are examples of self-injective Nakayama algebras. In [BLM] the authors present in contrast some examples of noninjective Nakayama algebras for which the product is trivial (in nonzero degree) and in particular the cohomology ring, which is infinite, is not finitely generated.

Since the Hochschild cohomology ring $H^*(A, A)$ is a graded commutative k-algebra, every homogeneous element of odd degree squares to zero (*char* $\mathrm{k} \neq 2$) and if $\mathcal{N}$ is the ideal generated by the homogeneous nilpotent elements, then $H^*(A,A)/\mathcal{N}$ is a commutative k-algebra. One expects to gain information for the full cohomology ring from this simpler one.

In [SS] it was conjectured that $H^*(A,A)/\mathcal{N}$ is finitely generated as a k-algebra for any finite dimensional algebra A. This was recently proved in [GSS] for monomial algebras and was already known for some other classes (see [GSS]). In [GS] the quotient $H^*(A,A)/\mathcal{N}$ was determined for the subclass of stacked monomial algebras, a class that contains TQA's. These results applied to noncycle TQA's yield $H^*(A,A)/\mathcal{N} \simeq \mathrm{k}$. However, since $H^*(A,A)$ is, in general, infinite dimensional over k, the structure of the full ring $H^*(A,A)$ remains open.

1.2. An overview of the main results

Given A an associative k-algebra with unit, the Hochschild cohomology groups $H^n(A,A)$ are, by definition, the groups $\mathrm{Ext}^n_{A^e}(A,A)$ where $A^e = A \otimes_{\mathrm{k}} A^{op}$. The natural identification between A-bimodules and left A^e-modules gives the definition of projective A-bimodule and A-bimodule homomorphism.

The standard bar resolution of A is the A^e-projective resolution

$$\cdots \to A \otimes A^{\otimes n} \otimes A \xrightarrow{b} A \otimes A^{\otimes(n-1)} \otimes A \cdots \xrightarrow{b} A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{\epsilon} A$$

where all tensors are over k and since $\mathrm{Hom}_{A^e}(A \otimes A^{\otimes n} \otimes A, A) \simeq \mathrm{Hom}_{\mathrm{k}}(A^{\otimes n}, A)$, the associated Hochschild complex is

$$A \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}(A,A) \xrightarrow{b} \cdots \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes(n-1)}, A\big) \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes n}, A\big) \xrightarrow{b} \cdots.$$

The cohomology group $H^*(A,A)$ has a ring structure given by the Yoneda product which coincides with the cup product defined as follows. Given two cochains,

$$f \in \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes m}, A\big), \qquad g \in \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes n}, A\big)$$

the cup product of f and g is the cochain $f \cup g \in \mathrm{Hom}_{\mathrm{k}}(A^{\otimes(m+n)}, A)$ defined by

$$f \cup g(\alpha_1 \otimes \cdots \otimes \alpha_{m+n}) = f(\alpha_1 \otimes \cdots \otimes \alpha_m) g(\alpha_{m+1} \otimes \cdots \otimes \alpha_{m+n}).$$

For TQA's, the bar resolution can be slightly simplified with the A-bimodule

$$\mathbf{Q}_n = A \otimes_{\Delta_0} A_+^{\otimes^n_{\Delta_0}} \otimes_{\Delta_0} A,$$

in place of $A \otimes A^{\otimes n} \otimes A$, where A_+ is the ideal $\bigoplus_{n=1}^{N-1} \mathrm{k}\Delta_n$ and $\otimes_{\Delta_0}$ means $\otimes_{\mathrm{k}\Delta_0}$. For the definition of the differential see Section 3.1.

In contrast to this resolution, there is the following minimal resolution $(\mathbf{P}, d)$, due to Bardzell [Ba] (cf. [AG,BK] and [Ha]), where the A^e-projective modules are

$$\mathbf{P}_n = \begin{cases} A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN} \otimes_{\Delta_0} A, & \text{if } n = 2k; \\ A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN+1} \otimes_{\Delta_0} A, & \text{if } n = 2k+1. \end{cases}$$

One has

$$\mathbf{P}^*_n = \mathrm{Hom}_{A^e}(\mathbf{P}_n, A) \simeq \begin{cases} \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN}, A), & \text{if } n = 2k; \\ \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN+1}, A), & \text{if } n = 2k+1. \end{cases}$$

The definition of the differential is in Section 3.2. For more details see Section 7.

We define the following A^e-morphisms between these two resolutions in both directions. See Section 4 for complete details.

First, let $\mathbf{F}:\mathbf{P}\to\mathbf{Q}$ be the A-bimodule extension of the map defined on $p_0=1\otimes v_1\dots v_{kN}\otimes 1\in\mathbf{P}_{2k}$ and $p_1=1\otimes v_1\dots v_{kN+1}\otimes 1\in\mathbf{P}_{2k+1}$, where v_i is an arrow for all i, by

$$\mathbf{F}_{2k}(p_0)=\sum 1[\underbrace{v_1\dots v_{x_1}}_{x_1}|\underbrace{v_{1+x_1}}_{1}|\underbrace{\dots v_{1+x_1+x_2}}_{x_2}|\underbrace{v_{2+x_1+x_2}}_{1}|\dots\dots|\underbrace{v_{k+\sum x_j}}_{1}]\underbrace{\dots\dots v_{kN}}_{kN-k-\sum x_j},$$

$$\mathbf{F}_{2k+1}(p_1)=\sum 1[\underbrace{v_1}_{1}|\underbrace{v_2\dots v_{1+x_1}}_{x_1}|\underbrace{v_{2+x_1}}_{1}|\underbrace{\dots v_{2+x_1+x_2}}_{x_2}|\dots\dots|\underbrace{v_{k+1+\sum x_j}}_{1}]\underbrace{\dots\dots v_{kN+1}}_{kN-k-1-\sum x_j},$$

where the sum is taken over all k-tuples $(x_1,\dots,x_k)\in\mathbb{Z}^k$ such that $1\leqslant x_i<N$.

Second, let $\mathbf{G}:\mathbf{Q}\to\mathbf{P}$ be the A-bimodule extension of the map defined on $q=1[\alpha_1|\dots|\alpha_n]1=a_1^1\dots a_{|\alpha_1|}^1 a_1^2\dots a_{|\alpha_2|}^2\dots\dots a_1^n\dots a_{|\alpha_n|}^n=v_1\dots v_{|q|}\in\mathbf{Q}_n$, by

$$\mathbf{G}_{2k}(q)=\begin{cases}1\otimes v_1\dots v_{kN}\otimes v_{kN+1}\dots v_{|q|}, & \text{if } \alpha_{2i-1}\alpha_{2i}=0 \text{ for } i=1,\dots,k;\\ 0, & \text{otherwise;}\end{cases}$$

$$\mathbf{G}_{2k+1}(q)=\begin{cases}\sum_{j=1}^{|\alpha_1|} v_1\dots v_{j-1}\otimes v_j\dots v_{kN+j}\otimes v_{kN+j+1}\dots v_{|q|}, & \text{if } \alpha_{2i}\alpha_{2i+1}=0 \text{ for } i=1,\dots,k;\\ 0, & \text{otherwise.}\end{cases}$$

Our first result is the following theorem (see Theorem 4.1).

Theorem. *The morphisms* $\mathbf{F}$ *and* $\mathbf{G}$ *between the resolutions* $\mathbf{P}$ *and* $\mathbf{Q}$ *are both comparison morphisms.*

The proofs are long, Sections 6 and 5 are exclusively devoted to them. They are subtle and give an insight on the nontrivial combinatorics underlying this problem.

When the field k is of characteristic zero, we use the comparison morphisms and the description of the cohomology given in [Lo] to describe the Yoneda product at the level of the minimal resolution and to determine the product in cohomology. The minimal resolution is naturally bigraded, but the product in this resolution is not compatible with this bigrading. However, the product at the cohomology level, which is essentially given by concatenation of paths, is compatible with the bigrading making $H^*(A,A)$ a bigraded ring.

More precisely, let $\vee$ be the product in $\mathbf{P}^*$ defined in the following way. For $(\alpha,\pi)\in\mathbf{P}^*_{n_1}$ and $(\beta,\tau)\in\mathbf{P}^*_{n_2}$,

$$(\alpha,\pi)\vee(\beta,\tau)=\begin{cases}(\alpha\beta,\pi\tau), & \text{if } n_1 \text{ or } n_2 \text{ is even}\\ 0, & \text{otherwise.}\end{cases}$$

We then have the following result.

Theorem. *Let A be a noncycle N-TQA over a field of characteristic zero. Then the $\vee$ product in* $\mathbf{P}^*$ *induces the Yoneda product in $H^n(A,A)$ and in particular*:

(i) *The product of two odd degree cohomology classes is zero.*
(ii) *If $f_1,\dots,f_N$ are cohomology classes of positive degree, then $f_1\dots f_N=0$.*
(iii) *$H^*(A,A)/\mathcal{N}=$ k, where $\mathcal{N}$ is the ideal generated by homogeneous nilpotent elements.*

This result extends the analogous result in [BLM] for truncated cycle algebras. Part (iii) can be deduced from the results in [GS].

This theorem allow us to derive a number of results on the structure of the full cohomology ring of TQA's. In this paper we investigate under which conditions is the cohomology ring trivial, meaning that the subalgebra $\bigoplus_{n\geqslant 1} H^n(A,A)$ has trivial product. It was believed that, generically, this was the case. Nevertheless examples of algebras with nontrivial product in cohomology appeared in [GMS]. Recently in [GS,GSS] examples within the class of monomial algebras are given. On the other hand, Bustamante and Gatica [BG] proved that the product is zero for monomial algebras with no oriented cycles.

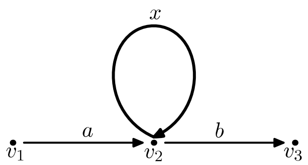
For the class of TQA's we prove in Section 8 the following theorem.

Theorem. *Let Δ be a quiver satisfying one of the following conditions.*

(i) *Δ has no oriented cycles.*
(ii) *Δ is not an oriented cycle and has neither sinks nor sources.*

Then the subalgebra $\bigoplus_{n\geqslant 1} H^n(A,A)$ with the Yoneda product is trivial.

On the other direction, we consider the cohomology ring of TQA'a associated with the quiver



and prove in Section 8.3 the following result.

Theorem. *Let A be an N-TQA associated with the above quiver Δ. Then, for all $n \in \mathbb{N}$, there exist nonzero cohomology classes $\omega_{n,j} \in H^n(A,A)$, $j = 1,\dots,N-1$, such that*

$$\omega_{n_1,j_1} \cup \omega_{n_2,j_2} = \begin{cases} \omega_{n_1+n_2,j_1+j_2}, & \textit{if } n_1 \textit{ or } n_2 \textit{ is even and } j_1+j_2<N; \\ 0, & \textit{otherwise}. \end{cases}$$

This theorem gives many examples of TQA's containing loops (and thus oriented cycles) whose cohomology ring contain nilpotent elements that are factorized as a product of two other nilpotent elements. On a full description of the cohomology ring of arbitrary TQA's this example should play a fundamental role.

At the end of the paper we use the comparison morphisms to construct explicit cohomology classes in the bar resolution. In particular, the truncated polynomial algebra in one variable is considered. This algebra has been widely studied, however, as far as we know, the comparison morphism has not been written down and a basis consisting of cohomology classes in the bar resolution can not be found in the literature. We give a full description of the cohomology ring in this case.

2. Preliminaries

2.1. Quiver algebras

Let Δ be a finite quiver, that is, a finite directed graph in which multiple arrows and loops are allowed. In this paper all quivers shall be assumed to be finite and connected.

The set of vertices and arrows of Δ are denoted by Δ_0 and Δ_1, respectively. To each arrow $a \in \Delta_1$ we associate its source vertex $o(a)$, and its end vertex $t(a)$. A path α is a single vertex or a sequence of arrows $\alpha = a_1 \dots a_n$ such that $t(a_i) = o(a_{i+1})$. The length $|\alpha|$ of a path α is the number of arrows of it and the set of paths of length n is denoted by Δ_n. We find it convenient to consider the vertices as paths of length zero. For a path $\alpha = a_1 \dots a_n \in \Delta_n$, we set $o(\alpha) = o(a_1)$ and $t(\alpha) = t(a_n)$.

Let k be any field of characteristic 0. Let $\mathrm{k}\Delta_n$ be the k-vector space with basis Δ_n and let $\mathrm{k}\Delta = \bigoplus_{n \geqslant 0} \mathrm{k}\Delta_n$. The quiver algebra associated to Δ is $\mathrm{k}\Delta$ with multiplication given by concatenation of paths. If $\alpha = a_1 \dots a_m \in \Delta_m$ and $\beta = b_1 \dots b_n \in \Delta_n$, then $\alpha\beta = a_1 \dots a_m b_1 \dots b_n \in \Delta_{m+n}$, if $t(\alpha) = o(\beta)$, or zero otherwise. It is clear that $\mathrm{k}\Delta$ is a graded algebra with unit $1 = \sum_{p \in \Delta_0} p$ and degree n component $\mathrm{k}\Delta_n$.

A *truncated quiver algebra* A is a quotient $A = \mathrm{k}\Delta / I^N$, where I is the ideal generated by Δ_1 and $N \geqslant 2$. Since I^N is an homogeneous ideal, truncated quiver algebras are graded.

Given a truncated quiver algebra A, we shall make no distinction between an element $\alpha \in \bigoplus_{n=0}^{N-1} \mathrm{k}\Delta_n \subset \mathrm{k}\Delta$ and its quotient projection in A. In particular, the set

$$\mathcal{B} = \bigcup_{n=0}^{N-1} \Delta_n$$

is a k-basis of A.

We finally point out that elements $\alpha, \beta \in A$ will frequently be assumed to be in $\mathcal{B}$ and, in these cases, $\alpha = a_1 \dots a_{|\alpha|}$ or $\beta = b_1 \dots b_{|\beta|}$ will be their arrow decomposition.

2.2. Hochschild cohomology

Given an associative k-algebra with unit A, the Hochschild cohomology groups of A with coefficients in the A-bimodule A, $H^n(A, A)$ for $n \geqslant 0$, are by definition, $\mathrm{Ext}^n_{A^e}(A, A)$ where $A^e = A \otimes_{\mathrm{k}} A^{op}$. The natural identification between A-bimodules and left A^e-modules gives the definition of projective A-bimodule and A-bimodule homomorphism.

We recall that the standard bar resolution of a k-algebra A with unit is the A^e-projective resolution of A,

$$\cdots \to A \otimes A^{\otimes n} \otimes A \xrightarrow{b} A \otimes A^{\otimes (n-1)} \otimes A \cdots \xrightarrow{b} A \otimes A \otimes A \xrightarrow{b} A \otimes A \xrightarrow{\epsilon} A$$

where $\epsilon(\alpha \otimes \beta) = \alpha\beta$ and the differential b in degree n is given by

$$\begin{aligned} & b_n(\alpha_0 \otimes \alpha_1 \dots \alpha_n \otimes \alpha_{n+1}) \\ & \quad = \alpha_0\alpha_1 \otimes \alpha_2 \dots \alpha_n \otimes \alpha_{n+1} + \sum_{i=1}^{n-1} (-1)^i \alpha_0 \otimes \alpha_1 \dots (\alpha_i \alpha_{i+1}) \dots \alpha_n \otimes \alpha_{n+1} \\ & \qquad + (-1)^n \alpha_0 \otimes \alpha_1 \dots \alpha_{n-1} \otimes \alpha_n \alpha_{n+1}. \end{aligned}$$

Since $\mathrm{Hom}_{A^e}(A \otimes A^{\otimes n} \otimes A, A) \simeq \mathrm{Hom}_{\mathrm{k}}(A^{\otimes n}, A)$, the associated Hochschild complex is

$$A \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}(A, A) \xrightarrow{b} \cdots \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes (n-1)}, A\big) \xrightarrow{b} \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes n}, A\big) \xrightarrow{b} \cdots.$$

The cohomology group $H^*(A, A)$ has a ring structure given by the Yoneda product which coincides with the cup product defined as follows. The cup product is graded commutative. Given two cochains,

$$f \in \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes m}, A\big), \qquad g \in \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes n}, A\big)$$

the cup product of f and g is the cochain $f \cup g \in \mathrm{Hom}_{\mathrm{k}}(A^{\otimes(m+n)}, A)$ defined by

$$f \cup g(\alpha_1 \otimes \cdots \otimes \alpha_{m+n}) = f(\alpha_1 \otimes \cdots \otimes \alpha_m) g(\alpha_{m+1} \otimes \cdots \otimes \alpha_{m+n}).$$

We finally recall that the Hochschild cohomology of the direct sum of two k-algebras is the direct sum of their Hochschild cohomologies. Thus we shall restrict ourselves to finite connected quivers.

3. Two projective resolutions

3.1. The (reduced) bar resolution $(\mathbf{Q}, b)$

When A is a truncated quiver algebra the bar resolution given above can be slightly simplified by tensoring over $\mathrm{k}\Delta_0$, as done in [Ci2]. More precisely, let us denote by A_+ the ideal $\bigoplus_{n=1}^{N-1} \mathrm{k}\Delta_n$ of A and let

$$\mathbf{Q}_n = A \otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^n} \otimes_{\Delta_0} A,$$

$\epsilon(\alpha \otimes \beta) = \alpha\beta$ and for $n > 0$ let

$$\begin{aligned} b_n\big(\alpha_0[\alpha_1|\ldots|\alpha_n]\alpha_{n+1}\big) = {} & \alpha_0\alpha_1[\alpha_2|\ldots|\alpha_n]\alpha_{n+1} \\ & + \sum_{i=1}^{n-1} (-1)^i \alpha_0[\alpha_1|\ldots|\alpha_i\alpha_{i+1}|\ldots|\alpha_n]\alpha_{n+1} \\ & + (-1)^n \alpha_0[\alpha_1|\ldots|\alpha_{n-1}]\alpha_n\alpha_{n+1}. \end{aligned}$$

Here we use the bar notation

$$\alpha_0[\alpha_1|\ldots|\alpha_n]\alpha_{n+1} = \alpha_0 \otimes_{\Delta_0} \alpha_1 \otimes_{\Delta_0} \ldots \otimes_{\Delta_0} \alpha_n \otimes_{\Delta_0} \alpha_{n+1} \in \mathbf{Q}_n.$$

It is not difficult to see that $\mathbf{Q}_n$ is A^e-projective, that b is well defined and $b^2 = 0$ (see [Ci2]).

A k-basis of $\mathbf{Q}_n$ is

$$\mathcal{B}_{\mathbf{Q}_n} = \left\{ \alpha_0[\alpha_1|\ldots|\alpha_n]\alpha_{n+1} \;\middle|\; \begin{array}{ll} \text{(i)} & \alpha_j \in \mathcal{B}, \text{ for all } j;\ |\alpha_j| \geqslant 1, \text{ for } j = 1, \ldots, n; \\ \text{(ii)} & t(\alpha_j) = o(\alpha_{j+1}), \text{ for } j = 0, \ldots, n \end{array} \right\}.$$

Let

$$\mathcal{B}'_{\mathbf{Q}_n} = \left\{ 1[\alpha_1|\ldots|\alpha_n]1 \;\middle|\; \begin{array}{ll} \text{(i)} & \alpha_j \in \mathcal{B} \text{ and } |\alpha_j| \geqslant 1, \text{ for } j = 1, \ldots, n; \\ \text{(ii)} & t(\alpha_j) = o(\alpha_{j+1}), \text{ for } j = 1, \ldots, n-1 \end{array} \right\}.$$

Since

$$1[\alpha_1|\ldots|\alpha_n]1 = o(\alpha_1)[\alpha_1|\ldots|\alpha_n]t(\alpha_n)$$

for every element of $\mathcal{B}'_{\mathbf{Q}_n}$, it follows that $\mathcal{B}'_{\mathbf{Q}_n} \subset \mathcal{B}_{\mathbf{Q}_n}$ and that the set $\mathcal{B}'_{\mathbf{Q}_n}$ generates $\mathbf{Q}_n$ as an A-bimodule.

As in the case of the bar resolution, it is straightforward to check that the map $s : \mathbf{Q}_n \to \mathbf{Q}_{n+1}$ defined by

$$s_n\big(\alpha_0[\alpha_1|\ldots|\alpha_n]\alpha_{n+1}\big) = \begin{cases} 1[\alpha_0|\ldots|\alpha_n]\alpha_{n+1}, & \text{if } |\alpha_0| > 0; \\ 0, & \text{if } |\alpha_0| = 0; \end{cases}$$

is a k-linear chain contraction of the identity, that is, $sb + bs = 1$. This shows that the complex $(\mathbf{Q}, b)$ is exact.

3.2. *The minimal resolution* $(\mathbf{P}, d)$

The Hochschild homology of truncated quiver algebras A was computed by Sköldberg in [Sk] and the Hochschild cohomology was computed by Locateli in [Lo]. In both papers, the authors used the minimal A^e-projective resolution $\mathbf{P}$ of A that we describe below. This minimal resolution was introduced in several earlier papers (see for instance [AG,Ba,BK] and [Ha]).

Let

$$\mathbf{P}_n = \begin{cases} A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN} \otimes_{\Delta_0} A, & \text{if } n = 2k; \\ A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN+1} \otimes_{\Delta_0} A, & \text{if } n = 2k+1. \end{cases}$$

In order to simplify the notation, the symbol $\otimes$ will always mean $\otimes_{\Delta_0}$ for elements in $\mathbf{P}$. Let $\epsilon(\alpha \otimes \beta) = \alpha\beta$ and, for $n > 0$, let $d_n : \mathbf{P}_n \to \mathbf{P}_{n-1}$ be defined by

$$\begin{aligned} d_{2k}(\alpha \otimes v_1 \ldots v_{kN} \otimes \beta) &= \sum_{j=0}^{N-1} \alpha v_1 \ldots v_j \otimes \underbrace{v_{j+1} \ldots v_t}_{(k-1)N+1} \otimes v_{t+1} \ldots v_{kN}\beta \\ &= \alpha \otimes v_1 \ldots v_{(k-1)N+1} \otimes v_{(k-1)N+2} \ldots v_{kN}\beta \\ &\quad + \cdots + \alpha v_1 \ldots v_{N-1} \otimes v_N \ldots v_{kN} \otimes \beta, \end{aligned}$$

$$d_{2k+1}(\alpha \otimes v_1 \ldots v_{kN+1} \otimes \beta) = \alpha v_1 \otimes v_2 \ldots v_{kN+1} \otimes \beta - \alpha \otimes v_1 \ldots v_{kN} \otimes v_{kN+1}\beta.$$

In particular

$$\begin{aligned} d_1(\alpha \otimes v \otimes \beta) &= \alpha v \otimes \beta - \alpha \otimes v\beta, \\ d_2(\alpha \otimes v_1 \ldots v_N \otimes \beta) &= \sum_{j=0}^{N-1} \alpha v_1 \ldots v_j \otimes v_{j+1} \otimes v_{j+2} \ldots v_N\beta. \end{aligned}$$

Again, it is easy to see that $\mathbf{P}_n$ is A^e-projective, d is well defined, $d^2 = 0$ and the set

$$\mathcal{B}'_{\mathbf{P}_n} = \left\{ 1 \otimes v_1 \ldots v_s \otimes 1 \;\middle|\; \begin{array}{ll} \text{(i)} & s = kN, \text{ if } n = 2k; \text{ or } s = kN+1, \text{ if } n = 2k+1; \\ \text{(ii)} & v_i \in \Delta_1 \text{ for } i = 1, \ldots, s \end{array} \right\}$$

generates $\mathbf{P}_n$ as A-bimodule.

The exactness of $(\mathbf{P}, d)$ is not obvious and a proof using a spectral sequence argument can be found in [Sk]. Alternatively, we give a chain contraction of the identity $r_n : \mathbf{P}_n \to \mathbf{P}_{n+1}$ in the following proposition.

Proposition 3.1. *Let $r_n : \mathbf{P}_n \to \mathbf{P}_{n+1}$ be the* k*-linear map defined on basis elements as follows*

$$
\begin{aligned}
r_{2k}(\alpha \otimes v_1 \dots v_{kN} \otimes \beta) &= \sum_{j=1}^{|\alpha|} a_1 \dots a_{j-1} \otimes \underbrace{a_j \dots a_{|\alpha|} v_1 \dots v_t}_{kN+1} \otimes v_{t+1} \dots v_{kN}\beta \\
&= 1 \otimes \alpha v_1 \dots v_{kN-|\alpha|+1} \otimes v_{kN-|\alpha|+2} \dots v_{kN}\beta \\
&\quad + \dots + a_1 \dots a_{|\alpha|-1} \otimes a_{|\alpha|} v_1 \dots v_{kN} \otimes \beta; \\
r_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) &= \begin{cases} 1 \otimes \alpha v_1 \dots v_{kN+1} \otimes \beta, & \text{if } |\alpha| = N-1; \\ 0, & \text{if } |\alpha| < N-1. \end{cases}
\end{aligned}
$$

Then $rd + dr = 1$ and therefore $(\mathbf{P}, d)$ is exact.

Remark 3.2. Notice that

$$
\begin{aligned}
r_0(\alpha \otimes \beta) &= \sum_{j=1}^{|\alpha|} a_1 \dots a_{j-1} \otimes a_j \otimes a_{j+1} \dots a_{|\alpha|}\beta \\
&= 1 \otimes a_1 \otimes a_2 \dots a_{|\alpha|}\beta + \dots + a_1 \dots a_{|\alpha|-1} \otimes a_{|\alpha|} \otimes \beta, \\
r_1(\alpha \otimes v \otimes \beta) &= \begin{cases} 1 \otimes \alpha v \otimes \beta, & \text{if } |\alpha| = N-1; \\ 0, & \text{if } |\alpha| < N-1. \end{cases}
\end{aligned}
$$

Proof. For $n = 2k$ we have

$$
\begin{aligned}
d_{2k+1} r_{2k}(\alpha \otimes v_1 \dots v_{kN} \otimes \beta) &= \sum_{j=1}^{|\alpha|} d_{2k+1}(a_1 \dots a_{j-1} \otimes \underbrace{a_j \dots a_{|\alpha|} v_1 \dots v_t}_{kN+1} \otimes v_{t+1} \dots v_{kN}\beta) \\
&= \sum_{j=1}^{|\alpha|} a_1 \dots a_j \otimes \underbrace{a_{j+1} \dots a_{|\alpha|} v_1 \dots v_t}_{kN} \otimes v_{t+1} \dots v_{kN}\beta \\
&\quad - \sum_{j=1}^{|\alpha|} a_1 \dots a_{j-1} \otimes \underbrace{a_j \dots a_{|\alpha|} v_1 \dots v_{t-1}}_{kN} \otimes v_t \dots v_{kN}\beta \\
&= \alpha \otimes v_1 \dots v_{kN} \otimes \beta - 1 \otimes \alpha v_1 \dots v_{kN-|\alpha|} \otimes v_{kN-|\alpha|+1} \dots v_{kN}\beta;
\end{aligned}
$$

and

$$
\begin{aligned}
r_{2k-1} d_{2k}(\alpha \otimes v_1 \dots v_{kN} \otimes \beta) &= \sum_{j=0}^{N-1} r_{2k-1}(\alpha v_1 \dots v_j \otimes \underbrace{v_{j+1} \dots v_t}_{(k-1)N+1} \otimes v_{t+1} \dots v_{kN}\beta) \\
&= r_{2k-1}(\alpha v_1 \dots v_{N-1-|\alpha|} \otimes v_{N-|\alpha|} \dots v_{kN-|\alpha|} \otimes v_{kN-|\alpha|+1} \dots v_{kN}\beta) \\
&= 1 \otimes \alpha v_1 \dots v_{kN-|\alpha|} \otimes v_{kN-|\alpha|+1} \dots v_{kN}\beta.
\end{aligned}
$$

Hence $d_{2k+1} r_{2k} + r_{2k-1} d_{2k}(\alpha \otimes v_1 \dots v_{kN} \otimes \beta) = \alpha \otimes v_1 \dots v_{kN} \otimes \beta$.

Similarly, for $n = 2k+1$ we have $d_{2k+2}r_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) = 0$ if $|\alpha| < N-1$ and

$$\begin{aligned} d_{2k+2}r_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) &= d_{2k+2}(1 \otimes \alpha v_1 \dots v_{kN+1} \otimes \beta) \\ &= \sum_{j=0}^{N-1} a_1 \dots a_j \otimes \underbrace{a_{j+1} \dots a_{N-1} v_1 \dots v_t}_{kN+1} \otimes v_{t+1} \dots v_{kN+1}\beta, \end{aligned}$$

if $|\alpha| = N-1$. On the other hand

$$\begin{aligned} &r_{2k}d_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) \\ &\quad = r_{2k}(\alpha v_1 \otimes v_2 \dots v_{kN+1} \otimes \beta - \alpha \otimes v_1 \dots v_{kN} \otimes v_{kN+1}\beta). \end{aligned}$$

If $|\alpha| < N-1$ then $r_{2k}(\alpha v_1 \otimes v_2 \dots v_{kN+1} \otimes \beta - \alpha \otimes v_1 \dots v_{kN} \otimes v_{kN+1}\beta)$ is a telescopic sum that adds up to $\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta$.

If $|\alpha| = N-1$ then $\alpha v_1 \otimes v_2 \dots v_{kN+1} \otimes \beta = 0$ and

$$\begin{aligned} r_{2k}d_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) &= -r_{2k}(\alpha \otimes v_1 \dots v_{kN} \otimes v_{kN+1}\beta) \\ &= -\sum_{j=1}^{N-1} a_1 \dots a_{j-1} \otimes \underbrace{a_j \dots a_{N-1} v_1 \dots v_t}_{kN+1} \otimes v_{t+1} \dots v_{kN+1}\beta. \end{aligned}$$

Hence $d_{2k+2}r_{2k+1} + r_{2k}d_{2k+1}(\alpha \otimes v_1 \dots v_{kN+1} \otimes \beta) = \alpha \otimes v_1 \dots v_{kN+1} \otimes \beta$. $\square$

4. The comparison morphisms

A comparison morphism between two projective resolutions of an algebra A is a morphism of chain complexes that lifts the identity map on A. Such a morphism induces a quasi-isomorphism between the derived complexes $\mathrm{Hom}_{A^e}(\cdot, A)$.

In this section we define maps

$$\mathbf{F} : \mathbf{P} \to \mathbf{Q} \quad \text{and} \quad \mathbf{G} : \mathbf{Q} \to \mathbf{P}$$

between these A-bimodule resolutions of A and we state in Theorem 4.1 that they are in fact comparison morphisms. This is one of the main results of the paper. The proofs, which we find nontrivial and subtle, are in Sections 5 and 6. The reader interested only in the main results in the paper may safely skip these two sections.

We define $\mathbf{F}$ and $\mathbf{G}$ as the A-bimodule extensions of maps defined on elements of $\mathcal{B}'_{\mathbf{P}_n}$ and $\mathcal{B}'_{\mathbf{Q}_n}$, respectively. As in the case of the differentials b and d one should check that these A-bimodule extensions are well defined, but this is straightforward since the tensor products in $\mathbf{Q}$ and $\mathbf{P}$ are both over $\mathrm{k}\Delta_0$.

4.1. The comparison morphism $\mathbf{F} : \mathbf{P} \to \mathbf{Q}$

Let $\mathbf{F}_0 = \mathrm{id}$ and, for $n \geqslant 1$, let $\mathbf{F}_n : \mathbf{P} \to \mathbf{Q}$ be the A-bimodule extension of the following map defined on elements of $\mathcal{B}'_{\mathbf{P}_n}$. If $n = 2k$ and $p = 1 \otimes v_1 \dots v_{kN} \otimes 1 \in \mathcal{B}'_{\mathbf{P}_{2k}}$, with v_i an arrow for all i, let

$$\mathbf{F}_{2k}(p) = \sum 1[\underbrace{v_1 \dots v_{x_1}}_{x_1} | \underbrace{v_{1+x_1}}_{1} | \underbrace{\dots v_{1+x_1+x_2}}_{x_2} | \underbrace{v_{2+x_1+x_2}}_{1} | \dots\dots | \underbrace{v_{k+\sum x_j}}_{1}] \underbrace{\dots\dots v_{kN}}_{kN-k-\sum x_j},$$

where the sum is taken over all k-tuples $(x_1,\dots,x_k)\in\mathbb{Z}^k$ such that $1\leqslant x_i<N$. If $n=2k+1$ and $p=1\otimes v_1\dots v_{kN+1}\otimes 1\in\mathcal{B}'_{\mathbf{P}_{2k+1}}$, with v_i an arrow for all i, let

$$\mathbf{F}_{2k+1}(p)=\sum 1[\underbrace{v_1}_{1}|\underbrace{v_2\dots v_{1+x_1}}_{x_1}|\underbrace{v_{2+x_1}}_{1}|\underbrace{\dots v_{2+x_1+x_2}}_{x_2}|\dots\dots|\underbrace{v_{k+1+\sum x_j}}_{1}]\underbrace{\dots\dots v_{kN+1}}_{kN-k-1-\sum x_j},$$

where the sum is taken over all k-tuples $(x_1,\dots,x_k)\in\mathbb{Z}^k$ such that $1\leqslant x_i<N$.

In order to prove that $\mathbf{F}$ is a comparison morphism, one should first check that the diagram

$$\begin{tikzcd}
A\otimes_{\Delta_0}A_+\otimes_{\Delta_0}A \arrow[r, "b_1"] & A\otimes A \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] & A \\
A\otimes_{\Delta_0}\mathrm{k}\Delta_1\otimes_{\Delta_0}A \arrow[r, "d_1"'] \arrow[u, "\mathbf{F}_1"] & A\otimes A \arrow[r, "\epsilon"] & A
\end{tikzcd}$$

is commutative. This is immediate since $\mathbf{F}_1(1\otimes v\otimes 1)=1[v]1$.

4.2. *The comparison morphism* $\mathbf{G}:\mathbf{Q}\to\mathbf{P}$

Let $\mathbf{G}_0=\mathrm{id}$ and, for $n\geqslant 1$, let $\mathbf{G}_n:\mathbf{Q}\to\mathbf{P}$ be the A-bimodule extension of the following map defined on elements of $\mathcal{B}'_{\mathbf{Q}_n}$. For $q=1[\alpha_1|\dots|\alpha_n]1\in\mathcal{B}'_{\mathbf{Q}_n}$, let $v(q)$ be the arrow decomposition of the path $\alpha_1\alpha_2\dots\alpha_n$ in $\mathrm{k}\Delta$ (not in A). Thus, if $\alpha_i=a^i_1\dots a^i_{|\alpha_i|}$, then

$$v(q)=v_1\dots v_{|q|}=a^1_1\dots a^1_{|\alpha_1|}a^2_1\dots a^2_{|\alpha_2|}\dots\dots a^n_1\dots a^n_{|\alpha_n|}\in\Delta_{|q|},$$

where $|q|=\sum_{i=1}^n|\alpha_i|$. Note that $v(q)\neq 0$ for all $q\in\mathcal{B}'_{\mathbf{Q}_n}$.

If $n=2k$ and $v(q)=v_1\dots v_{|q|}$, let

$$\mathbf{G}_{2k}(q)=\begin{cases}1\otimes v_1\dots v_{kN}\otimes v_{kN+1}\dots v_{|q|}, & \text{if } \alpha_{2i-1}\alpha_{2i}=0 \text{ for } i=1,\dots,k;\\ 0, & \text{otherwise.}\end{cases}$$

Note that the condition $\alpha_{2i-1}\alpha_{2i}=0$ for all $i=1,\dots,k$ implies that $|q|\geqslant kN$.

Similarly, if $n=2k+1$ and $v(q)=v_1\dots v_{|q|}$, let

$$\mathbf{G}_{2k+1}(q)=\begin{cases}\sum_{j=1}^{|\alpha_1|}v_1\dots v_{j-1}\otimes v_j\dots v_{kN+j}\otimes v_{kN+j+1}\dots v_{|q|}, & \text{if } \alpha_{2i}\alpha_{2i+1}=0 \text{ for } i=1,\dots,k;\\ 0, & \text{otherwise.}\end{cases}$$

Since $|\alpha_1|\geqslant 1$, then $|q|\geqslant kN+1$ provided that $\alpha_{2i}\alpha_{2i+1}=0$ for all $i=1,\dots,k$. Note also that only α_1 is involved in the sum.

In this case, the commutativity of the first diagram

$$\begin{tikzcd}
A\otimes_{\Delta_0}A_+\otimes_{\Delta_0}A \arrow[r, "b_1"] \arrow[d, "\mathbf{G}_1"'] & A\otimes A \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] & A \\
A\otimes_{\Delta_0}\mathrm{k}\Delta_1\otimes_{\Delta_0}A \arrow[r, "d_1"'] & A\otimes A \arrow[r, "\epsilon"] & A
\end{tikzcd}$$

follows immediately by evaluating the telescopic sum $d_1\circ\mathbf{G}_1$.

Theorem 4.1. *The following diagram is commutative for all $k \geqslant 1$ and therefore $\mathbf{F}$ and $\mathbf{G}$ are comparison morphisms between the A^e-projective resolutions $\mathbf{P}$ and $\mathbf{Q}$.*

$$
\begin{tikzcd}
A\otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k+1}} \otimes_{\Delta_0} A \arrow[d, equal] & A\otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k}} \otimes_{\Delta_0} A \arrow[d, equal] & A\otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k-1}} \otimes_{\Delta_0} A \arrow[d, equal] \\
\mathbf{Q}_{2k+1} \arrow[r, "b_{2k+1}"] \arrow[d, shift left, "\mathbf{G}_{2k+1}"] & \mathbf{Q}_{2k} \arrow[r, "b_{2k}"] \arrow[d, shift left, "\mathbf{G}_{2k}"] & \mathbf{Q}_{2k-1} \arrow[d, shift left, "\mathbf{G}_{2k-1}"] \\
\mathbf{P}_{2k+1} \arrow[r, "d_{2k+1}"'] \arrow[u, shift left, "\mathbf{F}_{2k+1}"] \arrow[d, equal] & \mathbf{P}_{2k} \arrow[r, "d_{2k}"'] \arrow[u, shift left, "\mathbf{F}_{2k}"] \arrow[d, equal] & \mathbf{P}_{2k-1} \arrow[u, shift left, "\mathbf{F}_{2k-1}"] \arrow[d, equal] \\
A\otimes_{\Delta_0} k\Delta_{kN+1}\otimes_{\Delta_0} A & A\otimes_{\Delta_0} k\Delta_{kN}\otimes_{\Delta_0} A & A\otimes_{\Delta_0} k\Delta_{(k-1)N+1}\otimes_{\Delta_0} A
\end{tikzcd}
$$

5. $\mathbf{G}: \mathbf{Q} \to \mathbf{P}$ is a comparison morphism

The proof is divided into two parts, (A) and (B), corresponding to the cases n even and n odd, respectively.

(A) Assume $n = 2k$. Let $q = 1[\alpha_1|\dots|\alpha_{2k}]1 \in \mathcal{B}'_{\mathbf{Q}_{2k}}$ and let $v(q) = v_1 \dots v_{|q|}$. Let

$$M = \bigl\{i \in \{1,\dots,k\}\colon \alpha_{2i-1}\alpha_{2i} \neq 0\bigr\}.$$

(A1) Case $M = \emptyset$. We have

$$\mathbf{G}_{2k}(q) = 1 \otimes v_1 \dots v_{kN} \otimes v_{kN+1} \dots v_{|q|}$$

and

$$d_{2k}\bigl(\mathbf{G}_{2k}(q)\bigr) = \sum_{j=0}^{N-1} v_1 \dots v_j \otimes \underbrace{v_{j+1} \dots v_t}_{(k-1)N+1} \otimes v_{t+1} \dots v_{|q|}.$$

On the other hand, $M = \emptyset$ implies

$$b_{2k}(q) = \alpha_1[\alpha_2|\dots|\alpha_{2k}]1 + \sum_{i=1}^{k-1} 1[\alpha_1|\dots|\alpha_{2i}\alpha_{2i+1}|\dots|\alpha_{2k}]1 + 1[\alpha_1|\dots|\alpha_{2k-1}]\alpha_{2k}$$

and, since $N \leqslant |\alpha_1| + |\alpha_2|$,

$$\begin{aligned}
\mathbf{G}_{2k-1}\bigl(b_{2k}(q)\bigr) = {} & \sum_{j=|\alpha_1|+1}^{N} v_1 \dots v_{j-1} \otimes v_j \dots v_{(k-1)N+j} \otimes v_{(k-1)N+1+j} \dots v_{|q|} \\
& + \sum_{i=1}^{k-1} \mathbf{G}_{2k-1}\bigl(1[\alpha_1|\dots|\alpha_{2i}\alpha_{2i+1}|\dots|\alpha_{2k}]1\bigr) + \mathbf{G}_{2k-1}\bigl(1[\alpha_1|\dots|\alpha_{2k-1}]\alpha_{2k}\bigr).
\end{aligned}$$

In the second line, all terms but one are zero, depending on which is the first j, $1 \leqslant j \leqslant k-1$, for which $\alpha_{2j}\alpha_{2j+1} = 0$. The nonzero term is

$$\sum_{j=1}^{|\alpha_1|} v_1 \dots v_{j-1} \otimes v_j \dots v_{(k-1)N+j} \otimes v_{(k-1)N+1+j} \dots v_{|q|}$$

and therefore $\mathbf{G}_{2k-1}(b_{2k}(q)) = d_{2k}(\mathbf{G}_{2k}(q))$.

(A2) Case $M \neq \emptyset$. We have now $d_{2k}(\mathbf{G}_{2k}(q)) = 0$. Let

$$M' = \bigl\{i \in \{1, \dots, k\}\colon \mathbf{G}_{2k-1}\bigl(1[\alpha_1| \dots |\alpha_{2i-1}\alpha_{2i}| \dots |\alpha_{2k}]1\bigr) \neq 0\bigr\} \subset M.$$

(A2a) Assume $M' \neq \emptyset$ and let i_0 be the smallest element in M'. We assume that $i_0 > 1$ since the case $i_0 = 1$ is easier. By the definition of $\mathbf{G}_{2k-1}$, it follows that

$$\begin{aligned} &\alpha_{2i}\alpha_{2i+1} = 0, \quad \text{for } i = 1, \dots, i_0 - 2, \quad \text{and} \\ &\alpha_{2i-1}\alpha_{2i} = 0, \quad \text{for } i = i_0 + 1, \dots, k. \end{aligned}$$

In particular i_0 is the largest element of M and

$$\begin{aligned} \mathbf{G}_{2k-1}\bigl(b_{2k}(q)\bigr) &= \mathbf{G}_{2k-1}\bigl(\alpha_1[\alpha_2| \dots |\alpha_{2k}]1\bigr) - \mathbf{G}_{2k-1}\bigl(1[\alpha_1|\alpha_2| \dots |\alpha_{2i_0-1}\alpha_{2i_0}| \dots |\alpha_{2k}]1\bigr) \\ &\quad + \sum_{i=i_0-1}^{k-1} \mathbf{G}_{2k-1}\bigl(1[\alpha_1| \dots |\alpha_{2i}\alpha_{2i+1}| \dots |\alpha_{2k}]1\bigr) + \mathbf{G}_{2k-1}\bigl(1[\alpha_1| \dots |\alpha_{2k-1}]\alpha_{2k}\bigr). \end{aligned}$$

As in the case (A1), all terms but one are zero in the second line, and this term cancels out with the first line.

(A2b) Assume $M' = \emptyset$. Thus $\mathbf{G}_{2k-1}(b_{2k}(q))$ contains only positive terms and we must prove that all of them are zero. Let i_0 be any element of M. Since $\alpha_{2i_0-1}\alpha_{2i_0} \neq 0$ the definition of $\mathbf{G}_{2k-1}$ implies that

$$\mathbf{G}_{2k-1}\bigl(1[\alpha_1| \dots |\alpha_{2i}\alpha_{2i+1}| \dots |\alpha_{2k}]1\bigr) = 0 \quad \text{for all } i = 1, \dots, i_0 - 1.$$

We now take $i_0 = \max(M)$. Since $i_0 \notin M'$, the maximality of i_0 implies that either $\alpha_{2(i_0-1)}\alpha_{2i_0-1}\alpha_{2i_0} \neq 0$ or there exists $j_0 < i_0 - 1$ such that $\alpha_{2j_0}\alpha_{2j_0+1} \neq 0$. In any case, there exists $j_0 < i_0$ such that $\alpha_{2j_0}\alpha_{2j_0+1} \neq 0$ (in particular $i_0 > 1$). Therefore $\mathbf{G}_{2k-1}(1[\alpha_1| \dots |\alpha_{2i}\alpha_{2i+1}| \dots |\alpha_{2k}]1) = 0$ for all $i = j_0 + 1, \dots, k-1$. Since the extreme cases $\mathbf{G}_{2k-1}(\alpha_1[\alpha_2| \dots |\alpha_{2k}]1)$ and $\mathbf{G}_{2k-1}(1[\alpha_1| \dots |\alpha_{2k-1}]\alpha_{2k})$ are clearly zero too, this completes the proof in case (A).

(B) Assume $n = 2k+1$. Let $q = 1[\alpha_1| \dots |\alpha_{2k+1}]1 \in \mathcal{B}'_{\mathbf{Q}_{2k}}$ and let $v(q) = v_1 \dots v_{|q|}$. Let

$$M = \bigl\{i \in \{1, \dots, k\}\colon \alpha_{2i}\alpha_{2i+1} \neq 0\bigr\}.$$

(B1) Case $M = \emptyset$. Then

$$\mathbf{G}_{2k+1}(q) = \sum_{j=1}^{|\alpha_1|} v_1 \dots v_{j-1} \otimes v_j \dots v_{kN+j} \otimes v_{kN+1+j} \dots v_{|q|}$$

and

$$d_{2k+1}\big(\mathbf{G}_{2k+1}(q)\big) = v_1 \dots v_{|\alpha_1|} \otimes v_{|\alpha_1|+1} \dots v_{|\alpha_1|+kN} \otimes v_{|\alpha_1|+kN+1} \dots v_{|q|} \\ - 1 \otimes v_1 \dots v_{kN} \otimes v_{kN+1} \dots v_{|q|}.$$

On the other hand, $M = \emptyset$ implies

$$b_{2k+1}(q) = \alpha_1[\alpha_2|\dots|\alpha_{2k+1}]1 \\ - \sum_{i=1}^{k} 1[\alpha_1|\dots|\alpha_{2i-1}\alpha_{2i}|\dots|\alpha_{2k+1}]1 - 1[\alpha_1|\dots|\alpha_{2k}]\alpha_{2k+1}$$

and

$$\mathbf{G}_{2k}\big(b_{2k+1}(q)\big) = v_1 \dots v_{|\alpha_1|} \otimes v_{|\alpha_1|+1} \dots v_{|\alpha_1|+kN} \otimes v_{|\alpha_1|+kN+1} \dots v_{|q|} \\ - \sum_{i=1}^{k} \mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2i-1}\alpha_{2i}|\dots|\alpha_{2k+1}]1\big) - \mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2k}]\alpha_{2k+1}\big).$$

As in case (A), all terms but one are zero in the second line, and the nonzero term is $1 \otimes v_1 \dots v_{kN} \otimes v_{kN+1} \dots v_{|q|}$. This yields $\mathbf{G}_{2k}(b_{2k+1}(q)) = d_{2k+1}(\mathbf{G}_{2k+1}(q))$.

(B2) Case $M \neq \emptyset$. We have $d_{2k}(\mathbf{G}_{2k}(q)) = 0$. Let

$$M' = \big\{i \in \{1, \dots, k\}\colon \mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2i}\alpha_{2i+1}|\dots|\alpha_{2k+1}]1\big) \neq 0\big\} \subset M.$$

(B2a) Assume $M' \neq \emptyset$ and let i_0 be the largest element in M'. By the definition of $\mathbf{G}_{2k}$, it follows that

$$\alpha_{2i-1}\alpha_{2i} = 0, \quad \text{for } i = 1, \dots, i_0 - 1, \quad \text{and} \\ \alpha_{2i}\alpha_{2i+1} = 0, \quad \text{for } i = i_0 + 1, \dots, k.$$

In particular i_0 is the smallest element of M and

$$\mathbf{G}_{2k+1}\big(b_{2k}(q)\big) = \mathbf{G}_{2k}\big(1[\alpha_1|\alpha_2|\dots|\alpha_{2i_0}\alpha_{2i_0+1}|\dots|\alpha_{2k+1}]1\big) \\ - \sum_{i=i_1}^{k-1} \mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2i-1}\alpha_{2i}|\dots|\alpha_{2k+1}]1\big) - \mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2k}]\alpha_{2k+1}\big);$$

where $i_1 = i_0 - 1$, if $i_0 > 1$; and $i_1 = 1$, if $i_0 = 1$. All terms but one are zero in the second line, and this term cancels out with the first line.

(B2b) Assume $M' = \emptyset$. Thus $\mathbf{G}_{2k}(b_{2k+1}(q))$ contains only negative terms and we must prove that all of them are zero. Let i_0 be any element of M. Then

$$\mathbf{G}_{2k}\big(1[\alpha_1|\dots|\alpha_{2i-1}\alpha_{2i}|\dots|\alpha_{2k+1}]1\big) = 0 \quad \text{for all } i = 1, \dots, i_0 - 1.$$

We now take $i_0 = \max(M)$. Since $i_0 \notin M'$, the maximality of i_0 implies that either $\alpha_{2i_0-1}\alpha_{2i_0}\alpha_{2i_0+1} \neq 0$ or there exist $j_0 < i_0$ such that $\alpha_{2j_0-1}\alpha_{2j_0} \neq 0$. In any case, there exist $j_0 \leqslant i_0$ such that $\alpha_{2j_0-1}\alpha_{2j_0} \neq 0$. Therefore $\mathbf{G}_{2k}(1[\alpha_1|\dots|\alpha_{2i-1}\alpha_{2i}|\dots|\alpha_{2k+1}]1) = 0$ for all $i = j_0, \dots, k$. Since the extreme case $\mathbf{G}_{2k}(1[\alpha_1|\dots|\alpha_{2k}]\alpha_{2k})$ is clearly zero too, this completes the case (B) and the proof.

6. $\mathbf{F} : \mathbf{P} \to \mathbf{Q}$ is a comparison morphism

We need some preliminary results.

6.1. A complex of compositions

Let $C_n(m)$ be the set of all the compositions (ordered partitions) of m in n parts in which only the first and last parts are allowed to be zero. That is,

$$C_n(m) = \left\{ [c_1, \ldots, c_n]\colon\ c_j \in \mathbb{N}_0,\ c_j > 0 \text{ for } j = 2, \ldots, n-1 \text{ and } \sum_{j=1}^{n} c_j = m \right\}.$$

Let $\mathbf{C}_n(m) = \mathrm{k}C_n(m)$ be the vector space with basis $C_n(m)$.

If $\alpha = [x_1, \ldots, x_n] \in C_n(m)$ and $\beta = [y_1, \ldots, y_{n'}] \in C_{n'}(m')$ then we shall denote by $[\alpha, \beta]$ the juxtaposition of α and β, that is

$$[\alpha, \beta] = [x_1, \ldots, x_n, y_1, \ldots, y_{n'}] \in C_{n+n'}(m + m').$$

Analogously, if $\alpha \in C_n(m)$ and $\beta \in \mathbf{C}_{n'}(m')$, with $\beta = \sum \beta_j$, $\beta_i \in C_{n'}(m')$, then $[\alpha, \beta] = \sum [\alpha, \beta_j]$.

Let $D : \mathbf{C}_n(m) \to \mathbf{C}_{n-1}(m)$ be the usual differential of compositions,

$$D\big([x_1, \ldots, x_n]\big) = \sum_{j=1}^{n-1} (-1)^{j+1} [x_1, \ldots, x_j + x_{j+1}, \ldots, x_n].$$

It is straightforward to see that $D^2 = 0$. Moreover, if $\mathbf{W}_n(m, N) \subset \mathbf{C}_n(m)$ is the subspace spanned by the compositions containing some part larger than or equal to N, then $D(\mathbf{W}_n(m, N)) \subset \mathbf{W}_n(m, N)$ and thus D factors through the quotient

$$\mathbf{C}_n(m, N) = \mathbf{C}_n(m) / \mathbf{W}_n(m, N).$$

For $\alpha \in \mathbf{C}_n(m)$, let $c_N \in \mathbf{C}_n(m, N)$ be its projection and define D_N by $D_N(c_N) = D(c)_N$. Thus $(\mathbf{C}_n(m, N), D_N)$ is again a complex.

Let $I_N = \{1, 2, \ldots, N-1\}$ and for $k \geqslant 1$ and $M \geqslant k(N-1)$ let

$$\alpha_M^k : I_N^k \to \mathbf{C}_{2k+1}(M + k, N), \qquad \alpha_M^k(x) = [x_1, 1, x_2, 1, \ldots, x_k, 1, M - \Sigma x_i]_N,$$
$$\beta_M^k : I_N^k \to \mathbf{C}_{2k+2}(M + k + 1, N), \qquad \beta_M^k(x) = [1, x_1, 1, x_2, 1, \ldots, x_k, 1, M - \Sigma x_i]_N.$$

The assumption $M \geqslant k(N-1)$ is necessary in order to ensure a nonnegative last part for any $x \in I_N^k$.

For $k \geqslant 1$ let

$$A_M^k = \sum_{x \in I_N^k} \alpha_M^k(x) \in \mathbf{C}_{2k+1}(M + k, N),$$
$$B_M^k = \sum_{x \in I_N^k} \beta_M^k(x) \in \mathbf{C}_{2k+2}(M + k + 1, N).$$

We define $B_M^0 = [1, M]_N$ and we need not define A_M^k for $k = 0$.

Lemma 6.1. *For all $k \geqslant 1$ and $M \geqslant k(N-1)$ we have*

$$B_M^k = \left[1, A_M^k\right]_N \quad \textit{and} \quad A_M^k = \sum_{j=1}^{N-1} \left[j, B_{M-j}^{k-1}\right]_N.$$

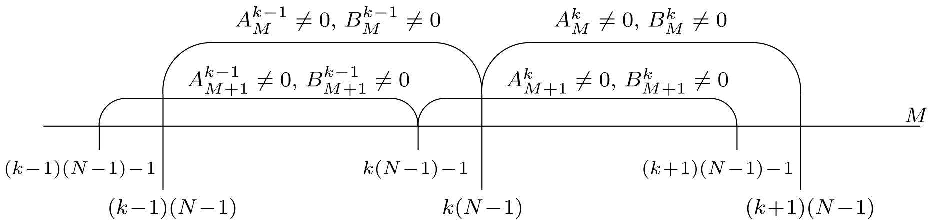
Moreover, $A_M^k \neq 0$ (resp. $B_M^k \neq 0$) if and only if $M \leqslant (k+1)(N-1)$.

Proof. The first part of the lemma is straightforward from the definition of $\alpha_M^k(x)$ and $\beta_M^k(x)$. The second part follows from the fact that the last part of the composition $\alpha_M^k(x)$ (resp. $\beta_M^k(x)$) is greater than or equal to N for all $x \in I_N^k$, if and only if $M > (k+1)(N-1)$. $\square$

Lemma 6.2. *For all $k \geqslant 1$ and $M \geqslant k(N-1)$ we have*

$$D_N\left(A_M^k\right) = -B_M^{k-1} \quad \textit{and} \quad D_N\left(B_M^k\right) = A_{M+1}^k.$$

Remark. The following picture shows the values of M for which A and B are different from zero. In particular it shows that the lemma is consistent with the fact that $D_N^2 = 0$.



Proof. We assume that M is fixed. We shall now prove simultaneously both equalities by induction on k for $1 \leqslant k \leqslant \frac{M}{N-1}$.

If $k = 1$ then

$$\begin{aligned} D_N\left(A_M^1\right) &= \sum_{x_1=1}^{N-1} [x_1+1, M-x_1]_N - \sum_{x_1=1}^{N-1} [x_1, M-x_1+1]_N \\ &= -[1, M]_N \\ &= -B_M^0. \end{aligned}$$

This completes the case $k = 1$ for the first equation. The case $k = 1$ for the second equation is analogous:

$$B_M^1 = \sum_{x_1=1}^{N-1} [1, x_1, 1, M-x_1]_N$$

and

$$D_N\left(B_M^1\right) = \sum_{x_1=1}^{N-1} [x_1+1, 1, M-x_1]_N - \sum_{x_1=1}^{N-1} [1, x_1+1, M-x_1]_N + \sum_{x_1=1}^{N-1} [1, x_1, M-x_1+1]_N.$$

The last two terms add up as a telescopic sum and the result is

$$[1,1,M]_N - [1,N,M-N+1]_N = [1,1,M]_N.$$

Therefore

$$\begin{aligned} D_N\big(B_M^1\big) &= \sum_{x_1=1}^{N-1} [x_1+1,1,M-x_1]_N + [1,1,M]_N \\ &= \sum_{x\in I_N^1} \alpha_{M+1}^1(x) \\ &= A_{M+1}^1. \end{aligned}$$

Now we assume that the lemma is true for $k-1$. From Lemma 6.1 we have

$$A_M^k = \sum_{j=1}^{N-1} \big[j, B_{M-j}^{k-1}\big]_N = \sum_{j=1}^{N-1} \big[j,1,A_{M-j}^{k-1}\big]_N.$$

Hence

$$\begin{aligned} D_N\big(A_M^k\big) &= \sum_{j=1}^{N-1} \big[1+j, A_{M-j}^{k-1}\big]_N - \sum_{j=1}^{N-1} \big[j, D_N\big(B_{M-j}^{k-1}\big)\big]_N \\ &= \sum_{j=2}^{N-1} \big[j, A_{M+1-j}^{k-1}\big]_N - \sum_{j=1}^{N-1} \big[j, A_{M+1-j}^{k-1}\big]_N \\ &= -\big[1, A_M^{k-1}\big]_N \\ &= -B_M^{k-1}. \end{aligned}$$

Similarly,

$$B_M^k = \big[1, A_M^k\big]_N = \sum_{j=1}^{N-1} \big[1,j,1,A_{M-j}^{k-1}\big]_N.$$

Hence

$$\begin{aligned} D_N\big(B_M^k\big) &= \sum_{j=1}^{N-1} \big[1+j,1,A_{M-j}^{k-1}\big]_N - \big[1, D_N\big(A_M^k\big)\big]_N \\ &= \sum_{j=1}^{N-2} \big[1+j,1,A_{M-j}^{k-1}\big]_N - \big[1, -B_M^{k-1}\big]_N \\ &= \sum_{j=1}^{N-1} \big[j, B_{M+1-j}^{k-1}\big]_N \\ &= A_{M+1}^k. \end{aligned}$$

This completes the inductive argument. $\square$

Now we define

$$\tilde{A}_M^k = \left[0, A_M^k\right]_N \in \mathbf{C}_{2k+2}(M+k, N), \quad \text{for } k \geqslant 1 \text{ and } M \geqslant k(N-1),$$
$$\tilde{B}_M^k = \left[0, B_M^k\right]_N \in \mathbf{C}_{2k+3}(M+k+1, N), \quad \text{for } k \geqslant 0 \text{ and } M \geqslant k(N-1).$$

Proposition 6.3. *Let $k \geqslant 1$ and $M \geqslant k(N-1)$ we have*

$$D_N\left(\tilde{B}_M^k\right) = \left[1, A_M^k\right]_N - \left[0, A_M^k\right]_N \quad \textit{and}$$
$$D_N\left(\tilde{A}_M^k\right) = \sum_{j=0}^{N-1} \left[j, B_{M-j}^{k-1}\right]_N.$$

Proof. Since $\tilde{B}_M^k = [0, B_M^k]_N = [0, 1, A_M^k]_N$, then Lemma 6.2 implies

$$\begin{aligned} D_N\left(\tilde{B}_M^k\right) &= \left[1, A_M^k\right]_N - \left[0, D_N\left(B_M^k\right)\right]_N \\ &= \left[1, A_M^k\right]_N - \left[0, A_M^k\right]_N. \end{aligned}$$

Similarly, since $\tilde{A}_M^k = [0, A_M^k]_N = \sum_{j=1}^{N-1} [0, j, B_{M-j}^{k-1}]_N$, then

$$\begin{aligned} D_N\left(\tilde{A}_M^k\right) &= \sum_{j=1}^{N-1} \left[j, B_{M-j}^{k-1}\right]_N - \left[0, D_N\left(A_M^k\right)\right]_N \\ &= \sum_{j=0}^{N-1} \left[j, B_{M-j}^{k-1}\right]_N. \qquad \square \end{aligned}$$

6.2. The final step

Each composition $\alpha = [x_1, \ldots, x_n] \in C_n(m)$ defines an A-bimodule morphism

$$\phi_\alpha : A \otimes_{\mathrm{k}\Delta_0} \mathrm{k}\Delta_m \otimes_{\mathrm{k}\Delta_0} A \to A \otimes_{\mathrm{k}\Delta_0} A_+^{\otimes_{\mathrm{k}\Delta_0}^{n-2}} \otimes_{\mathrm{k}\Delta_0} A$$

which is the A-bimodule extension of

$$\phi_\alpha(1 \otimes v_1 \ldots v_m \otimes 1) = \begin{cases} \underbrace{v_1 \ldots v_{s_1}}_{x_1} [\underbrace{\ldots v_{s_2}}_{x_2} | \ldots | \underbrace{\ldots v_{s_{n-1}}}_{x_{n-1}}] \underbrace{\ldots v_m}_{x_n}, & \text{if } c_j < N\ \forall j; \\ 0, & \text{otherwise.} \end{cases}$$

where $s_i = x_1 + \cdots + x_i$, $i = 1, \ldots, n$. Thus we obtain for all m a map

$$\phi : \mathbf{C}_n(m, N) \to \mathrm{Hom}_{A^e}(A \otimes_{\Delta_0} \mathrm{k}\Delta_m \otimes_{\Delta_0} A, \mathbf{Q}_{n-2}).$$

In this context, the comparison morphism $\mathbf{F} : \mathbf{P} \to \mathbf{Q}$ (see Section 4.1) is

$$\mathbf{F}_n = \begin{cases} \phi_{\tilde{A}_{k(N-1)}^k} : \mathbf{P}_{2k} = A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN} \otimes_{\Delta_0} A \to \mathbf{Q}_n, & \text{if } n = 2k, \\ \phi_{\tilde{B}_{k(N-1)}^k} : \mathbf{P}_{2k+1} = A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN+1} \otimes_{\Delta_0} A \to \mathbf{Q}_n, & \text{if } n = 2k+1. \end{cases}$$

Now the proof of Theorem 4.1 will be complete if we show that

$$b_{2k+1} \circ \phi_{\tilde{B}^k_{k(N-1)}} = \phi_{\tilde{A}^k_{k(N-1)}} \circ d_{2k+1} = \phi_{[0, A^k_{k(N-1)}]_N - [1, A^k_{k(N-1)}]_N}$$

and

$$b_{2k} \circ \phi_{\tilde{A}^k_{k(N-1)}} = \phi_{\tilde{B}^{(k-1)}_{(k-1)(N-1)}} \circ d_{2k} = \phi_{\sum_{j=0}^{N-1} [j, B^{k-1}_{k(N-1)-j}]_N},$$

which is the commutativity of the diagrams

$$\begin{tikzcd}
A \otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k+1}} \otimes_{\Delta_0} A \arrow[d, equal] & A \otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k}} \otimes_{\Delta_0} A \arrow[d, equal] & A \otimes_{\Delta_0} A_+^{\otimes_{\Delta_0}^{2k-1}} \otimes_{\Delta_0} A \arrow[d, equal] \\
\mathbf{Q}_{2k+1} \arrow[r, "b_{2k+1}"] & \mathbf{Q}_{2k} \arrow[r, "b_{2k}"] & \mathbf{Q}_{2k-1} \\
\mathbf{P}_{2k+1} \arrow[u, "\mathbf{F}_{2k+1}"] \arrow[r, "d_{2k+1}"'] & \mathbf{P}_{2k} \arrow[u, "\mathbf{F}_{2k}"] \arrow[r, "d_{2k}"'] & \mathbf{P}_{2k-1} \arrow[u, "\mathbf{F}_{2k-1}"] \\
A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN+1} \otimes_{\Delta_0} A \arrow[u, equal] & A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN} \otimes_{\Delta_0} A \arrow[u, equal] & A \otimes_{\Delta_0} \mathrm{k}\Delta_{(k-1)N+1} \otimes_{\Delta_0} A \arrow[u, equal]
\end{tikzcd}$$

This identities are proved in general in the following proposition (see also Proposition 6.3).

Proposition 6.4. *For all $n \geqslant 2$ and $m \geqslant 0$ the following diagram is commutative.*

$$\begin{tikzcd}
\mathbf{C}_{n+1}(m, N) \arrow[r, "\phi"] \arrow[d, "D_N"'] & \mathrm{Hom}_{A^e}(A \otimes_{\Delta_0} \mathrm{k}\Delta_m \otimes_{\Delta_0} A, \mathbf{Q}_{n-1}) \arrow[d, "b \circ _"] \\
\mathbf{C}_n(m, N) \arrow[r, "\phi"'] & \mathrm{Hom}_{A^e}(A \otimes_{\Delta_0} \mathrm{k}\Delta_m \otimes_{\Delta_0} A, \mathbf{Q}_{n-2})
\end{tikzcd}$$

In other words, $b \circ \phi_\alpha = \phi_{D_N(\alpha)}$ for all $\alpha \in \mathbf{C}_{n+1}(m, N)$.

Proof. It is sufficient to prove that $b(\phi_\alpha(T)) = \phi_{D_N(\alpha)}(T)$ for all the monomials T of the form $T = 1 \otimes v_1 \dots v_m \otimes 1 \in A \otimes_{\Delta_0} \mathrm{k}\Delta_m \otimes_{\Delta_0} A$ and for all the compositions $\alpha = [x_0, x_1, \dots, x_n] \in (C_N^m)_{n+1}$ with $x_j < N$ for all $j = 0, \dots, n$. If we denote by $s_i = x_0 + \dots + x_i$, $i = 0, \dots, n$, then both sides of the above equality are

$$\underbrace{v_1 \dots v_{s_1}}_{x_0+x_1} [\underbrace{v_{s_1+1} \dots v_{s_2}}_{x_2} | \dots | \underbrace{v_{s_{n-2}+1} \dots v_{s_{n-1}}}_{x_{n-1}}] \underbrace{v_{s_{n-1}+1} \dots v_m}_{x_n}$$
$$+ \sum_{i=1}^{n-1} (-1)^i \underbrace{v_1 \dots v_{s_0}}_{x_0} [\underbrace{\dots v_{s_1}}_{x_1} | \dots | \underbrace{v_{s_{i-1}+1} \dots v_{s_{i+1}}}_{x_i + x_{i+1}} | \dots | \underbrace{\dots v_{s_{n-1}}}_{x_{n-1}}] \underbrace{v_{s_{n-1}+1} \dots v_m}_{x_n}$$

$$+ (-1)^n \underbrace{v_1 \ldots v_{s_0}}_{x_0} [\underbrace{v_{s_0+1} \ldots v_{s_1}}_{x_1} | \ldots | \underbrace{v_{s_{n-3}+1} \ldots v_{s_{n-2}}}_{x_{n-2}}] \underbrace{v_{s_{n-2}+1} \ldots v_m}_{x_{n-1}+x_n},$$

as it follows from the definition of ϕ, D_N and b. $\square$

7. The Hochschild cohomology ring

In this section we will assume that the field k is of characteristic zero. We begin by recalling some definitions and notation following [Ci2]. As we said before, all quivers are assumed to be finite and connected.

A path γ of length $|\gamma| \geqslant 1$ in a quiver is said to be an *oriented cycle* if $o(\gamma) = t(\gamma)$.

Two paths α and β are *parallel*, if $o(\alpha) = o(\beta)$ and $t(\alpha) = t(\beta)$. Let for $i, j \geqslant 0$

$$\Delta_i /\!/ \Delta_j = \{(\alpha, \beta)\colon\ \alpha \in \Delta_i,\ \beta \in \Delta_j \text{ and } \alpha \text{ is parallel to } \beta\}.$$

A pair (α, β) of parallel paths is said to *start together* if they have the first arrow in common, and they are said to *end together* if they have the last arrow in common.

A vertex is called a *sink* (resp. a *source*) if it is not the source (resp. end) vertex of any arrow.

Parallel paths that start together and do not end at a sink can be pushed forward. More precisely, let (α, β) be a pair of parallel paths that start together. Then $\alpha = v\gamma$, $\beta = v\delta$ with $v \in \Delta_1$ and $t(\alpha) = t(\beta)$. Then any pair $(\tilde{\alpha}, \tilde{\beta})$ satisfying $\tilde{\alpha} = \gamma w$, $\tilde{\beta} = \delta w$, with $w \in \Delta_1$ and $o(w) = t(\alpha) = t(\beta)$ is called a $+$*movement* of (α, β). In an analogous way we define $-$*movements*.

A pair of parallel paths (α, β) is said to be a $+$*extreme* ($-$*extreme*) if it does not admit any $+$movement ($-$movement). Therefore, a pair of parallel paths (α, β) is a $+$extreme if and only if they end at a sink or do not start together (clearly both might occur simultaneously). An analogous characterization holds for $-$extremes. We shall call a pair of parallel paths (α, β) just an *extreme* if it is either a $+$extreme or a $-$extreme.

Finally, two pairs (α, β) and (γ, δ) in $\Delta_i /\!/ \Delta_j$ are said to be *equivalent*, and denoted by $(\alpha, \beta) \sim (\gamma, \delta)$, if there exists a finite sequence of $+$movements and $-$movements carrying (α, β) to (γ, δ).

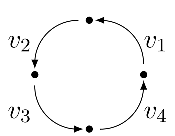
Definition 7.1. An equivalence class in $\Delta_i /\!/ \Delta_j$ is called a *medal* if all its $+$extremes end at a sink and all its $-$extremes start at a source. In particular, a class without extremes is a medal.

Examples. In the first example, let $\alpha = v_1 v_2$ and $\beta = v_1 v_2 v_3 v_4 v_1 v_2$. The class of (α, β) is a medal since it does not contain any extreme. In fact, any pair of parallel paths (α, β) in an oriented cycle can be pushed forward and pulled backwards and therefore there are no extremes. In particular, every class is a medal.

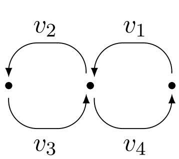
In the second example, let $\alpha = v_1 v_2$ and $\beta = v_1 v_2 v_3 v_4 v_1 v_2$. Although (α, β) could be pushed forward indefinitely, it is not a medal since

$$(\alpha, \beta) \sim (v_2 v_3, v_2 v_3 v_4 v_1 v_2 v_3) \sim (v_3 v_2, v_3 v_4 v_1 v_2 v_3 v_2) \sim (v_2 v_3, v_4 v_1 v_2 v_3 v_2 v_3)$$

and the last pair is a $+$extreme that does not end at a sink.



Example 1.



Example 2.

7.1. The Hochschild cohomology groups

The zero cohomology group $H^0(A, A)$ is the center of A as for any algebra. We now describe the cohomology groups $H^n(A, A)$, for $n > 0$, following [Lo]. Being

$$\mathbf{P}_n = \begin{cases} A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN} \otimes_{\Delta_0} A, & \text{if } n = 2k; \\ A \otimes_{\Delta_0} \mathrm{k}\Delta_{kN+1} \otimes_{\Delta_0} A, & \text{if } n = 2k+1; \end{cases}$$

we have

$$\mathrm{Hom}_{A^e}(\mathbf{P}_n, A) \simeq \begin{cases} \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN}, A), & \text{if } n = 2k; \\ \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN+1}, A), & \text{if } n = 2k+1. \end{cases} \tag{7.1}$$

Let $\mathbf{P}^*$ be the bigraded vector space

$$\mathbf{P}^* = \bigoplus_{n \geqslant 0} \bigoplus_{i=0}^{N-1} \mathbf{P}^*_{n,i}$$

where

$$\mathbf{P}^*_{2k,i} = \mathrm{k}\Delta_i /\!/ \Delta_{kN} \quad \text{and} \quad \mathbf{P}^*_{2k+1,i} = \mathrm{k}\Delta_i /\!/ \Delta_{kN+1}.$$

Since it is clear that $\mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_m, A) \simeq \bigoplus_{j=0}^{N-1} \mathrm{k}\Delta_j /\!/ \Delta_m$ for all $m \geqslant 0$, it follows that

$$\mathrm{Hom}_{A^e}(\mathbf{P}_n, A) \simeq \bigoplus_{i=0}^{N-1} \mathbf{P}^*_{n,i}. \tag{7.2}$$

The following theorem is proved in [Lo, §3] and it describes the cohomology of truncated quiver algebras. We note the word *j-extreme* is used instead of *medal* in Locateli's paper.

Theorem 7.2. *(See [Lo].) Let Δ be a quiver. Then the complex $\mathrm{Hom}_{A^e}(\mathbf{P}_n, A)$ has the following decomposition into subcomplexes*

$$\begin{tikzcd}
\mathrm{Hom}_{\Delta_0^e}(\mathrm{k}\Delta_{kN}, A) \arrow[r, "d_{2k}"] & \mathrm{Hom}_{\Delta_0^e}(\mathrm{k}\Delta_{kN+1}, A) \arrow[r, "d_{2k+1}"] & \mathrm{Hom}_{\Delta_0^e}(\mathrm{k}\Delta_{(k+1)N}, A) \\
\mathrm{k}\ \Delta_0 /\!/ \Delta_{kN} \arrow[dr, "D_0^{2k}"] & \mathrm{k}\ \Delta_0 /\!/ \Delta_{kN+1} \arrow[dddddr, bend left, "D_0^{2k+1}"] & \mathrm{k}\ \Delta_0 /\!/ \Delta_{(k+1)N} \\
\mathrm{k}\ \Delta_1 /\!/ \Delta_{kN} \arrow[dr, "D_1^{2k}"] & \mathrm{k}\ \Delta_1 /\!/ \Delta_{kN+1} & \mathrm{k}\ \Delta_1 /\!/ \Delta_{(k+1)N} \\
\mathrm{k}\ \Delta_2 /\!/ \Delta_{kN} & \mathrm{k}\ \Delta_2 /\!/ \Delta_{kN+1} & \mathrm{k}\ \Delta_2 /\!/ \Delta_{(k+1)N} \\
\vdots \quad\quad \vdots & \vdots & \vdots \\
\mathrm{k}\ \Delta_{N-2} /\!/ \Delta_{kN} \arrow[dr, "D_{N-2}^{2k}"] & \mathrm{k}\ \Delta_{N-2} /\!/ \Delta_{kN+1} & \mathrm{k}\ \Delta_{N-2} /\!/ \Delta_{(k+1)N} \\
\mathrm{k}\ \Delta_{N-1} /\!/ \Delta_{kN} & \mathrm{k}\ \Delta_{N-1} /\!/ \Delta_{kN+1} & \mathrm{k}\ \Delta_{N-1} /\!/ \Delta_{(k+1)N}
\end{tikzcd}$$

where

$$D_j^{2k}(\alpha,\pi)=\sum_{a\in\Delta_1}(a\alpha,a\pi)-\sum_{b\in\Delta_1}(\alpha b,\pi b),\quad k\geqslant 0 \text{ and } j=0,\dots,N-2,$$
$$D_0^{2k+1}(v,\pi)=\sum_{ab\in\Delta_{N-1}}(avb,a\pi b),\quad k\geqslant 0.$$

The following holds for the differentials.

(1) D_0^{2k+1} *is injective for all* $k\geqslant 0$,
(2) D_0^{2k} *is injective for all* $k>0$ *unless* Δ *is a cycle,*
(3) $\dim\ker D_j^{2k}$ *is equal to the number of medals in* $\Delta_j/\!\!/\Delta_{kN}$ *for all* $j=1,\dots,N-2$ *and for all* $k>0$. *More precisely, if for each medal* M *one considers*

$$\bar{M}=\sum_{(\alpha,\pi)\in M}(\alpha,\pi)\in \mathrm{k}\Delta_j/\!\!/\Delta_{kN},$$

then the set $\{\bar{M}\colon M \text{ is a medal in } \Delta_j/\!\!/\Delta_{kN}\}$ *is a basis of* $\ker D_j^{2k}$.

Definition 7.3. The cohomology class $\bar{M}$ corresponding to the medal M will be called the *medal cohomology class* associated to M.

Remark 7.4. The cohomology ring $H^*(A,A)$ inherits the bigrading of the subcomplex decomposition of $\mathbf{P}^*$ in Theorem 7.2. We now have

(1) $H^n(A,A)_0=0$, for all $n\geqslant 1$.
(2) $H^{2k}(A,A)_i$ is formed entirely of medals for all $1\leqslant i\leqslant N-2$.
(3) $H^{2k}(A,A)_{N-1}$ is a cokernel.
(4) $H^{2k+1}(A,A)_i$ is a cokernel for all $1\leqslant i\leqslant N-1$.

7.2. The Yoneda product

The Hochschild cohomology groups of A, $H^n(A,A)$ for $n\geqslant 0$, are by definition, $\mathrm{Ext}^n_{A^e}(A,A)$ and therefore $H^*(A,A)=\bigoplus_{n\geqslant 0}H^n(A,A)$ have a ring structure given by the multiplication induced by the Yoneda product.

It is well known that the Yoneda product of $H^*(A,A)$ coincides with the cup product defined on the cohomology of $\mathrm{Hom}_{A^e}(\mathbf{Q},A)$. More precisely, the cup product is originally defined in terms of the standard A^e-projective bar resolution $A\otimes A^{\otimes *}\otimes A$ of A as follows. Given two cochains,

$$f\in \mathrm{Hom}_{A^e}\big(A\otimes A^{\otimes m}\otimes A,A\big)\simeq \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes m},A\big),$$
$$g\in \mathrm{Hom}_{A^e}\big(A\otimes A^{\otimes n}\otimes A,A\big)\simeq \mathrm{Hom}_{\mathrm{k}}\big(A^{\otimes n},A\big)$$

the cup product of f and g is the cochain $f\cup g\in \mathrm{Hom}_{\mathrm{k}}(A^{\otimes(m+n)},A)$ defined by

$$f\cup g(\alpha_1\otimes\cdots\otimes\alpha_{m+n})=f(\alpha_1\otimes\cdots\otimes\alpha_m)g(\alpha_{m+1}\otimes\cdots\otimes\alpha_{m+n}).$$

The analogous definition works for the resolution $\mathbf{Q}$. For any $\mathrm{k}\Delta_0$-bimodule M, let $\mathrm{Hom}_{\Delta_0^e}(M,A)$ be the group of homomorphisms of $\mathrm{k}\Delta_0$-bimodules. Given two cochains,

$$f \in \operatorname{Hom}_{A^e}(\mathbf{Q}_m, A) \simeq \operatorname{Hom}_{\Delta_0^e}\big(A_+^{\otimes_{\Delta_0}^m}, A\big),$$

$$g \in \operatorname{Hom}_{A^e}(\mathbf{Q}_n, A) \simeq \operatorname{Hom}_{\Delta_0^e}\big(A_+^{\otimes_{\Delta_0}^n}, A\big)$$

the cup product of f and g is the cochain $f \cup g \in \operatorname{Hom}_{\Delta_0^e}(A_+^{\otimes_{\Delta_0}^{m+n}}, A)$ defined by

$$f \cup g\big([\alpha_1|\dots|\alpha_{m+n}]\big) = f\big([\alpha_1|\dots|\alpha_m]\big)g\big([\alpha_{m+1}|\dots|\alpha_{m+n}]\big). \tag{7.3}$$

We shall now use the comparison morphisms to describe the Yoneda product in the minimal resolution $\mathbf{P}$ of A.

Proposition 7.5. *Let $f \in \operatorname{Hom}_{A^e}(\mathbf{P}_m, A)$ and $g \in \operatorname{Hom}_{A^e}(\mathbf{P}_n, A)$. Then, in terms of the identification* (7.1)*, we have*

- *if $m = 2h$ and $n = 2k$ then*

$$f \cup g(v_1 \dots v_{hN} w_1 \dots w_{kN}) = f(v_1 \dots v_{hN}) g(w_1 \dots w_{kN}),$$

- *if $m = 2h$ and $n = 2k+1$ then*

$$f \cup g(v_1 \dots v_{hN} w_1 \dots w_{kN}) = f(v_1 \dots v_{hN}) g(w_1 \dots w_{kN+1}),$$

- *if $m = 2h+1$ and $n = 2k+1$ then*

$$\begin{aligned} &f \cup g(u_1 \dots u_{(h+k+1)N}) \\ &\quad = \sum_{0<i<j<N} u_1 \dots f(u_i \dots u_{i+hN}) u_{i+hN+1} \dots g(u_{j+hN} \dots u_{j+(h+k)N}) \dots u_{(h+k+1)N}. \end{aligned}$$

Proof. By definition $f \cup g = ((f \circ \mathbf{G}) \cup (g \circ \mathbf{G})) \circ \mathbf{F}$.

Assume that $m = 2h$ and $n = 2k$ and let $u_1 \dots u_{lN} = v_1 \dots v_{hN} w_1 \dots w_{kN}$, $l = h+k$. Then

$$\begin{aligned} \mathbf{F}(u_1 \dots u_{lN}) &= \mathbf{F}(1 \otimes u_1 \dots u_{lN} \otimes 1) \\ &= \sum 1[\underbrace{u_1 \dots u_{x_1}}_{x_1} | \underbrace{u_{1+x_1}}_{1} | \underbrace{\dots u_{1+x_1+x_2}}_{x_2} | \underbrace{u_{2+x_1+x_2}}_{1} | \dots\dots | \underbrace{u_{l+\sum x_j}}_{1}] \underbrace{\dots u_{lN}}_{lN-l-\sum x_j}, \end{aligned}$$

where the sum is over all l-tuples $(x_1, \dots, x_l) \in \mathbb{Z}^l$ with $1 \leqslant x_i < N$. Being $f \circ \mathbf{G} \in \operatorname{Hom}_{\Delta_0^e}(A_+^{\otimes_{\Delta_0}^{2h}}, A)$ and $g \circ \mathbf{G} \in \operatorname{Hom}_{\Delta_0^e}(A_+^{\otimes_{\Delta_0}^{2k}}, A)$, and identifying $[\alpha_1|\dots|\alpha_m]$ with $1[\alpha_1|\dots|\alpha_m]1$, we have (see (7.3))

$$\begin{aligned} &\big((f \circ \mathbf{G}) \cup (g \circ \mathbf{G})\big) \circ \mathbf{F}(u_1 \dots u_{lN}) \\ &\quad = \sum (f \circ \mathbf{G})\big(1[\underbrace{u_1 \dots u_{x_1}}_{x_1} | \underbrace{u_{1+x_1}}_{1} | \dots\dots | \underbrace{\dots u_{h-1+\sum_{j=1}^h x_j}}_{x_h} | \underbrace{u_{h+\sum_{j=1}^h x_j}}_{1}]1\big) \\ &\qquad \times (g \circ \mathbf{G})\big(1[\underbrace{\dots u_{h+\sum_{j=1}^{h+1} x_j}}_{x_{h+1}} | \underbrace{u_{h+1+\sum_{j=1}^{h+1} x_j}}_{1} | \dots\dots | \underbrace{u_{l+\sum x_j}}_{1}]1\big) \underbrace{\dots u_{lN}}_{lN-l-\sum x_j}. \end{aligned}$$

By the definition of $\mathbf{G}$, the only nonzero term of this sum is that one corresponding to $(x_1,\dots,x_l) = (N-1, N-1,\dots,N-1)$. This yields

$$\begin{aligned} f \cup g(u_1 \dots u_{lN}) &= f(u_1 \dots u_{hN}) g(u_{hN+1} \dots u_{lN}) \\ &= f(v_1 \dots v_{hN}) g(w_1 \dots w_{kN}). \end{aligned}$$

The proof is analogous if $m = 2h$ and $n = 2k+1$. Finally assume that $m = 2h+1$ and $n = 2k+1$. Let $l = h+k+1$. Then

$$\begin{aligned} &\big((f \circ \mathbf{G}) \cup (g \circ \mathbf{G})\big) \circ \mathbf{F}(u_1 \dots u_{lN}) \\ &\quad = \sum (f \circ \mathbf{G})\big(1[\underbrace{u_1 \dots u_{x_1}}_{x_1} | \underbrace{u_{1+x_1}}_{1} | \dots\dots | \underbrace{u_{h+\sum_{j=1}^{h} x_j}}_{1} | \underbrace{\dots u_{h+\sum_{j=1}^{h+1} x_j}}_{x_{h+1}}]1\big) \\ &\qquad \times (g \circ \mathbf{G})\big(1[\underbrace{u_{h+1+\sum_{j=1}^{h+1} x_j}}_{1} | \underbrace{\dots u_{h+1+\sum_{j=1}^{h+2} x_j}}_{x_{h+2}} | \dots\dots | \underbrace{u_{l+\sum x_j}}_{1}]1\big) \underbrace{\dots u_{lN}}_{lN-l-\sum x_j} . \end{aligned}$$

Again, by the definition of $\mathbf{G}$, the only nonzero terms of this sum are those corresponding to $(x_1,\dots,x_l) = (x_1, N-1, N-1,\dots,N-1)$ for all $x_1 = 1,\dots,N-1$. Therefore

$$f \cup g(u_1 \dots u_{lN}) = \sum_{x_1=1}^{N-1} \sum_{j=1}^{x_1} u_1 \dots f(\underbrace{u_j \dots}_{hN+1}) u_{j+hN+1} \dots g(\underbrace{u_{x_1+1+hN} \dots}_{kN+1}) \dots u_{(h+k+1)N}$$

as claimed. $\square$

Recall that

$$\mathbf{P}^*_{2k,i} = \mathrm{k}\Delta_i /\!\!/ \Delta_{kN} \quad \text{and} \quad \mathbf{P}^*_{2k+1,i} = \mathrm{k}\Delta_i /\!\!/ \Delta_{kN+1}$$

and that we have an isomorphism $\mathrm{Hom}_{A^e}(\mathbf{P}_n, A) \simeq \bigoplus_{i=0}^{N-1} \mathbf{P}^*_{n,i}$ (see (7.2)).

Theorem 7.6. *Let A be a truncated quiver algebra. Then, in terms of the above isomorphism we have that*

- *if $(\alpha,\pi) \in \mathbf{P}^*_{2h,i}$ and $(\beta,\tau) \in \mathbf{P}^*_{2k,j}$, then*

$$(\alpha,\pi) \cup (\beta,\tau) = (\alpha\beta, \pi\tau) \in \mathbf{P}^*_{2(h+k),i+j},$$

- *if $(\alpha,\pi) \in \mathbf{P}^*_{2h,i}$ and $(\beta,\tau) \in \mathbf{P}^*_{2k+1,j}$, then*

$$(\alpha,\pi) \cup (\beta,\tau) = (\alpha\beta, \pi\tau) \in \mathbf{P}^*_{2(h+k)+1,i+j},$$

- *if $(\alpha,\pi) \in \mathbf{P}^*_{2h+1,i}$ and $(\beta,\tau) \in \mathbf{P}^*_{2k+1,j}$, then*

$$(\alpha,\pi) \cup (\beta,\tau) = \sum_{\mu} (\gamma_\mu, \mu) \in \mathbf{P}^*_{2(h+k)+2,N-2+i+j},$$

where the sum is over all paths μ containing π and τ as a subpath, and γ_μ is the result of substituting π and τ by α and β respectively in μ. In particular, $(\alpha,\pi) \cup (\beta,\tau) = 0$, if $i + j > 1$.

Remark 7.7. The third line of the previous theorem shows that the Yoneda product on the complex $\mathbf{P}^*$ is not compatible with the bigrading.

Let us define the following product in $\mathbf{P}^*$. If $(\alpha,\pi)\in\mathbf{P}^*_{n_1}$ and $(\beta,\tau)\in\mathbf{P}^*_{n_2}$ let

$$(\alpha,\pi)\vee(\beta,\tau)=\begin{cases}(\alpha\beta,\pi\tau), & \text{if } n_1 \text{ or } n_2 \text{ are even};\\ 0, & \text{otherwise}.\end{cases}$$

The following theorem extends the results of Sections 3 and 4 of [BLM], proved for truncated cycle algebras, to any truncated quiver algebra.

Theorem 7.8. *Let A be a truncated quiver algebra. Then the product $\vee$ in $\mathbf{P}^*$ induces the Yoneda product in the Hochschild cohomology group $H^*(A,A)$, and thus it is a bigraded commutative ring (see Remark 7.4). In particular the Yoneda product of two odd degree cohomology classes is zero.*

Proof. If n_1 or n_2 are even, then the result is straightforward from the previous theorem. If $n_1 = hN+1$ and $n_2 = kN+1$ are odd numbers then $(\alpha,\pi)\in \mathrm{k}\Delta_i /\!\!/ \Delta_{hN+1}$ and $(\beta,\tau)\in \mathrm{k}\Delta_j /\!\!/ \Delta_{kN+1}$ with $1\leqslant i,j\leqslant N-1$ (see Theorem 7.2). Thus $i+j\geqslant 2$ and, according to the previous theorem, we obtain $(\alpha,\pi)\cup(\beta,\tau)=0$. $\square$

Corollary 7.9. *Let A be a noncycle truncated quiver algebra. If the Yoneda product of two cohomology classes of positive degree is not zero, then at least one of them is a medal cohomology class.*

Proof. Let $f\in \mathrm{k}\Delta_i /\!\!/ \Delta_{m_1}$ and $g\in \mathrm{k}\Delta_j /\!\!/ \Delta_{m_2}$ be representatives of cohomology classes $\bar f$ and $\bar g$ of cohomological degrees $n_1>0$ and $n_2>0$ respectively and assume that $\bar f\cup\bar g\neq 0$. Combining Theorems 7.2 and 7.8 we obtain that either n_1 or n_2 is even and that $i+j\leqslant N-1$, $i>0$, $j>0$. Thus $i,j\leqslant N-2$ and assuming that n_1 is even it follows that $\bar f$ is a medal cohomology class. $\square$

Corollary 7.10. *Let A be a noncycle truncated quiver algebra. If $f_1,\dots,f_N$ are cohomology classes of positive degree, then $f_1\dots f_N=0$. In particular $H^*(A,A)/\mathcal{N}\simeq \mathrm{k}$, where $\mathcal{N}$ is the ideal generated by homogeneous nilpotent elements.*

Proof. The result follows directly from Theorems 7.2 and 7.8. $\square$

8. Applications

Even though it does not appear in the literature as a conjecture, many people believed that for TQA's A the cohomology ring should be trivial, except for cycle algebras, meaning more precisely that the product in the subring $\bigoplus_{n\geqslant 1} H^n(A,A)$ should be trivial.

After Corollary 7.9, the understanding of medals in Δ is highly relevant to determine whether the product in cohomology is trivial or not.

A full classification of TQA's with trivial cohomology ring requires a deeper understanding of the spaces of paths, parallel paths and medals of quivers. We intend to carry out this classification in a future work.

We present here two large classes of quivers whose associated TQA's have trivial cohomology rings. Namely the class of quivers with no oriented cycles and the class of quiver with neither sinks nor sources. On the other hand we present an interesting example of a small quiver yielding TQA's with a nontrivial cohomology ring.

8.1. Quivers with no oriented cycles

We recall that if Δ has no oriented cycles, then A has finite global dimension, that is $H^i(A,A)=0$ for all $i>i_0$ for a sufficiently large i_0. This follows, for example, directly from Bardzell's resolution.

Theorem 8.1. *Let Δ be a quiver without any cycle. Then the Yoneda product in $\bigoplus_{n\geqslant 1} H^n(A, A)$ is zero.*

Proof. The set of vertices Δ_0 of a quiver without any cycle is a partial ordered set: $v_1 \preccurlyeq v_2$ if and only if there exists a path α such that $o(\alpha) = v_1$ and $t(\alpha) = v_2$.

We first prove that for any nonzero medal cohomology class M of positive cohomological degree, there exist vertices $v_1 \prec v_2$ such that $o(\beta) \preccurlyeq v_1$ and $v_2 \preccurlyeq t(\beta)$ for any pair $(\beta, \tau) \in M$. Let (α, π) be any pair in M. We know that $1 \leqslant |\alpha| \leqslant N-2 < N \leqslant |\pi|$ (cf. Theorem 7.2). Assume that $\alpha = a_1 \dots a_{|\alpha|}$ and $\pi = p_1 \dots p_{|\pi|}$. Let $l \geqslant 0$ be the largest integer for which $a_i = p_i$ for $i = 1, \dots, l$. If $l = |\alpha|$ then $p_{l+1} \dots p_{|\pi|}$ would be an oriented cycle, which is impossible, and thus $l < |\alpha|$. Similarly, if $r \geqslant 0$ is the smallest integer for which $a_i = p_{|\pi|-|\alpha|+i}$ for $i = r, \dots, |\alpha|$ then $r > 1$. Clearly $l < r$ and in fact $l < r-1$ for if $l = r-1$ then $p_{l+1} \dots p_{|\pi|-|\alpha|+r-1}$ would be an oriented cycle. Therefore, if $v_1 = t(a_l)$ and $v_2 = o(a_r)$, then $v_1 \prec v_2$. Since $a_{l+1} \neq p_{l+1}$ and $a_{r-1} \neq p_{|\pi|-|\alpha|+r-1}$ it follows that for any pair $(\beta, \tau) \in M$, β must contain the path $a_{l+1} \dots a_{r-1}$. Therefore $o(\beta) \preccurlyeq v_1$ and $v_2 \preccurlyeq t(\beta)$.

We now prove that the Yoneda product in $\bigoplus_{n\geqslant 1} H^n(A, A)$ is zero. Assume, on the contrary, that there are two cohomology classes of positive cohomological degree M_1 and M_2 such that $M_1 \cup M_2 \neq 0$. By Corollary 7.9 we may assume that M_1 is a medal cohomology class. Let $v_1 \prec v_2$ as above. We now must consider two possibilities:

(1) M_2 is also a medal cohomology class. Let $w_1 \prec w_2$ as above. Since $M_1 \cup M_2 \neq 0$ there exist $(\alpha, \pi) \in M_1$ and $(\beta, \tau) \in M_2$ such that $v_2 \preccurlyeq t(\alpha) = o(\beta) \preccurlyeq w_1$. Additionally, since the cup product is commutative, there exist $(\alpha', \pi') \in M_1$ and $(\beta', \tau') \in M_2$ such that $w_2 \preccurlyeq t(\beta') = o(\alpha') \preccurlyeq v_1$. This is a contradiction since $v_1 \prec v_2$ and $w_1 \prec w_2$.

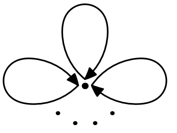
(2) M_2 is a cohomology class of odd degree. Since $M_2 \cup M_1 = M_1 \cup M_2 \neq 0$ there exist a representative (β, τ) of M_2 and pairs $(\alpha, \pi), (\alpha', \pi') \in M_1$ such that $v_2 \preccurlyeq t(\alpha) = o(\beta) \preccurlyeq t(\beta) \preccurlyeq o(\alpha) \prec v_1$, which is a contradiction.

This completes the proof. $\square$

Remark 8.2. Quivers in this class might have lots of medals.

8.2. Quivers with neither sinks nor sources

Truncated tensor algebras are particular cases of truncated quiver algebras associated to quivers without sinks and sources. Indeed, let V be a finite dimensional k-vector space and let $A_N = T(V)/(V^{\otimes N})$ the N-truncated tensor algebra. Then A_N is a truncated quiver algebra corresponding to the quiver



with $\dim_k V$ loops. If $\dim_k V \geqslant 2$ then it is not difficult to see that this quiver has no medals in $\Delta_j /\!\!/ \Delta_{kN}$ for $k > 0$. According to Corollary 7.9 the Yoneda product in positive cohomology degrees is zero. In Theorem 8.6 below we extend this result to all quivers without sinks and sources that are not an oriented cycle. We point out that if $\dim_k V = 1$ then the N-truncated tensor algebra A_N is a truncated polynomial algebra and its ring structure is described in Section 8.4.

Lemma 8.3. *Let Δ be a quiver. If (α, β) is a pair of parallel paths such that $|\alpha| < |\beta|$ and its class contains no $+$extremes, then there exists an oriented cycle γ such that $\beta = \alpha\gamma$. Similarly, if (α, β) is a pair of parallel paths such that $|\alpha| < |\beta|$ and its class contains no $-$extremes, then there exists an oriented cycle γ such that $\beta = \gamma\alpha$.*

Proof. Let (α,β) be a pair of parallel paths such that $|\alpha| < |\beta|$. Then $\alpha = a_1 \dots a_{|\alpha|}$ and $\beta = b_1 \dots b_{|\alpha|} \dots b_{|\beta|}$. If the class of (α,β) contains no $+$extremes then, in particular, (α,β) can be pushed forward $|\alpha|$ times. Therefore $b_j = a_j$ for all $j = 1,\dots,|\alpha|$. Let $\gamma = b_{|\alpha|+1} \dots b_{|\beta|}$. Then $o(\gamma) = t(\alpha) = t(\beta) = t(\gamma)$ and thus γ is an oriented cycle.

The proof of the second statement is analogous. □

Lemma 8.4. *Let Δ be a quiver. Then Δ is an oriented cycle if and only if there exists an oriented cycle γ and a pair of parallel paths (α,β), with α and β subpaths of γ and $|\alpha| < |\beta|$, such that the class of (α,β) does not have any extreme.*

Proof. If Δ is an oriented cycle then it is clear that there are no extremes at all.

Conversely, assume that Δ is not an oriented cycle. We shall prove that given an oriented cycle γ and a pair of parallel paths (α,β), with α and β subpaths of γ and $|\alpha| < |\beta|$, the class of (α,β) has an extreme.

Let γ be the oriented cycle. Since Δ is not an oriented cycle there must exist a vertex p in γ and two different arrows $u, v \in \Delta_1$ (not necessarily in γ) such that either $t(u) = t(v) = p$ or $o(u) = o(v) = p$.

Now let (α,β) be a pair of parallel paths such that α and β are subpaths of γ and $|\alpha| < |\beta|$.

In the case $o(u) = o(v) = p$ we shall see that (α,β) can be pushed forward until we obtain a $+$extreme. We first push (α,β) forward in order to reach a pair (α',β') with $t(\alpha') = t(\beta') = p$. Since γ is an oriented cycle this is possible, unless we reach a $+$extreme before. If (α',β') is a $+$extreme we are done. Otherwise α' and β' must start together and hence $\alpha' = a'_1 \dots a'_{|\alpha|}$ and $\beta' = b'_1 \dots b'_{|\beta|}$ with $a'_1 = b'_1$. Since $|\alpha| + 1 \leqslant |\beta|$ and $t(\alpha') = p$ we can assume that $u \neq b'_{|\alpha|+1}$. Next, we push (α',β') forward obtaining

$$(\alpha',\beta') \sim \left(a'_2 \dots a'_{|\alpha|} u, b'_2 \dots b'_{|\alpha|} b'_{|\alpha|+1} \dots b'_{|\beta|} u\right).$$

Now, if we keep pushing forward, since $u \neq b'_{|\alpha|+1}$ it is clear that in at most $|\alpha| - 1$ times we shall reach a $+$extreme.

In the case $t(u) = t(v) = p$ an analogous argument shows how to pull (α,β) backwards until a $-$extreme is reached. □

Theorem 8.5. *Let Δ be a finite connected quiver that is not an oriented cycle, and let (α,β) be a pair of parallel paths such that $|\alpha| < |\beta|$. Then the class of (α,β) has an extreme. In particular, if in addition Δ has neither sinks nor sources then $\Delta_i /\!\!/ \Delta_j$ does not have any medal for all $i \neq j$.*

Proof. Let (α,β) be a pair of parallel paths such that $|\alpha| < |\beta|$ and assume its class of (α,β) contains no $+$extremes. Then, by Lemma 8.3 there exists an oriented cycle $\gamma = v_1 \dots v_k$, $k \geqslant 1$, such that $\beta = \alpha\gamma$. Let $v_{k+j} = v_j$ for all $j > 0$. Thus, by pushing forward,

$$(\alpha,\beta) = (\alpha, \alpha v_1 \dots v_k) \sim (v_{k+1} \dots v_{k+|\alpha|}, v_1 \dots v_k v_{k+1} \dots v_{k+|\alpha|}) = (\tilde{\alpha}, \tilde{\beta}).$$

Now $(\tilde{\alpha},\tilde{\beta})$ is a pair of parallel paths contained in the oriented cycle γ. Since $|\tilde{\alpha}| = |\alpha| < |\beta| = |\tilde{\beta}|$ and Δ is not an oriented cycle, Lemma 8.4 implies that there exist an extreme pair in the class of $(\tilde{\alpha},\tilde{\beta})$.

In particular, if Δ has no sinks and no sources and $(\alpha,\beta) \in \Delta_i /\!\!/ \Delta_j$ with $i \neq j$ then (α,β) is equivalent to either a $+$extreme not ending at a sink or a $-$extreme not starting at a source. Thus the class of (α,β) is not a medal. □

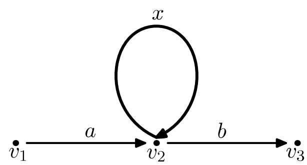
Theorem 8.6. *Let Δ be a quiver that is not an oriented cycle and with neither sinks nor sources. Then the Yoneda product in $\bigoplus_{n \geqslant 1} H^n(A, A)$ is zero.*

Proof. From Theorem 8.5 we know that $\Delta_i /\!/ \Delta_j$ contains no medals when $i \neq j$ and therefore Corollary 7.9 implies that the Yoneda product in $\bigoplus_{n \geqslant 1} H^n(A, A)$ is zero. $\square$

8.3. A distinguished example

We exhibit a noncycle TQA with nontrivial cohomology ring. It turns out that this example is a fundamental piece to understand and describe the full structure of the cohomology ring of any TQA. This will be done in a forthcoming paper.

Let Δ be the following quiver



and let $A = \mathrm{k}\Delta/(\Delta_N)$ be a TQA with $N \geqslant 3$. It is clear that $\Delta_0 /\!/ \Delta_0 = \{(v_1, v_1), (v_2, v_2), (v_3, v_3)\}$ and it is not difficult to see that for $j > 0$ or $M > 0$

$$\Delta_j /\!/ \Delta_M = \left\{ \left(ax^{j-1}, ax^{M-1}\right), \left(x^j, x^M\right), \left(x^{j-1}b, x^{M-1}b\right), \left(ax^{j-2}b, ax^{M-2}b\right) \right\}$$

with the conventions that $x^0 = v_2$ and a pair containing x^m with $m < 0$ does not appear. Observe that for this quiver we have that for any $(\alpha, \pi) \in \Delta_j /\!/ \Delta_M$ the path π is determined by α. Thus, in order to have a clearer notation we shall denote the pair (α, π) by α. Now, using this notation, we have

$$\Delta_0 /\!/ \Delta_0 = \{v_1, v_2, v_3\}, \quad \text{and}$$
$$\Delta_j /\!/ \Delta_M = \left\{ax^{j-1}, x^j, x^{j-1}b, ax^{j-2}b\right\}, \quad \text{for } j > 0 \text{ or } M > 0.$$

The associated matrices of the differentials of the complex of Theorem 7.2 are described below:

for $k = 0$,

$$\begin{array}{ccccc}
\mathrm{Hom}_{\Delta_0}(\mathrm{k}\Delta_0, A) & \xrightarrow{d_0} & \mathrm{Hom}_{\Delta_0}(\mathrm{k}\Delta_1, A) & \xrightarrow{d_1} & \mathrm{Hom}_{\Delta_0}(\mathrm{k}\Delta_2, A) \\
\hline
\mathrm{k}\ \Delta_0/\!/\Delta_0 & & \mathrm{k}\ \Delta_0/\!/\Delta_1 & \xrightarrow{D_0^1} & \mathrm{k}\ \Delta_0/\!/\Delta_2 \\
 & \searrow^{D_0^0} & & & \\
\mathrm{k}\ \Delta_1/\!/\Delta_0 & & \mathrm{k}\ \Delta_1/\!/\Delta_1 & & \mathrm{k}\ \Delta_1/\!/\Delta_2 \\
 & \searrow^{D_1^0} & & & \\
\mathrm{k}\ \Delta_2/\!/\Delta_0 & & \mathrm{k}\ \Delta_2/\!/\Delta_1 & & \mathrm{k}\ \Delta_2/\!/\Delta_2 \\
\vdots & \vdots & \vdots & & \vdots \\
\mathrm{k}\ \Delta_{N-2}/\!/\Delta_0 & & \mathrm{k}\ \Delta_{N-2}/\!/\Delta_1 & & \mathrm{k}\ \Delta_{N-2}/\!/\Delta_2 \\
 & \searrow^{D_{N-2}^0} & & & \\
\mathrm{k}\ \Delta_{N-1}/\!/\Delta_0 & & \mathrm{k}\ \Delta_{N-1}/\!/\Delta_1 & & \mathrm{k}\ \Delta_{N-1}/\!/\Delta_2
\end{array}$$

we have

$$\left[D_0^0\right] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}; \qquad \left[D_j^0\right] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad 2 \leqslant j \leqslant N-2; \qquad \left[D_0^1\right] = \begin{bmatrix} N-1 \\ N \\ N-1 \\ N-2 \end{bmatrix},$$

and for $k \geqslant 1$,

$$\mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN}, A) \xrightarrow{d_{2k}} \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{kN+1}, A) \xrightarrow{d_{2k+1}} \mathrm{Hom}_{(\mathrm{k}\Delta_0)^e}(\mathrm{k}\Delta_{(k+1)N}, A)$$

$$\begin{tikzcd}
\mathrm{k}\,\Delta_0/\!/\Delta_{kN} \arrow[dr, "D_0^{2k}"] & \mathrm{k}\,\Delta_0/\!/\Delta_{kN+1} \arrow[dddddr, bend left, "D_0^{2k+1}"] & \mathrm{k}\,\Delta_0/\!/\Delta_{(k+1)N} \\
\mathrm{k}\,\Delta_1/\!/\Delta_{kN} \arrow[dr, "D_1^{2k}"] & \mathrm{k}\,\Delta_1/\!/\Delta_{kN+1} & \mathrm{k}\,\Delta_1/\!/\Delta_{(k+1)N} \\
\mathrm{k}\,\Delta_2/\!/\Delta_{kN} & \mathrm{k}\,\Delta_2/\!/\Delta_{kN+1} & \mathrm{k}\,\Delta_2/\!/\Delta_{(k+1)N} \\
\vdots & \vdots & \vdots \\
\mathrm{k}\,\Delta_{N-2}/\!/\Delta_{kN} \arrow[dr, "D_{N-2}^{2k}"] & \mathrm{k}\,\Delta_{N-2}/\!/\Delta_{kN+1} & \mathrm{k}\,\Delta_{N-2}/\!/\Delta_{(k+1)N} \\
\mathrm{k}\,\Delta_{N-1}/\!/\Delta_{kN} & \mathrm{k}\,\Delta_{N-1}/\!/\Delta_{kN+1} & \mathrm{k}\,\Delta_{N-1}/\!/\Delta_{(k+1)N}
\end{tikzcd}$$

we have

$$\left[D_0^{2k}\right] = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \left[D_0^{2k+1}\right] = \begin{bmatrix} N-1 \\ N \\ N-1 \\ N-2 \end{bmatrix},$$

$$\left[D_1^{2k}\right] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \left[D_j^{2k}\right] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \quad 2 \leqslant j \leqslant N-2.$$

Therefore a basis of the cohomology is described by Table 1.

The elements of this basis have been chosen so that the product becomes more transparent. For $n \geqslant 1$ and $1 \leqslant j \leqslant N-1$ let $\omega_{n,j}$ be the basis element of $\mathrm{k}\Delta_j/\!/\Delta_M \subset H^n(A, A)$ placed at the top of each row in Table 1, that is

$$\omega_{n,j} = \begin{cases} x^j + ax^{j-1}, & \text{if } n \text{ is odd;} \\ ax^{j-1} + x^j + x^{j-1}b + ax^{j-2}b, & \text{if } n \text{ is even.} \end{cases}$$

Note that $\omega_{2k,j}$ is a sum of two different medal cohomology classes (see Definition 7.3) $\bar{M}_1 + \bar{M}_2$ where $M_1 = \{ax^{j-1}, x^j, x^{j-1}b\}$ and $M_2 = \{ax^{j-2}b\}$. From Theorem 7.8 it follows that

$$\omega_{n_1,j_1} \cup \omega_{n_2,j_2} = \begin{cases} \omega_{n_1+n_2, j_1+j_2}, & \text{if } n_1 \text{ or } n_2 \text{ is even and } j_1 + j_2 < N; \\ 0, & \text{otherwise.} \end{cases}$$

Table 1

	$k\Delta_0/\Delta_M$	$k\Delta_1/\Delta_M$	$k\Delta_2/\Delta_M, \ldots, k\Delta_{N-2}$	$k\Delta_{N-1}/\Delta_M$
$H^0(A, A)$ $\dim = 2$	1	$\emptyset$	$\emptyset$	x^{n-1}
$H^1(A, A)$ $\dim = 2N-3$	$\emptyset$	$x+a$	$x^j + ax^{j-1}$ ax^{j-1}	$x^{N-1} + ax^{N-2}$ ax^{N-2}
coboundaries		a, b	$ax^{j-1} - x^{j-1}b$	$ax^{N-2} - x^{N-2}b$
$H^{2k}(A, A)$ $k \geqslant 1$ $\dim = 2N-2$	$\emptyset$	$a+x+b$	$ax^{j-1} + x^j + x^{j-1}b + ax^{j-2}b$ $ax^{j-2}b$	$ax^{N-2} + x^N + x^{N-2}b + ax^{N-3}b$ $ax^{N-3}b$ x^{N-1}
coboundaries		$\emptyset$	$\emptyset$	$(N-1)ax^{N-2} + Nx^{N-1} + (N-1)x^{N-2}b + (N-2)ax^{N-3}b$
$H^{2k+1}(A, A)$ $k \geqslant 1$ $\dim = 2N-2$	$\emptyset$	$x+a$ a	$x^j + ax^{j-1}$ ax^{j-1}	$x^{N-1} + ax^{N-2}$ ax^{N-2}
coboundaries		$a-b$	$ax^{j-1} + ax^{j-2}b$, $x^{j-1}b + ax^{j-2}b$	$ax^{N-2} + ax^{N-3}b$, $x^{N-2}b + ax^{N-3}b$

8.4. *Nonzero cohomology classes in the bar complex*

In this subsection we use the comparison morphisms to construct explicit nonzero cohomology classes in the bar complex. In the first example we consider the group $H^{2k}(A, A)_{N-1}$ of any N-TQA and in the last one we give a full description of the cohomology ring of truncated polynomial algebras in one variable.

8.4.1. *Nonzero cohomology classes in $H^{2k}(A, A)_{N-1}$*

Recall that $H^{2k}(A, A)_{N-1}$ is the cokernel of the injective map

$$D_0^{2k-1} : \Delta_0 /\!\!/ \Delta_{(k-1)N+1} \to \Delta_{N-1} /\!\!/ \Delta_{kN},$$
$$(v, \pi) \mapsto \sum_{ab \in \Delta_{N-1}} (avb, a\pi b)$$

(see Theorem 7.2 and Remark 7.4). In particular, a pair of parallel paths $(\beta, \tau) \in \Delta_{N-1} /\!\!/ \Delta_{kN}$ with the property that they neither start together nor end together is not in the image of D_0^{2k-1} and hence corresponds to a nonzero cohomology class.

Assume that there exists such a pair $(\beta, \tau) \in \Delta_{N-1} /\!\!/ \Delta_{kN}$. According to the identification (7.1) it corresponds to the element $g_{(\beta,\tau)} \in \mathrm{Hom}_{(k\Delta_0)^e}(k\Delta_{kN}, A)$ given by

$$g_{(\beta,\tau)}(\pi) = \begin{cases} \beta, & \text{if } \pi = \tau; \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that

$$f_{(\beta,\tau)} = g_{(\beta,\tau)} \circ \mathbf{G} \in \mathrm{Hom}_{A^e}\big(A \otimes A^{\otimes 2k} \otimes A, A\big) \simeq \mathrm{Hom}_k\big(A^{\otimes 2k}, A\big)$$

is given by

$$f_{(\beta,\tau)}\big(1[\alpha_1|\dots|\alpha_{2k}]1\big) = \begin{cases} \beta, & \text{if } \alpha_{2i-1}\alpha_{2i} = 0 \text{ in } A \text{ for } i = 1,\dots,k, \\ & \text{and } \alpha_1\dots\alpha_{2k} = \tau \text{ in } \mathrm{k}\Delta; \\ 0 & \text{otherwise.} \end{cases}$$

8.4.2. Truncated polynomial algebra in one variable

This case has been deeply studied. However, as far as we know, even in this case the comparison morphism has not been written down and a basis consisting of cohomology classes in the bar resolution can not be found in the literature. We present such a basis in this subsection.

If $A = \mathrm{k}[x]/(x^N)$ we have

$$\mathbf{P}^*_{2k,i} = \mathrm{k}\big(x^i, x^{kN}\big) \quad \text{and} \quad \mathbf{P}^*_{2k+1,i} = \mathrm{k}\big(x^i, x^{kN+1}\big)$$

and the only nonzero differentials are D_0^{2k+1} for all k. Thus

$$H^{2k}(A, A) = \mathrm{k}\big\{\big(x^0, x^{kN}\big), \big(x^1, x^{kN}\big), \dots, \big(x^{N-2}, x^{kN}\big)\big\}$$

and

$$H^{2k+1}(A, A) = \mathrm{k}\big\{\big(x^1, x^{kN+1}\big), \big(x^2, x^{kN+1}\big), \dots, \big(x^{N-1}, x^{kN+1}\big)\big\}.$$

Using the comparison morphism $\mathbf{G}$ it is not difficult to give a basis of the cohomology in the (reduced) bar resolution. Indeed, let

$$q = 1\big[x^{r_1}|\dots|x^{r_n}\big]1 \in \mathbf{Q}_n = A \otimes_{\Delta_0} A_+^{\otimes^n_{\Delta_0}} \otimes_{\Delta_0} A, \quad r_i > 0;$$

and let

$$f_{2k,i}\big(1\big[x^{r_1}|\dots|x^{r_{2k}}\big]1\big) = \begin{cases} x^{i+\sum r_j - kN}, & \text{if } r_{2j-1} + r_{2j} \geqslant N \text{ for } j = 1,\dots,k; \\ 0, & \text{otherwise;} \end{cases}$$

$$f_{2k+1,i}\big(1\big[x^{r_1}|\dots|x^{r_{2k+1}}\big]1\big) = \begin{cases} x^{i+\sum r_j - kN-1}, & \text{if } r_{2j} + r_{2j+1} \geqslant N \text{ for } j = 1,\dots,k; \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$H^{2k}(A, A) = \mathrm{k}\{f_{2k,0}, f_{2k,1}, \dots, f_{2k,N-2}\}$$

and

$$H^{2k+1}(A, A) = \mathrm{k}\{f_{2k+1,1}, f_{2k+1,2}, \dots, f_{2k+1,N-1}\}.$$

The cup product is given by

$$f_{m,i} \cup f_{n,j} = \begin{cases} f_{m+n,i+j} & \text{if either } m \text{ or } n \text{ is even and } i + j < N; \\ 0, & \text{otherwise;} \end{cases}$$

and $\{f_{0,1}, f_{1,1}, f_{2,0}\}$ is a set of generators of $H^*(A, A)$ as an algebra.

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