JOURNAL OF DIFFERENTIAL EQUATIONS 52, 91–115 (1984)

# The Existence, Uniqueness, and Instability of Spherically Symmetric Solutions of a System of Reaction–Diffusion Equations

GENE A. KLAASEN*

*Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916*

AND

WILLIAM C. TROY†

*Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260*

Received September 23, 1980; revised October 26, 1981

The system $\partial x/\partial t = \Delta x + F(x, y)$, $\partial y/\partial t = G(x, y)$ is investigated, where $x$ and $y$ are scalar functions of time $(t \geqslant 0)$, and $n$ space variables $(\xi_1, ..., \xi_n)$, $\Delta x \equiv \sum_{i=1}^{n} \partial^2 x/\partial \xi_i^2$, and $F$ and $G$ are nonlinear functions. Under certain hypotheses on $F$ and $G$ it is proved that there exists a unique spherically symmetric solution $(x(r), y(r))$, where $r = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$, which is bounded for $r \geqslant 0$ and satisfies $x(0) > x_0$, $y(0) > y_0$, $x'(0) = 0$, $y'(0) = 0$, and $x' < 0$, $y' > 0$, $\forall r > 0$. Thus, $(x(r), y(r))$ represents a time independent equilibrium solution of the system. Further, the linearization of the system restricted to spherically symmetric solutions, around $(x(r), y(r))$, has a unique positive eigenvalue. This is in contrast to the case $n = 1$ (i.e., one space dimension) in which zero is an eigenvalue. The uniqueness of the positive eigenvalue is used in the proof that the spherically symmetric solution described is unique.

## 1. INTRODUCTION

In this paper we investigate the existence, uniqueness, and stability properties of spherically symmetric solutions of a system of equations of the form

$$\frac{\partial x}{\partial t} = \Delta x + F(x, y), \tag{1.1}$$

$$\frac{\partial y}{\partial t} = G(x, y). \tag{1.2}$$

* Supported in part by NSF Grant MCS–8002948.

† Supported in part by NSF Grant MCS–8002948 and NIH Career Development Award 00306–01.



0022-0396/84 $3.00

Copyright © 1984 by Academic Press, Inc.
All rights of reproduction in any form reserved.

Here $x$ and $y$ are scalar functions of time $t$, and $n$ space variables $\xi_1, \ldots, \xi_n$; $\Delta x \equiv \sum_{i=1}^{n} \partial^2 x/\partial \xi_i^2$ and $F$ and $G$ are nonlinear functions of $x$ and $y$. Models of the form (1.1), (1.2) arise in biology, neurophysiology, and chemistry. For example, the Fitzhugh–Nagumo [4, 5] nerve conduction equations and the Field–Noyes [1] model of the Belousov–Zhabotinskii reactions both are of the form given in Eqs. (1.1), (1.2).

Jones [3] has recently analyzed the scalar equation

$$\frac{\partial x}{\partial t} = \Delta x + f(x), \tag{1.3}$$

with the assumptions

(H1) $f: \mathbb{R} \to \mathbb{R}$ is smooth, $f(0) = f(1) = 0$, and there is a unique $\alpha$ between 0 and 1 such that $f(\alpha) = 0$. Furthermore, $f'(0) < 0, f'(1) < 0$, and $\int_0^1 f(\mu)\, d\mu > 0$.

(H2) $f''(\beta) \leqslant 0$ for all $\beta \in [\alpha, 1]$.

A spherically symmetric solution of Eq. (1.3) is a solution of the form $x = x(r)$, where $r = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$. Thus, Eq. (1.3) becomes

$$x'' + \frac{(n-1)}{r} x' + f(x) = 0. \tag{1.4}$$

The boundary conditions associated with a bounded spherically symmetric solution are given by

$$0 < x(0) < 1, \qquad x'(0) = 0, \qquad x(\infty) = 0. \tag{1.5}$$

Under hypothesis (H1) Jones [3] proves that the problem (1.4), (1.5) has a solution which we denote by $\bar{x}(r)$. In order to investigate the linear stability of $\bar{x}(r)$ he linearizes Eq. (1.4) around $\bar{x}(r)$ and investigates the system

$$x'' + \frac{(n-1)}{r} x' + f'(\bar{x})\, x = \lambda x. \tag{1.6}$$

The solution $x(r)$ is said to be linearly unstable if there exists $\lambda$ with $\mathrm{Re}(\lambda) > 0$ for which (1.6) admits a solution $\bar{x}(r)$ bounded on $[0, \infty)$. Under hypothesis (H2) Jones proves that there is a unique $\lambda > 0$ for which Eq. (1.6) has a bounded solution. Thus the solution $\bar{x}(r)$ is linearly unstable. Further, hypothesis (H2) allows him to prove that the solution $\bar{x}(r)$ of the problem (1.4), (1.5) is the unique one for which $0 < x < 1$ and $x' < 0\ \forall r > 0$.

In this paper we extend the results of Jones to the system (1.1), (1.2). We make several reasonable assumptions on the functions $F$ and $G$ and prove that Eqs. (1.1), (1.2) have a unique spherically symmetric solution which is linearly unstable.

In Section 2 we state our assumptions on $F$ and $G$ and make several preliminary mathematical comments necessary for the statement of our main results. Section 3 contains the statement and a discussion of our main results together with an outline of their proofs. The proofs appear in Section 4. In Appendix A we show that over an appropriate range of parameters the Fitzhugh–Nagumo nerve conduction equations and the Field–Noyes model of the Belousov–Zhabotinskii reaction fall within the class of equations which we are considering. Neither of these models satisfy hypothesis (H2) of Jones.

## 2. Mathematical Preliminaries

We investigate Eqs. (1.1), (1.2) for the existence of solutions of the form $(x(r), y(r))$, where $r = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$. Then Eqs. (1.1), (1.2) become

$$x'' + \frac{(n-1)}{r} x' + F(x, y) = 0, \tag{2.1}$$

$$G(x, y) = 0. \tag{2.2}$$

We assume

(i) $F(x, y)$, $G(x, y)$ are $C^1$ functions on an open rectangle $(X_1, X_2) \times (Y_1, Y_2)$. Also, $G_y < 0$ and $1 + F_y G_x/(G_y^2) > 0$ on $(X_1, X_2) \times (Y_1, Y_2)$.

(ii) There exist $(\alpha, \beta) \subset (X_1, X_2)$ and a function $k \in C^1((\alpha, \beta))$ such that $G(x, y) = 0 \Leftrightarrow y = k(x)\ \forall x \in (\alpha, \beta)$.

Substituting $y = k(x)$ into Eq. (2.1) we obtain

$$x'' + \frac{(n-1)}{r} x' + f(x) = 0, \tag{2.3}$$

where

$$f(x) \equiv F(x, k(x)), \qquad x \in (\alpha, \beta). \tag{2.4}$$

Assumption (ii) and the first part of (i) are standard and are useful in proving the existence and uniqueness of solutions of the initial value problem for Eq. (2.3). The second part of (i) is a technical assumption satisfied by both the Fitzhugh–Nagumo and Field–Noyes models. It is used to show that the function $b(r, \lambda)$ (see Section 4) is monotone increasing, which in turn is crucial in the proof that there is a unique positive eigenvalue of the linear stability problem.

(iii) $f$ depends continuously on a parameter $a$ (which we suppress throughout for ease of notation) and there exist numbers $a^*$ and $\delta > 0$, such that for each $a \in (a^* - \delta, a^*]$ there are exactly three values $x_0(a)$, $x_1(a)$, $x_2(a)$ satisfying $\alpha < x_0(a) < x_1(a) < x_2(a) < \beta$, $\forall a \in [a^* - \delta, a^*]$; $f(x_i(a)) = 0$ $(i = 0, 1, 2)$; and $f'(x_i) < 0$, $i = 0, 2$; $f'(x_1(a)) > 0$ and $f' < 0$, $\forall x \in (x_2(a), \beta)$.

(iv) $\int_{x_0(a)}^{x_2(a)} f(\mu)\, d\mu > 0$, $\forall a \in (a^* - \delta, a^*)$ and $\int_{x_0(a^*)}^{x_2(a^*)} f(\mu)\, d\mu = 0$.

Assumption (iii) states that $f$ is dependent on the parameter $a$ and that if $a$ is close to a critical value $a^*$ then $f$ has three distinct zeros. These zeros represent constant solutions of Eq. (2.3). The derivative conditions given in the last line of (iii) guarantee that two of the constant solutions are stable and one unstable. Assumption (iv) is an integral condition which appears to be well known to the neurophysiologists but whose physical significance is not entirely clear. As shown in the Appendix both the Fitzhugh–Nagumo and the Field–Noyes models satisfy (iii) and (iv).

Since $f'(x_1) > 0$ and $f'(x_2) < 0$ then it is reasonable to make the final assumption

(v) for each $a \in (a^* - \delta, a^*]$ there is a unique $x^* = x^*(a)$, $x_1(a) < x^*(a) < x_2(a)$, such that $f'(x^*(a)) = 0$ and $f''(x^*(a)) < 0$.

This last assumption makes the analysis simpler.

The main difference between our assumptions and those of Jones [3] is that we omit his (H2) and repalce it with the integral condition in (iv). Assumptions (iv), (v) occur more naturally in the applications than does his (H2). (See, e.g., Appendix A.)

## 3. Statement of Main Results

We assume throughout that $n > 1$ and consider the problem

$$\bar{x}'' + \frac{(n-1)}{r}\bar{x}' + f(\bar{x}) = 0, \tag{3.1}$$

$$x_0 < \bar{x}(0) < x_2, \qquad \bar{x}'(0) = 0, \qquad \bar{x}(\infty) = x_0, \tag{3.2}$$

where $f$, $x_0$, $x_2$ satisfy assumptions (i)–(iv). Recall the definitions of $a^*$ and $\delta$ given in Section 2. Then we state

THEOREM 1. *For each* $a \in (a^* - \delta, a^*)$ *the problem* (3.1), (3.2) *has a solution* $\bar{x}(r)$ *which satisfies*

$$x_0 < \bar{x}(r) < x_2, \qquad \bar{x}'(r) \leqslant 0, \qquad \forall r \geqslant 0. \tag{3.3}$$

Further, if $\delta > 0$ is sufficiently small then the solution of (3.1)–(3.3) is unique.

Having found the solution $\bar{x}(r)$ of the problem (3.1), (3.2), we observe that the pair $(\bar{x}(r), \bar{y}(r))$, where $\bar{y}(r) \equiv k(\bar{x}(r))$, solves the problem

$$x'' + \frac{(n-1)}{r} x' + F(x, y) = 0, \tag{3.4}$$

$$G(x, y) = 0, \tag{3.5}$$

$$x_0 < x(0) < x_2, \qquad x'(0) = 0, \qquad x(\infty) = x_0. \tag{3.6}$$

Next, to determine the linear stability properties of the solution we linearize Eqs. (3.4), (3.5) around $(\bar{x}(r), \bar{y}(r))$ and obtain the linear system

$$x'' + \frac{(n-1)}{r} x' + F_x(\bar{x}, \bar{y})\, x + F_y(\bar{x}, \bar{y})\, y = \lambda x, \tag{3.7}$$

$$G_x(\bar{x}, \bar{y})\, x + G_y(\bar{x}, \bar{y})\, y = \lambda y. \tag{3.8}$$

The solution $(\bar{x}, \bar{y})$ is said to be linearly unstable if there exists a $\lambda$ with $\mathrm{Re}(\lambda) > 0$ and a solution of Eq. (3.7), (3.8) which is bounded for $r \geqslant 0$. We now state

THEOREM 2. *If $\delta > 0$ is sufficiently small then there exist a unique $\lambda \geqslant 0$ and a corresponding solution of Eqs.* (3.7), (3.8) *which is bounded for $r \geqslant 0$. In fact, $\lambda > 0$.*

*Comments.* It is interesting to note that if $n = 1$ then $\lambda = 0$ is an eigenvalue for the problem (3.7), (3.8) with corresponding eigenfunction $(x, y) = (\bar{x}', \bar{x}'')$. However, for the case $n > 1$ we observe in Theorem 2 that this is not the case.

In Section 4 we give the proofs of our main theorems. First, we follow Jones [3] and use a "backwards shooting" argument to prove the existence of a spherically symmetric solution of the problem (3.1), (3.2). Next, we prove that the solution is linearly unstable. That is, we investigate Eqs. (3.7), (3.8) and prove that there is at least one nonnegative eigenvalue and corresponding eigenfunction solution. We then prove Lemmas 1–4 which are technical lemmas necessary for the proof that there is exactly one nonnegative eigenvalue and that is positive. Finally, we prove that the solution of (3.1), (3.2) is unique. The proof uses Lemmas 1–4, as well as Lemmas 5–10.

## 4. Proofs

### A. *Proof of Theorem* 1

A spherically symmetric solution of (1.1), (1.2) is a solution of the form $x = x(r)$, $y = y(r)$, $(r = (\xi_1^2 + \cdots + \xi_n^2)^{1/2})$ of

$$x'' + \frac{(n-1)}{r} x' + F(x, y) = 0, \tag{4.1}$$

$$G(x, y) = 0, \tag{4.2}$$

such that $x'(0) = y'(0) = 0$, $x(\infty) = x_0$, $y(\infty) = y_0 \equiv k(x_0)$. As shown in Section 2 this is equivalent to solving the problem

$$x'' + \frac{(n-1)}{r} x' + f(x) = 0,$$

where $x'(0) = 0$, $x(\infty) = x_0$. In system form this becomes

$$x' = y,$$

$$y' = -\frac{(n-1)}{r} y - f(x), \tag{4.3}$$

$$x(\infty) = x_0, \qquad y(0) = 0. \tag{4.4}$$

Let $\rho = r/(r+1)$. Then (4.3) can be rewritten as the system

$$x' = y, \tag{4.5}$$

$$y' = \frac{-(n-1)(1-\rho)}{\rho} y - f(x), \tag{4.6}$$

$$\rho' = (1-\rho)^2. \tag{4.7}$$

Since the solution $(x, y, \rho)$ which we seek must satisfy $(x(r), y(r), \rho(r)) \to (x_0, 0, 1)$ as $r \to \infty$, we examine the stable manifold of the system (4.5)–(4.7) at $(x_0, 0, 1)$. Let $A$ be the Jacobian matrix for the linearized system associated with (4.5)–(4.7) evaluated at $(x_0, 0, 1)$. Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -f'(x_0) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and its eigenvalues $\lambda$ satisfy $-\lambda^3 - f'(x_0)\lambda = 0$ or $\lambda = \pm\sqrt{-f'(x_0)}$. The eigenvector $(\tilde{x}, \tilde{y}, \tilde{\rho})$ associated with the negative eigenvalue $-\sqrt{-f'(x_0)}$ is determined by

$$\begin{pmatrix} \sqrt{-f'(x_0)} & 1 & 0 \\ -f'(x_0) & \sqrt{-f'(x_0)} & 0 \\ 0 & 0 & \sqrt{-f'(x_0)} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{\rho} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or $\tilde{y} = -\sqrt{-f'(x_0)}\, \tilde{x}$, $\tilde{\rho} = 0$ and, hence, $\tilde{y}/\tilde{x} = -2\sqrt{-f'(x_0)} < 0$. Moreover, since $x \equiv x_0$, $y \equiv 0$, $\rho = r/(r+1)$ satisfies (4.5)–(4.7), we conclude that this system has a $C^1$-local center-stable manifold at $(x_0, 0, 1)$ which we denote by $W^{cs}_{\text{loc}}$ and which is tangent to the vector span of $(0, 0, -1)$ and $(1, -\sqrt{-f'(x_0)}, 0)$ at $(x_0, 0, 1)$. The phase portrait (Fig. 1) of solutions in the section $\rho = 1$ can be determined by examining the system

$$x' = y, \tag{4.8}$$

$$y' = -f(x), \tag{4.9}$$

whose solutions $(x, y)$ satisfy (see Fig. 1)

$$H(x, y) \equiv \frac{y^2}{2} + \int_{x_0}^{x} f(s)\, ds = \text{constant}. \tag{4.10}$$

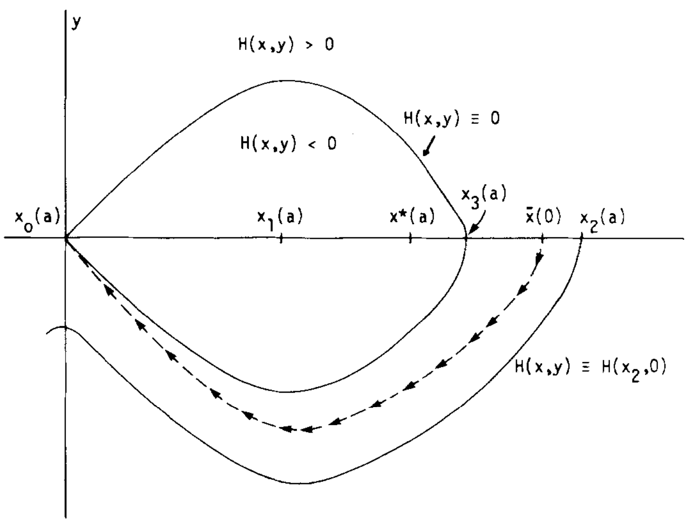


FIG. 1. The solid curves represent $H \equiv$ constant. The dotted curve denotes the spherically symmetric solution $(\bar{x}(r), \bar{y}(r))$ which must remain between the two curves $H(x, y) \equiv 0$ and $H(x, y) \equiv H(x_2, 0)$ for all $r > 0$.

The "fish" $H(x, y) \equiv 0$ is a portion of the stable manifold in the plane $\rho = 1$. Let $T(r)$ be the solution operator of (4.5)–(4.7) for each $r$; that is, $T(r)(\hat{x}, \hat{y}, \hat{\rho}) = (x(r), y(r), \rho(r))$, where $(x, y, \rho)$ is the solution of (4.5)–(4.7) which satisfies $(x(0), y(0), \rho(0)) = (\hat{x}, \hat{y}, \hat{\rho})$. Let $W^{cs} = \bigcup_{r \leqslant 0} T(r)\, W^{cs}_{loc}$ and $W = \{(\tilde{x}, \tilde{y}, \tilde{\rho}) |$ the solution $(x(r), y(r), \rho(r)) \to (x_0, 0, 1)$ as $r \to \infty$ and $(x(0), y(0), \rho(0)) = (\tilde{x}, \tilde{y}, \tilde{\rho})\}$. Then $W^{cs} \subset W$ since $T(r)\, W^{cs}_{loc} \subset W$.

To further discuss the local center-stable manifold at $(x_0, 0, 1)$ we make an appropriate affine change of coordinates to transform Eqs. (4.5)–(4.7) into an equation of the form

$$y' = Dy + g(y), \tag{4.11}$$

where $y = (y_1, y_2, y_3)^T$,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\lambda_1 > 0$, $\lambda_2 < 0$, and $g'(0) = 0$. Let $C_0 = \{(y_1, y_2, y_3) \mid |y_1| \geqslant |(y_2, y_3)|\}$ and for any $y \in R^3$ set $C_y = y + C_0$. Throughout the following lemma we adopt the notation that $\phi(r)$ and $\psi(r)$ are solutions of (4.11) with $\phi(0) = y$ and $\psi(0) = z$. Finally, let $\pi_1(y) = y_1$, where $y = (y_1, y_2, y_3)^T$.

PROPOSITION 1 (Jones [3, p. 27]. *There exists a neighborhood U of* $(0, 0, 0)$ *so that if* $y, z \in U$ *and* $z \in C_y$ *then* $|\pi_1(\psi(r) - \phi(r))|$ *is bounded away from zero as long as* $\psi(r)$ *and* $\phi(r)$ *are in U.*

No confusion should arise if we let $W^{cs}_{loc}$ also denote the $C^1$ local center-stable manifold associated with Eq. (4.11) at $(0, 0, 0)$ whose tangent space is generated by $(0, 1, 0)$ and $(0, 0, 1)$. Let $W^{cs}$ and $W$ denote sets associated with Eq. (4.11) similar to their characterizations for Eqs. (4.5)–(4.7).

COROLLARY. *If* $\hat{\psi}(r)$ *is a solution of* (4.11) *such that* $\hat{\psi}(r) \in U$ *for all* $r \geqslant 0$ *and* $\hat{\psi}(r) \to 0$ *and* $r \to \infty$ *then* $\hat{\psi}(r) \in W^{cs}_{loc}$ *for all* $r \geqslant 0$.

*Proof.* We assume, for the sake of contradiction that the corollary is false. Since $\hat{\psi}(r) \to 0$ as $r \to \infty$ there exist $r_0 > 0$ and $y \in W^{cs}_{loc}$ such that $\psi(r_0) \in C_y$. Let $\psi(r) = \hat{\psi}(r + r_0)$ and $\phi(0) = y$ then Proposition 1 implies that $|\pi_1(\psi(r) - \phi(r))|$ is bounded away from zero, which contradicts $\psi(r) \to 0$ as $r \to \infty$. Thus, $\hat{\psi}(r) \in W^{cs}_{loc}$ for all $r \geqslant 0$.

Also as a result of this corollary we see that $W = W^{cs}$. The existence of a spherically symmetric solution is established by showing that there exists a point $(x, y, \rho) \in W^{cs}$ such that $x > x_0$, $y = 0$, $\rho = 0$.

First, examining the Hamiltonian $H$ of (4.10) along solutions of (4.5)–(4.7), we observe that $\dot{H}(x, y) = -((n-1)(1-\rho)/\rho)\, y^2$ and so if $0 < \rho < 1$ and $y \neq 0$ then $\dot{H} < 0$ and solutions of (4.5)–(4.7) cross the solution curves of (4.8), (4.9) with decreasing energy. If $(x, y, \rho)$ is any solution on $W^{cs}$ with $0 < \rho < 1$ then $\dot{H}(x(r), y(r)) < 0$ except when $y(r) = 0$, and $H(x(r), y(r)) \to H(x(\infty), y(\infty)) = H(x_0, 0) = 0$ as $r \to \infty$, hence, $H(x(r), y(r)) > 0$ for all $0 \leqslant r < \infty$ and $(x(r), y(r))$ remains outside the "fish" of Fig. 1.

The observations made above are crucial to the proofs of Theorems 1 and 2. The remainder of the proof of the existence of a solution of the problem (4.3), (4.4) follows exactly as that given by Jones [3, pp. 13–17] and therefore the details are omitted. The uniqueness of the solution is proved following the proof of Theorem 2.

B. *Proof of Theorem* 2

We seek a bounded (for $r \geqslant 0$) solution $(x(r), y(r))$ of

$$x'' + \frac{(n-1)}{r} x' + F_x(\bar{x}, \bar{y})\, x + F_y(\bar{x}, \bar{y})\, y = \lambda x, \tag{4.12}$$

$$G_x(\bar{x}, \bar{y})\, x + G_y(\bar{x}, \bar{y})\, y = \lambda y, \tag{4.13}$$

for some $\lambda \geqslant 0$ such that $x'(0) = 0$, $y'(0) = 0$. Solving (4.13) for $y$ and substituting into (4.12), we obtain the equations

$$x'' + \frac{(n-1)}{r} x' + \left[ F_x(\bar{x}, \bar{y}) - \frac{F_y(\bar{x}, \bar{y})\, G_x(\bar{x}, \bar{y})}{G_y(\bar{x}, \bar{y}) - \lambda} - \lambda \right] x = 0 \tag{4.14}$$

and

$$y = \frac{-G_x(\bar{x}, \bar{y})\, x}{G_y(\bar{x}, \bar{y}) - \lambda}. \tag{4.15}$$

Thus, it suffices to prove the existence of a bounded solution $x(r)$ of (4.14) such that $x'(0) = 0$. Writing (4.14) as a system we obtain

$$x' = y, \tag{4.16}$$

$$y' = \frac{-(n-1)}{r} y + \left( \lambda - F_x + \frac{F_y G_x}{G_y - \lambda} \right) x. \tag{4.17}$$

Define the function

$$b(r, \lambda) = \left( \lambda - F_x + \frac{F_y G_x}{G_y - \lambda} \right) \Bigg|_{(x, y) = (\bar{x}(r), \bar{y}(r))}$$

The following properties of $b(r, \lambda)$ shall be used in the ensuing analysis:

(P1) $b(\infty, 0) = -f'(x_0) > 0$,

(P2) $(\partial b / \partial \lambda)(r, \lambda) > 0$, $\forall r \geqslant 0$, $\lambda \geqslant 0$,

(P3) $(\partial b / \partial \lambda)(\infty, \lambda) > 0$, $\forall \lambda \geqslant 0$.

Properties (P1) (P2) follow from hypothesis (i) and (iii); (P1)–(P3) imply that $b(\infty, \lambda) > 0$ for all $\lambda > 0$. Since $b(\infty, 0) > 0$ it is easy to see that a bounded solution of Eqs. (4.16), (4.17) goes to the origin as $r \to \infty$. Thus, we consider the stable manifold of (4.16), (4.17) at $r = \infty$ and $(x, y) = (0, 0)$. Its linearized system is

$$x' = y, \tag{4.18}$$

$$y' = \left(\lambda - F_x + \frac{G_x F_y}{G_y - \lambda}\right) x = b(\infty, \lambda)\, x, \tag{4.19}$$

which has eigenvalues $\mu = \pm\sqrt{b(\infty, \lambda)}$. The eigenvector $(\hat{x}, \hat{y})$ associated with the negative eigenvalue $-\sqrt{b(\infty, \lambda)}$ satisfies the matrix equation

$$\begin{pmatrix} \sqrt{b(\infty,\lambda)} & 1 \\ b(\infty,\lambda) & \sqrt{b(\infty,\lambda)} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, equivalently, $\sqrt{b(\infty, \lambda)}\, \hat{x} = -\hat{y}$. We choose $\hat{x} > 0$, $\hat{y} < 0$. Then $\pi/2 < \arctan(\hat{y}/\hat{x}) < 0$ for all $\lambda \geqslant 0$. Let $\theta_\lambda(r) = \arctan(y(r)/x(r))$, where $(x(r), y(r))$ is a solution of (4.16), (4.17). Then $\theta_\lambda$ satisfies the equation

$$\theta' = -\frac{(n-1)}{r}\sin(\theta)\cos(\theta) + \left(\lambda - F_x + \frac{F_y G_x}{(G_y - \lambda)}\right)\cos^2(\theta) - \sin^2(\theta). \tag{4.20}$$

The solution of Eqs. (4.16), (4.17) which we seek must satisfy $(x(\infty), y(\infty)) = (0, 0)$ and $y(0) = 0$. It is not difficult to show that if Eqs. (4.16), (4.17) have a bounded solution then the associated solution $\theta_\lambda(r)$ of Eq. (4.20) must satisfy

$$\theta_\lambda(\infty) = \bar{\theta}_\lambda \equiv \arctan(-\sqrt{b(\infty, \lambda)}). \tag{4.21}$$

Next we need to determine $\theta_\lambda(0)$. From (4.20) we observe that $\theta' = -1$ whenever $\theta = (\pi/2) + k\pi$ ($k$ an integer). Let $\alpha \in (0, \pi/2)$ be fixed. Then Eq. (4.20) implies that there exists $r_0 > 0$ such that if $0 < r < r_0$ and $\theta = \alpha + k\pi$ ($k$ an integer) then $\theta' < 0$, while if $\theta = \alpha + (\pi/2) + k\pi$ then $\theta' > 0$. Therefore, $\lim_{r \to 0^+} \theta_\lambda(r)$ exists and is finite. If $\theta_\lambda(0) \neq m\pi$ for some integer $m$ then a contradiction is easily arrived at by observing, from (4.20), that $\theta_\lambda'$ becomes unbounded as $r \to 0^+$. Therefore,

$$\theta_\lambda(0) = m\pi \qquad (m \text{ an integer}). \tag{4.22}$$

Next, let $\bar{\theta}_\lambda(r)(\mathrm{mod}(\pi))$ denote the unique solution of Eq. (4.20) which satisfies (4.21). We determine those values $\lambda$ and $m$ such that condition (4.22) is satisfied.

First, suppose that $m < 0$. Then $\theta_\lambda(0) \leqslant -\pi$. If $\theta_\lambda(r)$ satisfies (4.21) then $\theta_\lambda(r_\lambda) = -(\pi/2)$ for some first $r_\lambda > 0$. Thus, $\theta'_\lambda(r_\lambda) \geqslant 0$. However, it is clear from Eq. (4.20) that $\theta'_\lambda(r_\lambda) = -1 < 0$, a contradiction. Therefore $m \geqslant 0$.

Second, suppose that $(\lambda_i, m_i)$ $(i = 1, 2)$ satisfy condition (4.22) with $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, and $m_1 < m_2$. We claim that $\lambda_1 > \lambda_2$. If not, and $\lambda_1 < \lambda_2$, then $\bar{\theta}_{\lambda_1} > \bar{\theta}_{\lambda_2}$ and $\bar{\theta}_{\lambda_1}(r) - \bar{\theta}_{\lambda_2}(r) > 0$ for $r \gg 1$. However, $m_1 < m_2 \Rightarrow \bar{\theta}_{\lambda_1}(r) - \bar{\theta}_{\lambda_2}(r) < 0$ for $0 < r \ll 1$. Therefore, there exists $R > 0$ such that $\bar{\theta}_{\lambda_1}(R) - \bar{\theta}_{\lambda_2}(R) = 0$ and $\bar{\theta}'_{\lambda_1}(R) - \bar{\theta}'_{\lambda_2}(R) \geqslant 0$. If $\bar{\theta}_{\lambda_1}(R)$ is not an odd multiple of $\pi/2$ then the monotonicity of $b(r, \lambda)$ implies that $\bar{\theta}'_{\lambda_1}(R) - \bar{\theta}'_{\lambda_2}(R) = (b(R, \lambda_1) - b(R, \lambda_2)) \cos^2(\theta_{\lambda_1}(R)) < 0$, which is a contradiction. If $\bar{\theta}_{\lambda_1}(R)$ is an odd multiple of $\pi/2$ then it follows that $\bar{\theta}'_{\lambda_1}(R) - \bar{\theta}'_{\lambda_2}(R) = \bar{\theta}''_{\lambda_1}(R) - \bar{\theta}''_{\lambda_2}(R) = 0$. However, from Eq. (4.20) it follows that $\bar{\theta}'''_{\lambda_1}(R) - \bar{\theta}'''_{\lambda_2}(R) < 0$, a contradiction. If $\lambda_1 = \lambda_2$ then the uniqueness of solutions satisfying (4.21) implies that $\bar{\theta}_{\lambda_1}(r) \equiv \bar{\theta}_{\lambda_2}(r)$ $\forall r \geqslant 0$ hence $m_1 = m_2$, a contradiction. Thus, we conclude that $m_1 < m_2$ implies $\lambda_2 < \lambda_1$. This in turn implies that the set of $\lambda$ for which (4.22) holds is bounded above.

Following Jones [3], we separate the remainder of the proof of Theorem 2 into three parts, namely,

(i) for a given $m \geqslant 0$ there is at most one $\lambda$ satisfying (4.22),

(ii) there exists $\lambda > 0$ such that $\bar{\theta}_\lambda(0) = 0$ (i.e., $m = 0$),

(iii) $0 < \bar{\theta}_0(0) < \pi$.

From (ii) we see that there exists $\lambda_0 > 0$ for which a solution of (4.20) exists which satisfies $\bar{\theta}_{\lambda_0}(0) = 0$. Since $\lambda_1 < \lambda_0$ implies $m_1 < 0$ then there is no eigenvalue greater than $\lambda_0$. Suppose there is an eigenvalue $\lambda_1 \in (0, \lambda_0)$. A comparison of $\bar{\theta}_{\lambda_1}(r)$ with $\bar{\theta}_0(r)$ rules this out. Therefore, the uniqueness of a positive eigenvalue is assured.

The proofs of (i), (ii), and the first part of (iii), i.e., $\bar{\theta}_0(0) > 0$, are identical to those given by Jones [3, pp. 20–22]. However, the proof that $\bar{\theta}_0(0) < \pi$, and also the uniqueness of solutions, relies heavily on assumption (iv) together with a few of the details of the proof of (ii). Thus, for the sake of simplicity and completeness, we omit the details of the proof of (i) and include the proof of (ii).

To prove part (ii), observe that for $\lambda > 0$ sufficiently large $b(r, \lambda) > 1$ for all $r \geqslant 0$. Also, $-\pi/2 < \bar{\theta}_\lambda < 0 \Rightarrow -\pi/4 < \bar{\theta}_\lambda/2 < 0$ and hence $\tan^2(\bar{\theta}_\lambda/2) < 1$. Thus, if $\theta = \bar{\theta}_\lambda/2$ and $\lambda > 0$ is sufficiently large then

$$\theta' = -\frac{(n-1)}{r} \sin\theta\cos\theta + b(r, \lambda)\cos^2\theta - \sin^2\theta > b(r, \lambda)\cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta[b(r, \lambda) - \tan^2\theta] > 0.$$

Hence, for large $\lambda > 0$, $\bar{\theta}_\lambda(r) < \bar{\theta}_\lambda/2 < 0$ for all $r \geqslant 0$.

Next, consider the $\lambda = 0$ case. Let $\tilde{\theta} = \arctan(\bar{x}''/\bar{x}')$. Then $\tilde{\theta}$ satisfies the equation

$$\tilde{\theta}' = -\frac{(n-1)}{r}\sin\tilde{\theta}\cos\tilde{\theta} + \left(\frac{n-1}{r^2} + b(r,0)\right)\cos^2\tilde{\theta} - \sin^2\tilde{\theta}. \quad (4.23)$$

Since

$$\bar{x}'' = -\frac{(n-1)}{r}\bar{x}' - f(\bar{x}) \quad (4.24)$$

and

$$\bar{x}''' = \frac{(n-1)}{r^2}\bar{x}' - \frac{(n-1)}{r}\bar{x}'' + b(r,0)\,\bar{x}', \quad (4.25)$$

we see from Eq. (4.24) that for $r$ sufficiently large $\bar{x}'' > 0$ and from Eq. (4.25) that $\bar{x}''' < 0$. Hence, $(\bar{x}'(r), \bar{x}''(r)) \to (0,0)$ as $r \to \infty$ and by analyzing the stable manifold of (4.24) at (0, 0) we conclude that $\tilde{\theta}(\infty) = \bar{\theta}_0$. On the other hand, since $\bar{w}' < 0$ for $r > 0$ and $\bar{x}'(0) = 0$ then it follows from Eq. (4.24) that $\bar{x}'' < 0$ for small $r > 0$. Therefore, $\tilde{\theta}(r) \in (0, \pi/2]$ for sufficiently small $r > 0$. It easily follows from Eq. (4.23) that $\tilde{\theta}$ can oscillate finitely often as $r \to 0^+$. Therefore, $\tilde{\theta}(0)$ exists and satisfies $0 \leqslant \tilde{\theta}(0) \leqslant \pi/2$. Since $\tilde{\theta}(\infty) = \bar{\theta}_0(\infty)$, for $r > 0$ sufficiently large $-\pi/2 < \tilde{\theta}(r)$, $\bar{\theta}_0(r) < 0$, and by the mean value theorem, $(\bar{\theta}_0(r) - \tilde{\theta}(r))' = [-((n-1)/r)\cos^2\hat{\theta} + (-1 - b(r,0))\, 2\sin\hat{\theta}\cos\hat{\theta}] \times (\bar{\theta}_0 - \tilde{\theta})(r) - ((n-1)/r^2)\cos^2\tilde{\theta}(r)$, where $-\pi/2 - \hat{\theta}$, $\hat{\theta} < 0$. Hence, $[-((n-1)/r)\cos^2\hat{\theta} + (-1 - b(r,0)\, 2\sin\hat{\theta}\cos\hat{\theta}] > 0$ as long as $\bar{\theta}_0(r) < \tilde{\theta}(r)$, $\bar{\theta}_0'(r) < \tilde{\theta}'(r)$. Thus, we conclude that for $r > 0$ sufficiently large, $\bar{\theta}_0(r) > \tilde{\theta}(r)$. By a comparison argument $\bar{\theta}_0(r) > \tilde{\theta}(r)$ for all $r > 0$ and, hence, $\bar{\theta}_0(0) \geqslant \tilde{\theta}(0) \geqslant 0$. Suppose $\bar{\theta}_0(0) = \tilde{\theta}(0) = 0$. Then, as before, $(\bar{\theta}_0(r) - \tilde{\theta}(r))' < 0$ for $r > 0$ sufficiently small and, hence, we have a contradiction. Thus, $\bar{\theta}_0(0) > 0$.

Thus, it follows from a straightforward shooting argument (Jones [3, p. 23]) that there exists a $\lambda > 0$ such that $\bar{\theta}_\lambda(0) = 0$. This finishes the proof of part (ii) and the proof that $0 < \bar{\theta}_0(0)$ in part (iii).

*Completion of proof of* (iii). In the previous subsection we showed that $\bar{\theta}_0(0) > 0$. Thus, it remains to prove that $\bar{\theta}_0(0) < \pi$. We do this using a sequence of four auxiliary lemmas. Basically, these lemmas show that if $a$ is close to $a^*$ then $\bar{\theta}_0(r)$ cannot exceed the slope of the curve $H(x, y) \equiv 0$.

Let $x_3(a)$ be defined by $\int_{x_0(a)}^{x_3(a)} f(u)\, du = 0$. Then the "fish" of Fig. 1 crosses the $x$ axis at $(x_3(a), 0)$ (see Fig. 1).

LEMMA 1. *Let* $(\bar{x}(r), \bar{y}(r))$ *denote a solution of the problem* (4.3), (4.4). *Then* $\lim_{a \to a^*, a < a^*} H(\bar{x}(r), \bar{y}(r)) = 0$ *uniformly for* $r \in [0, \infty)$.

*Proof.* From assumptions (iii) and (iv) it follows that $x_0(a^*) < x_1(a^*) < x_3(a^*) = x_2(a^*)$. Also, since $H(\bar{x}(r), \bar{y}(r)) > 0$ for all $r \geqslant 0$ then $x_3(a) < \bar{x}(0) < x_2(a)$ for all $a \in (a^* - \delta, a^*)$. These observations lead to the conclusion that $\lim_{a \to a^*, a < a^*} H(\bar{x}(0), \bar{y}(0)) = 0$. Thus, since $dH/dr = -((n-1)/r)(\bar{y}(r))^2$ then the proof of the lemma easily follows.

For our next lemma recall that $x^*(a)$ denotes the unique local maximum value of $f$ between $x_1(a)$ and $x_2(a)$.

LEMMA 2. *For each* $a \in (a^* - \delta, a^*)$ *let* $R_0 = R_0(a) > 0$ *satisfy* $\bar{x}(R_0(a)) = x^*(a)$. *Then* $\lim_{a \to a^*, a < a^*} R_0(a) = +\infty$.

*Proof.* The value $R_0(a)$ is well defined since $\bar{x}(0) > x^*$, $\bar{x}(\infty) = x_0 < x^*$, and $\bar{x}'(r) < 0$, $\forall r > 0$. If the lemma is false then there is an increasing sequence $\{a_i\}_{i \in N}$ with $\lim_{i \to \infty} a_i = a^*$, and a value $M_1 > 0$ such that for each $i \in N$,

$$R_0(a_i) < M_1. \tag{4.26}$$

Define $\sigma(x, a) = -(-2 \int_{x_0}^{x} f(\mu)\, d\mu)^{1/2}$ for $x \in [x_0(a), x_3(a)]$ and $a^* - \delta < a \leqslant a^*$. Then $H(x, \sigma) \equiv 0$, $\forall x \in [x_0(a), x_3(a)]$. Further, $\sigma_x(x, a) < 0$, $\forall x \in (x_0, x_1)$, $\sigma_x(x_1(a), a) = 0$, and $\sigma_x(x, a) > 0$, $\forall x \in (x_1, x_3(a))$. From assumption (v) it follows that

$$x_0(a) < x_1(a) < x^*(a) < x_3(a) \leqslant x_2(a), \tag{4.27}$$

$\forall a \in [a^* - \delta, a^*]$ and $\delta > 0$ sufficiently small. Therefore, since $\sigma(x, a)$ is continuous on the set $[x_0(a), x_3(a)] \times [a^* - \delta, a^*]$ then it follows from the observations made above that for $\delta > 0$ sufficiently small there exists $m < 0$ independent of $a$ such that

$$\sigma(x, a) < m, \qquad \forall (x, a) \in [(x_0(a) + x^*(a))/2, x^*] \times [a^* - \delta, a^*]. \tag{4.28}$$

Since $(\bar{x}(r), \bar{y}(r))$ cannot intersect the curve $H(x, y) \equiv 0$ in the region $x > x_0$, $y < 0$ then it follows from (4.28) that

$$\bar{y}(r) < m, \tag{4.29}$$

for $r \geqslant R_0(a)$ as long as $\bar{x}(r) \geqslant x_1$. Similarly, there exists $M < m$ independent of $a$ such that

$$\bar{y}(r) > M, \tag{4.30}$$

for $r \geqslant R_0(a)$ as long as $\bar{x}(r) \geqslant x_1$. Let $R_1 = R_1(a) > R_0(a)$ denote the unique value of $r$ such that

$$\bar{x}(R_1) = x_1. \tag{4.31}$$

Let $\varepsilon = x^*(a) - x_1(a)$ and integrate (4.30). Then

$$R_1(a_i) - R_0(a_i) \geqslant -\varepsilon/M > 0, \tag{4.32}$$

for all sufficiently large $i$. Since $(d/dr)\, H(\bar{x}(r), \bar{y}(r)) = -((n-1)/r)\, \bar{y}^2(r)$ then (4.29) implies that

$$\frac{dH}{dr}(\bar{x}(r), \bar{y}(r)) \leqslant -\frac{(n-1)\, m^2}{r}, \tag{4.33}$$

for all $r \in [R_0(a_i), R_1(a_i)]$. Integrating (4.33) from $r = R_0$ to $r = R_i^1 \equiv R_0(a_i) - \varepsilon/M$, we obtain

$$\begin{aligned} &H(\bar{x}(R_i^1), \bar{y}(R_i^1)) - H(\bar{x}(R_0(a_i)), \bar{y}(R_0(a_i))) \\ &\qquad \leqslant -(n-1)\, m^2 \ln(1 - \varepsilon/(M_1 M)) < 0, \end{aligned} \tag{4.34}$$

for all $i$. However, since $R_0(a_i) < M_1$ for all $i$ it follows from Lemma 1 that $\lim_{i \to \infty} H(\bar{x}(R_0(a_i)), \ \bar{y}(R_0(a_i))) = 0$. Therefore, $H(\bar{x}(R_i^1), \bar{y}(R_i^1)) < 0$ for $i$ sufficiently large, a contradiction since $(\bar{x}, \bar{y})$ cannot intersect the curve $H(x, y) = 0$ for $r \geqslant 0$.

In the next two technical lemmas our goal is to show that if $a$ is sufficiently close to $a^*$ then $\bar{\theta}_0(r)$ is close to the slope of the curve $H(x, y) \equiv 0$ whenever $x_1(a) \leqslant \bar{x}(r) \leqslant x^*(a)$. Thus, for such values of $r$ it follows that $\bar{\theta}_0(r) < \pi/2$. Appropriate comparison arguments then let us extend this inequality to all nonnegative $r$.

First, we need some notation. Define the function $f_* \equiv f(x)|_{a=a^*}$. Let $(x, y, \rho)$ denote the unique solution of the problem

$$\begin{aligned} x' &= y, \\ y' &= -f_*(x), \\ \rho' &= (1-\rho)^2, \end{aligned} \tag{4.35}$$

where

$$x(0) = x^*(a^*),$$

$$y(0) = y^*(a) \equiv -\left(-2 \int_{x_0(a)}^{x^*(a^*)} f_*(s)\, ds\right)^{1/2}, \qquad \rho(0) = 1. \tag{4.36}$$

Further, for each $a \in (a^* - \delta, a^*)$ we let $(\bar{x}_a, \bar{y}_a)$ denote a solution of (4.3), (4.4). Recall from assumptions (iii) and (iv) that

$$x_1(a) < x^*(a) < x_2(a), \qquad \forall a \in (a^* - \delta, a^*],$$

for small $\delta > 0$. Next, since the solution of (4.35), (4.36) satisfies $x' < 0$, $\forall r > 0$ then there is a unique value $R_1^* > 0$ such that $x(R_1^*) = x_1(a^*)$. Finally, since $\bar{\theta}_0(0) > 0$ and $\bar{x}' < 0$ for all $r > 0$ then it follows from Eq. (4.20) and assumptions (iii), (iv) that there is a unique value $r(a) > 0$ such that $\bar{\theta}_0(R_0(a) + r(a)) = 0$ for each $a \in (a^* - \delta, a^*)$. We now state

LEMMA 3. $\lim_{a \to a^*, a < a^*} r(a) = R_1^*$. *Also,*

$$\lim_{\substack{a \to a^* \\ a < a^*}} (\bar{x}_a(r + R_0(a)) - x(r), \bar{y}_a(r + R_0(a)) - y(r)) = (0, 0)$$

*uniformly for* $r \in [0, R_1^*]$.

*Proof.* Define $\hat{x}_a(r) \equiv \bar{x}_a(r + R_0(a))$ and $\hat{y}_a(r) \equiv \bar{y}_a(r + R_0(a))$, and let $\hat{\rho}_a(r)$ solve the equation $\hat{\rho}_a' = (1 - \hat{\rho}_a)^2$ with $\hat{\rho}_a(0) = R_0(a)/(1 + R_0(a))$. Then $(\hat{x}_a, \hat{y}_a, \hat{\rho}_a)$ solves the initial value problem

$$\begin{aligned} \hat{x}_a' &= \hat{y}_a, \\ \hat{y}_a' &= -p_a(r)\,\hat{y}_a - f(\hat{x}_a), \end{aligned} \tag{4.37}$$

where

$$\hat{x}_a(0) = x^*(a), \qquad \hat{y}_a(0) = \bar{y}_a(R^*(a)), \qquad \hat{\rho}_a(0) = \bar{\rho}(R^*(a)) \tag{4.38}$$

and

$$p_a(r) \equiv \frac{(n-1)}{r + R^*(a)}, \qquad \forall r \geqslant 0. \tag{4.39}$$

Recall that $x_1(a) < \bar{x}_a(0) < x_2(a)$, $\bar{y}_a(0) = 0$, and that the solution $(\bar{x}_a(r), \bar{y}_a(r))$ lies between the curves $H(x, y) \equiv 0$ and $H(x, y) \equiv H(x_2(a), 0)$ for all $r \geqslant R_0(a)$. As shown in Lemma 1 these curves converge to each other as $a \to a^*$. This, and assumption (v) imply that

$$\lim_{\substack{a \to a^* \\ a < a^*}} \bar{y}_a(R_0(a)) = y^*(a^*).$$

From these observations, a comparison of (4.37) with (4.36), and continuity of solutions with respect to initial conditions and parameters, it follows that

$$\lim_{\substack{a \to a^* \\ a < a^*}} (\hat{x}_a(r) - x(r), \hat{y}_a(r) - y(r)) = (0, 0) \tag{4.40}$$

uniformly for $r \in [0, R_1^*]$. The second part of the lemma is now complete. It remains to be shown that $\lim_{a \to a^*, a < a^*} r(a) = R_1^*$. Let $\varepsilon$ be chosen such that $0 < \varepsilon < R_1^*$. Since $y' < 0$ for $r < R_1^*$, and $y' > 0$ for $r > R_1^*$ then

$$y(R_1^* - \varepsilon) > y(R_1^*) \qquad \text{and} \qquad y(R_1^* + \varepsilon) > y(R_1^*). \tag{4.41}$$

Further, since $(\bar{x}_a(r), \bar{y}_a(r))$ lies between the curves $H(x, y) \equiv 0$ and $H(x, y) \equiv H(x_2, 0)$ for all $r \geqslant R_0(a)$ then it follows that

$$\lim_{\substack{a \to a^* \\ a < a^*}} (\bar{x}_a(R_0(a) + r(a)), \bar{y}_a(R_0(a) + r(a))) = (x_1(a^*), y^*), \tag{4.42}$$

where $y^* = -(-\int_{x_0}^{x_1} f_*(\mu)\, d\mu)^{1/2}$. From (4.40)–(4.42) it then follows that

$$\bar{y}_a(R_1^* + \varepsilon + R_0(a)) > \bar{y}_a(R_0(a) + r(a)), \tag{4.43}$$

$$\bar{y}_a(R_1^* - \varepsilon + R_0(a)) > \bar{y}_a(R_0(a) + r(a)), \tag{4.44}$$

for $a^* - a > 0$ sufficiently small. Since $\bar{y}_a$ has only one minimum value then we conclude that

$$R_1^* - \varepsilon + R_0(a) + r(a) < R_1^* + \varepsilon + R_0(0),$$

if $a^* - a > 0$ is small. The lemma now follows.

LEMMA 4. *$\bar{\theta}_0(R_0(a)) < \pi/2$ if $a^* - a > 0$ is sufficiently small.*

*Proof.* Let $\theta_a(r) \equiv \bar{\theta}_0(r + R_0(a))$ for all $r \geqslant 0$. Then $\theta_a(r)$ satisfies

$$\theta_a' = p_a(r)\cos(\theta_a)\sin(\theta_a) - f'(\hat{x}_a(r))\cos^2(\theta_a) - \sin^2(\theta_a), \tag{4.45}$$

$$\theta_a(r(a)) = 0, \tag{4.46}$$

where $p_a(r)$, $r(a)$, and $\hat{x}_a(r)$ are as in the proof of Lemma 3. Next, we let $\phi(r)$ solve the problem

$$\phi' = -f_*'(x(r))\cos^2(\phi) - \sin^2(\phi), \tag{4.47}$$

$$\phi(R_1^*) = 0, \tag{4.48}$$

where $x(r)$, $R_1^*$, and $f_*$ are as defined following the proof of Lemma 2. We note that $\phi(r)$ is the slope of the curve $H(x, y) \equiv 0$ evaluated at the point $(x(r), y(r))$. Therefore,

$$\phi(r) < \frac{\pi}{2}, \qquad \forall r \in (0, R_1^*). \tag{4.49}$$

Thus, from Lemma 3, (4.45)–(4.49), and continuity of solutions with respect to initial conditions and parameters we conclude that $\theta_a(r) - \phi(r) \to 0$ as

$a \to a^*$ uniformly for $r \in [0, R_1^*]$. Thus, the lemma follows for $a$ sufficiently close to $a^*$.

We are now prepared to complete the proof that $\bar{\theta}_0(0) < \pi$. Define

$$\psi_a(r) = \arctan(\bar{x}_a'(r)/(\bar{x}_a(r) - x_1(a))).$$

Then $\psi(r)$ satisfies

$$\psi' = -\frac{(n-1)}{r}\sin(\psi)\cos(\psi) - f'(\bar{x}_a)\cos^2(\psi)$$
$$- \sin^2(\psi) + g(\bar{x}_a), \tag{4.50}$$

where

$$g(x) \equiv (x - x_1)(f'(\bar{x}_a)(\bar{x}_a - x_1) - f(\bar{x}_a)). \tag{4.51}$$

From assumptions (iii), (iv) it follows that

$$g(x) < 0, \qquad \forall x \in [x^*(a), x_2(a)]. \tag{4.52}$$

Next, we set $\hat{\theta} = \bar{\theta}_0(r) - \pi$ and note that $\hat{\theta}(r)$ also satisfies Eq. (4.20). Using (4.47)–(4.49), we may compare $\hat{\theta}(r)$ with $\psi(r)$ and easily show that $\hat{\theta}(r) < \psi(r)$, $\forall r \in [0, R_0(a)]$. Therefore, $\theta_0(r) \leqslant \pi/2$, $\forall r \in [0, R_0(0)]$ and the proof is complete.

*Uniqueness*

We now complete the proof that the problem (4.3)–(4.4) has a unique solution satisfying $x_0 < x(0) < x_2$, and $x' < 0$, $\forall r > 0$. For the sake of notation we replace $\theta_0(r)$ with $\theta(r)$, where $\theta_0(r)$ corresponds to a solution $(x(r), y(r))$ of (4.3), (4.4) and $\theta_0(r)$ solves (4.20) for $\lambda = 0$, with $\theta_0(\infty) = \bar{\theta}_0$.

We first prove six technical lemmas necessary for the completion of the proof of uniqueness.

LEMMA 5. *Let* $(x(r), y(r))$ *solve* (4.3), (4.4) *with* $x(0) \in (x_3, x_2)$, $y(0) = 0$, *and* $y(r) < 0$, $\forall r > 0$. *Then* $\theta(0) = \pi/2$.

*Proof.* From the proof of Theorem 2 it follows that $0 < \theta(0) < \pi$. Suppose, for the sake of contradiction, that $\theta(0) = \eta \in (0, \pi/2)$. Recall that $\theta(r)$ satisfies

$$\theta' = -\frac{(n-1)}{r}\sin(\theta)\cos(\theta) - f'(x(r))\cos^2(\theta) - \sin^2(\theta). \tag{4.53}$$

Since $x(r) \in (x_0, x_2)$, $\forall r \geqslant 0$, and $f'(x)$ is bounded $\forall x \in [x_0, x_2]$ then there exists $\hat{R} > 0$ such that

$$\theta' < -\frac{(n-1)}{2r} \sin(\eta) \cos(\eta), \tag{4.54}$$

$\forall r \in (0, \hat{R})$. Integrating (4.54) from $r/2$ to $r$, where $r \in (0, \hat{R})$ is arbitrarily chosen, we obtain

$$\theta(r) - \theta(r/2) < -\frac{(n-1)}{2} \sin(\eta) \cos(\eta) \ln (2) < 0,$$

$\forall \Omega \in (0, \hat{R})$. However, this leads to a contradiction since $\lim_{r \to 0} \theta(r) = \lim_{r \to 0} \theta(r/2)$. We reach a similar contradiction if we assume that $\theta(0) \in (\pi/2, \pi)$. Similar arguments eliminate the possibility that $\theta$ oscillates as $r \to 0$. This completes the proof of Lemma 5.

LEMMA 6. *Let $(x(r), y(r))$ satisfy the hypotheses of Lemma 5. Then there is a value $\tilde{R} > 0$ such that $\theta(r) > \pi/4$, $\forall r \in [0, \tilde{R}]$.*

*Proof.* From Lemma 5 we conclude that there exists $\tilde{R} > 0$ such that $\theta(\tilde{R}) > \pi/4$, and further,

$$-\frac{(n-1)}{r} - \frac{f'(x)}{2} - \frac{1}{2} < 0, \qquad \forall r \in (0, \tilde{R}), \quad \forall x \in [x_0, x_2]. \tag{4.55}$$

Suppose that there is a positive value $\tilde{r} \in (0, \tilde{R})$ such that $\theta(\tilde{r}) = \pi/4$. Then

$$\theta'(\tilde{r}) \geqslant 0. \tag{4.56}$$

However, from (4.53) and (4.55) we obtain $\theta'(\tilde{r}) < 0$, contradicting (4.56), and completing the proof of the lemma.

LEMMA 7. *Let $(x(r), y(r))$, $(\hat{x}(r), \hat{y}(r))$ satisfy the hypotheses of Lemma 5 with $x(0) < \hat{x}(0)$. Then $\hat{x}' > x'$ for $r > 0$ sufficiently small.*

*Proof.* Define $l(r) = \hat{x}(r) - x(r)$, $\forall r \geqslant 0$. Then $l(r) > 0$ for all small $r \geqslant 0$. Suppose that $l'(r) \leqslant 0$, $\forall r \in (0, \bar{r})$, for some $\bar{r} > 0$. Then there is a positive sequence $\{r_i\}_{i \in N}$ with $\lim_{i \to \infty} r_i = 0$ and such that for each $i$,

$$l''(r_i) \leqslant 0. \tag{4.57}$$

However, since $f'(x) < 0$, $\forall x \in (x_3, x_2)$, and $x_3 < x < \hat{x} < x_2$ for small $r > 0$ then from Eq. (4.1) we obtain, for large $i$,

$$l''(r_i) = -\frac{(n-1)}{r_i} l'(r_i) - f(\hat{x}(r_i)) + f(x(r_i)) > 0, \tag{4.58}$$

contradicting (4.57). Therefore, there exists $\bar{R} > 0$ such that $l'(\bar{R}) > 0$ and

$$x_3 < x(r) < \hat{x}(r) < x_2, \qquad \forall r \in [0, \bar{R}]. \tag{4.59}$$

If there is a first positive $R_1 < \bar{R}$ for which $l'(R_1) = 0$ then $l''(R_1) > 0$ and $l' < 0$ on an interval to the left of $R_1$. However, since $l'$ cannot be negative for all $r \in (0, R_1)$ then there is a first positive $R_2 < R_1$ for which $l'(R_2) = 0$, and therefore,

$$l''(R_2) \leqslant 0. \tag{4.60}$$

Again, since $x_3 < x(R_2) < \hat{x}(R_2) < x_2$ it follows as (4.58) that $l''(R_2) > 0$, contradicting (4.60). Therefore, $l' > 0$, $\forall r \in (0, R_2]$, and the lemma is proved. Recall the definition of $W$ and let $W_r = \{(\tilde{x}, \tilde{y}, \tilde{\rho}) \in W \mid \tilde{\rho} = r/(r+1)\}$.

LEMMA 8. *$W_0 \cap \{(x, y) \mid x_3 < x < x_2, y = 0\}$ is finite.*

*Proof.* First, we suppose that there exists an interval $[a, b] \subset (x_3, x_2)$ such that $a < b$ and $[a, b] \times \{0\} \subseteq W_0$. Let $(x_a(r), y_a(r))$, $(x_b(r), y_b(r))$ denote solutions of Eq. (4.1) with $x_a(0) = a$, $x_b(0) = b$, $y_a(0) = 0$, $y_b(0) = 0$. Let $\theta_a$, $\theta_b$ denote the corresponding solutions of Eq. (4.53). From Lemmas 5 and 7 it follows that there is a value $R > 0$ such that for each $r \in (0, R]$.

$$x_a(r) < x_b(r) \qquad \text{and} \qquad y_a(r) < y_b(r), \tag{4.61}$$

and

$$\theta_a(r) > \pi/4 \qquad \text{and} \qquad \theta_b(r) > \pi/4. \tag{4.62}$$

Let $\Gamma_R$ denote the continuous arc of $W_R$ leading from $(x_a(R), y_a(R))$ to $(x_b(R), y_b(R))$. Then it follows from (4.62) that there is a decreasing sequence $\{R_i\}_{i \in N}$ with $\lim_{i \to \infty} R_i = 0$ and corresponding solutions $(x_i(r), y_i(r))$, $\theta_i(r)$ of Eqs. (4.3) and (4.53), respectively, such that

(i) $(x_i(R_i), y_i(R_i)) \in \Gamma_{R_i}$, $\forall i \in N$,

(ii) $\lim_{i \to \infty} (x_i(R_i), y_i(R_i)) = (\hat{a}, 0) \in [a, b] \times \{0\}$,

(iii) $\theta_i(R_i) > 0$, $\forall i \in N$,

(iv) $\lim_{i \to \infty} \theta_i(R_i) = 0$.

Since $(x_i(R), y_i(R)) \in \Gamma_R$, $\forall i \in N$ then, by considering subsequences if necessary, we may assume since $\Gamma_R$ is bounded that $\lim_{i \to \infty} (x_i(R), y_i(R)) = (x^0, y^0) \in \Gamma_R$. Let $(x(r), y(r))$, and correspondingly, $\theta(r)$, denote the solutions of (4.3) and (4.53), respectively, such that $(x(R), y(R)) = (x^0, y^0)$. From Lemmas 5 and 7 and the uniqueness of solutions, it follows that $\lim_{r \to 0} (x(r),$

$y(r)) \in [a, b] \times \{0\}$, and $\lim_{r \to 0^+} \theta(r) = \pi/2$. Therefore, $\theta(\hat{R}) > \pi/4$ for some small $\hat{R} > 0$. Thus, it follows that $\theta_i(\hat{R}) > \pi/4$ for all large $i$. But then Lemma 6 implies that $\theta_i > \pi/4$, $\forall r \in [0, \hat{R}]$, for all large $i$, contradicting (iv) above. If $W_0 \cap \{(x, y) \mid y = 0, \quad x \in (x_3, x_2)\}$ is infinite then $W_0 \cap \{(x, y) \mid y = 0, \quad x \in (x_3, x_2)\}$ has an accumulhtion point $(\hat{a}, 0) \in [x_3, x_2] \times \{0\}$. This case can be eliminated using the same arguments as above and we omit the details.

LEMMA 9. *Let* $(x(r), y(r))$, $(\hat{x}(r), \hat{y}(r))$ *denote solutions of* (4.3), (4.4). *For each* $R > 0$ *let* $T_R$ *denote the continuous arc of* $W_R$ *leading from* $(x(R), y(R))$ *to* $(\hat{x}(R), \hat{y}(R))$. *If there exists* $R > 0$ *such that* $y(r) < 0$ *and* $\hat{y}(r) < 0$, $\forall r \in (0, R]$ *then* $T_R \subseteq \{(x, y) \mid x > 0$ *and* $y < 0\}$ *and for each solution* $(\tilde{x}(r), \tilde{y}(r))$ *with* $(\tilde{x}(R), \tilde{y}(R)) \in T_R$ *then* $\tilde{y}(r) < 0$, $\forall r \in (0, R]$.

*Proof.* An analysis of the stable manifold close to the steady state solution $(x, x', \rho) \equiv (0, 0, 1)$ that $T_R \subseteq \{(x, y) \mid x > 0, y < 0\}$ for all large $R > 0$. If $T_r$ intersects the region $y > 0$, for some $r > 0$, then there is a value $\hat{r} > 0$ such that $T_r \subseteq \{(x, y) \mid y < 0\}$, $\forall r > \hat{r}$ while $T_{\hat{r}}$ is tangent to the line $y = 0$ at a point $x^0$. If $x^0 < x_0$ then the solution passing through $(x^0, 0)$ at $r = \hat{r}$ must satisfy $x' < 0$ for all $r > \hat{r}$ and therefore $(x^0, 0) \notin W_{\hat{r}}$. Similarly, we may eliminate the possibility that $x^0 > x_2$. Therefore, since $W_{\hat{r}}$ cannot intersect the curve $H(x, y) \equiv 0$ then it follows that $x_3 < x^0 < x_2$. Since $T_{\hat{r}}$ is tangent to the line $y = 0$ at $x = x^0$ then the solutions $(\tilde{x}(r), \tilde{y}(r))$ and $\theta(r)$ of (4.2) and (4.53), respectively, with $(\tilde{x}(\hat{r}), \tilde{y}(\hat{r})) = (x^0, 0)$ must satisfy

$$\theta(\hat{r}) = 0 \tag{4.63}$$

and

$$\tilde{x}' < 0, \qquad \tilde{x}'' < 0, \qquad \forall r \in (\hat{r}, \hat{r} + \varepsilon), \tag{4.46}$$

for some $\varepsilon > 0$. Setting $\tilde{\theta}(r) = \arctan(\tilde{x}''/\tilde{x}')$, we conclude from (4.64) that $\tilde{\theta}(\hat{r}) \geqslant 0$. However, in the discussion following (4.25) we proved that $\tilde{\theta}(\hat{r}) < \theta(\hat{r})$, a contradiction of (4.63).

LEMMA 10. *If* $(x(r), y(r))$ *is a solution of* (4.3) *then* $y(r)$ *cannot have a relative minimum in the set* $x > x_3$, $y < 0$.

*Proof.* In the set $x > x_3$, $y < 0$ it follows from Eq. (4.1) and assumption (iii) that $y'' < 0$ whenever $y' = 0$. Thus, $y$ cannot have a relative minimum.

*Completion of the Proof of Uniqueness*

Suppose, for the sake of contradiction, that the problem (4.3), (4.4) has two solutions $(x(r), y(r))$ and $(\hat{x}(r), \hat{y}(r))$ with $x_3 < x(0) < \hat{x}(0) < x_2$ such that $y < 0$, $\hat{y} < 0$, $\forall r > 0$. Let $\theta(r)$, $\hat{\theta}(r)$ be the corresponding solutions of Eq.

(4.53). Lemma 5 implies that $\theta(r) \to \pi/2$ and $\hat{\theta}(r) \to \pi/2$ as $r \to 0^+$. Recall that $T_R$ denotes the continuous arc of $W_R$ leading from $(x(R), y(R))$ to $(\hat{x}(R), \hat{y}(R))$. Continuity implies that for $R > 0$ sufficiently small there exists a connected arc $C_R \subseteq T_R$, with $(x(R), y(R)) \in \bar{C}_R$, such that if $(x^*(r), y^*(r))$, and $\theta^*(r)$ are solutions of Eqs. (4.3) and (4.53), respectively, with $(x^*(R), y^*(R)) \in C_R$ then

$$\theta^*(R) > \pi/4, \qquad x(R) < x^*(R), \qquad y(R) < y^*(R). \tag{4.65}$$

We wish to show that (4.65) is preserved for all $r \in (0, R)$. Lemma 6 implies that $\theta^*(r) > \pi/4$ for $r < R$ as long as $x \geqslant x^*$ and $y \leqslant y^*$. If there exists $R_1 \in (0, R)$ such that $y(R_1) = y^*(R_1)$ and $x(r) < x^*(r)$, $\forall r \in [R_1, R]$ then $\theta^*(R_1) < 0$, a contradiction. Suppose that there exists $R_2 \in (0, R)$ such that $x(R) = x^*(R)$ and $y(r) < y^*(r)$, $\forall r \in [R_2, R]$. Then $y^*(R_2) > y(R_2)$ and from Eq. (4.3) and assumptions (iii) we conclude that $(y^*)'(R_2) < y'(R_2) < 0$. It then easily follows from (4.3) and assumption (iii) that $y^* > y$, $\forall r \in (0, R_2)$, hence, $x^*(0) < x(0)$ and $y^*(0) = 0$. This contradicts the fact that $W_0$ does not intersect the line $y = 0$ in the interval $x_3 < x < x(0)$.

Since $C_R$ is a continuous connected arc then it follows from Eq. (4.3) and assumptions (ii) and (iii) that $\lim_{r \to 0^+}(x^*(r), y^*(r)) = (x(0), 0)$. Similarly, there exists a connected arc $\hat{C}_R \subseteq T_R$, with $(\hat{x}(R), \hat{y}(R)) \in \hat{C}_R$ such that if a solution $(\bar{x}(r), \bar{y}(r))$ of Eq. (4.3) satisfies $(\bar{x}(R), \bar{y}(R)) \in \hat{C}_R$ then $\lim_{r \to 0^+}(\bar{x}(r), \bar{y}(r)) = (\hat{x}(0), 0)$. Define the set

$$A = \{(q, p) \in T_R \mid \text{the solution } (x^*(r), y^*(r)) \text{ of Eq. (4.3) with } (x^*(R), y^*(R)) = (q, p) \text{ satisfies } \lim_{r \to 0^+} (x(r), y^*(r)) = (x(0), 0)\}.$$

Then our previous discussion shows that $A \neq \varnothing$ and $A \neq T_R$. We need to show that $A$ is relatively open in $T_R$. Let $(x^*, y^*) \in A$ and let $\{(x_i^*, y_i^*)\}_{i \in N} \subseteq T_R$ converge to $(x^*, y^*)$. If $(x_i^*(r), y_i^*(r))$ is the solution with $(x_i^*(R), y_i^*(R)) = (x_i^*, y_i^*)$, and $\theta_i^*(r)$ the associated solution of Eq. (4.53), then for large $i$ it follows from continuity that $\theta_i^*(R) > \pi/4$. It now follows as above that $\lim_{r \to 0^+}(x_i^*(r), y_i^*(r)) = (x(0), 0)$, hence, $A$ is relatively open. Next we show that $A$ is relatively closed. Let $\{(x_i^*, y_i^*)\}_{i \in N} \subseteq A$ approach $(x^*, y^*)$ as $i \to \infty$. Suppose that $(x^*, y^*) \notin A$. Consider the solution $(x^*(r), y^*(r))$ with $(x^*(R), y^*(R)) = (x^*, y^*)$ and corresponding solution $\theta^*(r)$ of (4.53). If $(y^*)' = 0$ for some $\tilde{R} \in (0, R]$ then $(y^*)'' < 0$ and it follows that $y_i^{*\prime}$ has a zero at some $R_i \in (0, R)$ for large $i$. But then (4.3) and assumption (iii) imply that $y_i^{*\prime} > 0$, $\forall r \in (0, R_i)$ and $\lim_{r \to 0^0} y_i^*(r) < 0$, a contradiction. Therefore, $y^{*\prime} < 0$, $\forall r \in (0, R]$ and it follows that $\lim_{r \to 0^+} y^*(r) = 0$, and $x(0) \leqslant x^*(0) \leqslant \hat{x}(0)$. If $x(0) \neq x^*(0)$ then $x^*(0) = \hat{x}(0)$ and there exists $R_0 \in (0, R)$ such that $x^*(R) > x(0)$. But $x_i^*(r) < x(0)$, $\forall i$ and $\forall r \in (0, R]$.

Therefore, $x_i^*(R_0) \to x^*(R)$ as $i \to \infty$, a contradiction. Our conclusion is that $(x^*, y^*) \in A$, hence, $A$ is relatively open ans closed in $T_R$. Since $T_R$ is connected, this is a contradiction. This completes the proof of uniqueness.

## Appendix A

### A.1. *The Fitzhugh–Nagumo Equations*

The Fitzhugh–Nagumo [5] model consists of the system

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial \xi^2} + g(v) - w, \qquad \frac{\partial w}{\partial t} = \varepsilon(v - \gamma w), \tag{5.1}$$

where $\varepsilon > 0$, $\gamma > 0$, $g(v) = v(v-a)(1-v)$, $a \in (0, 1)$. Equations (5.1) were developed as a simplification of the Hodgkin–Huxley nerve conduction equations with $v$ playing the role of transmembrane potential and $w$ representing the recovery variable. Due to its simplicity the equations have served as a prototype of reaction–diffusion mechanisms, in general. A summary of recent results obtained for (5.1) may be found in [4]. If we extend (5.1) to $n$ space dimensions then we obtain

$$\frac{\partial v}{\partial t} = \Delta v + g(v) - w, \qquad \frac{\partial w}{\partial t} = \varepsilon(v - \gamma w), \tag{5.2}$$

where $\Delta v \equiv \sum_{j=1}^{n} \partial^2 v / \partial \xi_j^2$. Assuming that (5.2) has a solution $(v, w) = (v(r), w(r))$, where $r = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$ we obtain the system

$$v'' + \frac{(n-1)}{r} v + g(v) - w = 0, \tag{5.3}$$

$$v - \gamma w = 0. \tag{5.4}$$

We solve (5.4) for $w$ as a function of $v$ and then (5.3) must become

$$v'' + \frac{(n-1)}{r} v + f(v) = 0, \tag{5.5}$$

where $f(v) = v(v-a)(1-v) - (v/\gamma)$. The equation $f(v) = 0$ has three roots, $v_0 = 0$, $v_1 = (a + 1 + ((a+1)^2) - 4(a + 1/\gamma))^{1/2})/2$, and $v_2 = (a + 1 -$

$((a+1)^2 - 4(a+1/\gamma))^{1/2})/2$. The physically reasonable range of values of $a$ is $0 < a < \frac{1}{2}$. It is easy to show that for each $a \in (0, \frac{1}{2})$ there exists $\gamma(a) > 0$ (with $\lim_{a \to 1/2} \gamma(a) = \infty$) such that

$$\int_0^{v_2} \left( v(1-v)(a-v) - \frac{v}{\gamma(a)} \right) dv = 0,$$

while for each $a \in (0, \frac{1}{2})$ and $\gamma \in (0, \gamma(a))$,

$$\int_0^{v_2} v \left( (1-v)(a-v) - \frac{1}{\gamma} \right) dv > 0.$$

Thus, if we let $a^* \in (0, \frac{1}{2})$ then

$$\int_0^{v_2} \left( v(1-v)(v-a) - \frac{v}{\gamma(a^*)} \right) dv > 0, \qquad \forall a \in (0, a^*).$$

It is clear that hypotheses (i)–(v) are satisfied for this system.

A.2. *The Field–Noyes Model*

The Field–Noyes [1] model of the Belousov–Zhabotinskii reaction in a capillary tube consists of the system

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial \xi^2} + S(y - xy + x - qx^2) \equiv \frac{\partial^2 x}{\partial \xi^2} + F(x, y), \tag{5.6}$$

$$\frac{\partial y}{\partial t} = \frac{1}{s}(-y - xy + fz) \equiv G(x, y, z, f), \tag{5.7}$$

$$\frac{\partial z}{\partial t} = \varepsilon(x - z), \tag{5.8}$$

where $0 < \varepsilon \ll 1$, $s = 77.27$, $q = 8.375 \times 10^{-6}$, and $x\alpha[\mathrm{HBrO_2}]$, $y\alpha[\mathrm{Br}]$, $z\alpha[\mathrm{Ce(IV)}]$. Extending (5.6)–(5.8) to $n$ space dimensions, we obtain

$$\frac{\partial x}{\partial t} = \Delta x + F(x, y), \tag{5.9}$$

$$\frac{\partial y}{\partial t} = G(x, y, z, f), \tag{5.10}$$

$$\frac{\partial z}{\partial t} = \varepsilon(x - z), \tag{5.11}$$

where $\Delta x \equiv \sum_{i=1}^n \partial^2 x / \partial \xi_i^2$. There is a unique rest state (of Eqs. (5.9)–(5.11)) which lies in the region $x > 0$, $y > 0$, $z > 0$ and which is given by $x_0 = 1 - f - q + ((1 - f - q)^2 + 4q(1 + f))^{1/2})/2q$, $z_0 = x_0$, $y_0 = f x_0/(1 - x_0)$.

If we set $\varepsilon = 0$ and $z \equiv x_0(f)$, its rest state, then Eqs. (5.9)–(5.11) become

$$\frac{\partial x}{\partial t} = \Delta x + F(x, y), \tag{5.12}$$

$$\frac{\partial y}{\partial t} = G(x, y, x_0, f). \tag{5.13}$$

Spherically symmetric solution of (5.12), (5.13) satisfy

$$x'' + \frac{(n-1)}{r} x' + F(x, y) = 0, \tag{5.14}$$

$$G(x, y, x_0, f) = 0. \tag{5.15}$$

Solving (5.15) for $y$ as a function of $x$ it follows that

$$G(x, y, x_0, f) = 0 \Leftrightarrow y = \frac{f x_0}{1 + x} \equiv k(x). \tag{5.16}$$

Also, from (5.6) it follows that

$$F(x, y) = 0 \Leftrightarrow y = \frac{q x^2 - x}{1 - x} \equiv h(x). \tag{5.17}$$

Thus, substituting $y = k(x)$ into (5.15) we obtain

$$x'' + \frac{(n-1)}{r} x' + l(x) = 0, \tag{5.18}$$

where

$$l(x) \equiv (1 - x)(k(x) - h(x)). \tag{5.19}$$

We define $\alpha = 1$ and $\beta = 1/q$.

Field and Troy [2] have proved that there is an interval $(f_1, f_2) \subset (1, \infty)$ such that if $f \in (f_1, f_2]$ then the equation $l(x) = 0$ has three roots $x_0(f)$, $x_1(f)$, $x_2(f)$ satisfying $l(x_i) = 0$ $(i = 0, 1, 2)$, $l'(x_0) < 0$, $l'(x_2) < 0$, $1 < x_0(f) < x_1(f) < x_2(f) < 1/q$. Further

$$\int_{x_0(f)}^{x_2(f)} l(\mu)\, d\mu > 0, \qquad \forall f \in (f_1, f_2)$$

and

$$\lim_{f \to f_2^-} \int_{x_0(f)}^{x_2(f)} l(\mu)\, d\mu = 0.$$

For the Field–Noyes model the parameter $f$ corresponds to the parameter $a$ in the statement of Theorems 1 and 2. Also $f_2$ plays the role of $a^*$. The first part of assumption (i) is obviously satisfied by $F$ and $G$ as defined in Eqs. (5.6), (5.7) with $(X_1, X_2) = (1, 1/q)$ and $(Y_1, Y_2) = (0, \infty)$. We now discuss the second part of (i). We need to show that $F_y G_x > 0$. From Eqs. (5.6)–(5.7) it follows that

$$F_y G_x = (1 - x)(-y/s) > 0,$$

for $(x, y) \in (1, 1/q) \times (0, \infty)$. We have chosen $(X_1, X_2) \times (Y_1, Y_2) = (1, 1/q) \times (0, \infty)$ since this rectangle is invariant for the systems (5.6)–(5.8) and (5.9)–(5.11). It is easily verified that Eqs. (5.9)–(5.11) also satisfy assumptions (ii)–(v).

## Acknowledgment

The authors thank Chris Jones for his valuable comments.

## References

1. R. J. Field and R. M. Noyes, Limit cycle oscillations in a model of a real chemical reaction, *J. Chem. Phys.* **60** (1974), 1877–1884.
2. R. J. Field and W. C. Troy, The existence of solitary travelling wave solutions of a model of the Belousov–Zhabotinskii reaction, *SIAM J. App. Math.* (1979), in press.
3. C. K. R. T. Jones, Ph.D. thesis, University of Wisconsin, 1979; MRC Technical Report No. 2046, 1980.
4. S. P. Hastings, Single and multiple pulse waves for the Fitzhugh–Nagumo equations, preprint.
5. J. Nagumo, S. Artimoto, and S. Yoshizawa, An active pulse transmission line simulating nerve axon, *Proc. Inst. Radio Engr.* **50** (1962), 2061–2070.
6. M. Protter and H. Weinberger, "Maximum Principles in Differential Equations," Prentice–Hall, Englewood Cliffs, N. J., 1967.