Topology Vol. 9, pp. 153—154. Pergamon Press, 1970. Printed in Great Britain

ON FIBERING CERTAIN FOLIATED MANIFOLDS OVER S^1*

D. TISCHLER

(*Received* 19 *May* 1969)

IT IS shown in Rosenberg [2] that a closed 3-dimensional manifold foliated by 2-dimensional planes must be a torus. In so doing, use is made of Stallings theorem [4] in order to show first that the manifold is fibered over the circle, S^1. In this paper we give a theorem which takes the place of Stallings theorem and allows us to generalize the fibering over S^1 to higher dimensional manifolds. The manifolds and foliations are assumed throughout to be of class C^2.

THEOREM 1. *Let M^n be a closed n-dimensional manifold. Suppose M^n admits a non-vanishing closed 1-form. Then M^n is a fiber bundle over S^1.*

Of course, a non-vanishing closed 1-form defines a foliation. On the other hand, Sacksteder [3] p. 96, shows that an orientable co-dimension one foliation on M^n, which has no holonomy, can be defined by a closed non-vanishing 1-form. Thus we have the following corollary.

COROLLARY 1. *If M^n has an orientable co-dimension one foliation without holonomy, in particular, if all the leaves are simply connected, then M^n is a fiber bundle over S^1.*

COROLLARY 2. *Let M^n be a closed n-dimensional manifold. If M^n has m orientable co-dimension one foliations, each without holonomy, which intersect transversally, then M^n is a fiber bundle over the m-dimensional torus. (By transversal intersection we mean that the 1-forms defining the foliations are independent).*

Proof. The proof of this corollary is a direct application of the method used in proving Theorem 1.

THEOREM 2. *Let M^n be a closed n-dimensional manifold. If the dimension of $H^1(M^n; R)$ is one, and there is an orientable co-dimension one foliation on M^n, without holonomy, then it is isotopic to the foliation induced by the fibering of Corollary 1.*

Proof of Theorem 1. Let ω be the closed 1-form in the hypothesis of Theorem 1. Let x^i, $1 \le i \le k$, be closed 1-forms on M^n which define a basis for the deRham cohomology $H^1(M^n; R)$. Then $\omega = \sum_{i=1}^{k} r_i x^i + dg$, for g a real valued function. Let $S^1 = \{t \in R$ with t identified with $t+1\}$. Let ψ: $\mathrm{Hom}(H_1(X), R) \to H^1(X; R)$ be the deRham isomorphism where $H_1(X)$ is singular homology. Then ψ is natural with respect to differentiable maps from

*This research was supported by NSF-GU-2061.

X to Y. Let $\mu \in \text{Hom}(H_1(S^1), R)$ be defined by $\mu(\theta) = 1$ where θ is a generator of $H_1(S^1)$. Then $\psi(\mu) = [dt]$, where dt is the obvious l-form on S^1. Since S^1 is an Eilenberg–Maclane space there exist functions $f_i: M^n \to S^1$, such that $f_i^*(\mu) = \psi^{-1}([x^i])$. By the naturality of ψ we deduce that there are differentiable functions $f_i: M^n \to S^1$ such that $f_i^*(dt) = x^i + dg_i$, for real valued functions g_i. Thus $\omega = \sum_{i=1}^k r_i f_i^*(dt) + \sum_{i=1}^k r_i dg_i + dg$. We can assume without loss of generality that the last two summands are zero in the preceding equation. This follows since $f_1^*(dt) + dh = (f_1 + \Pi \circ h)^*(dt)$, where $\Pi: R \to S^1$ is the natural projection, h is a real valued function, and the right hand side addition is induced by the group structure on S^1. Then for appropriate choice of rational numbers n^i/d, $1 \leq i \leq k$, we can make $\|\omega - 1/d(\sum_{i=1}^k n^i f_i^*(dt))\|$ arbitrarily small, where the norm comes from a Riemannian metric on M^n. Thus $\sum_{i=1}^k n^i f_i^*(dt)$ is a non-vanishing l-form on M^n. Since the n^i are integers we may define $f: M^n \to S^1$ by the formula $f = \sum_{i=1}^k n^i f_i$. It follows that $f^*(dt)$ is non-vanishing and that f is a submersion. Since M^n is compact f is a fiber map. This proves Theorem 1.

The proof of Theorem 2 requires a lemma.

LEMMA 1. *Let M^n be a closed n-manifold. Let M^n have a foliation defined by a closed form ω. Let p be the projection of the fibering of M^n onto S^1. Suppose the class of ω is a real multiple of the class of $p^*(dt)$ in $H^1(M^n; R)$. Then there is an isotopy of M^n taking the foliation defined by ω into the foliation defined by $p^*(dt)$.*

Proof. We have that $r\omega = p^*(dt) + dg$, where r is a non-zero real number. We may assume $r = 1$ since $r\omega$ defines the same foliation as ω. Put a Riemannian metric on M^n. Let ϕ_u be the l-parameter family of diffeomorphisms associated with the vector field $\mathbf{v}$ perpendicular to the foliation defined by ω. We can lift our foliation to the covering space $F \times R$ of M where F is the fiber of p. Then $\omega = dh$ where $h(x, t) = t + g(\Pi(x, t))$. Consider the isotopy K_s, $0 \leq s \leq 1$, of $F \times R$ given by $K_s(x, t) = \phi_{s \cdot \beta(x, t)}(x, t)$ where $\beta(x, t)$ is the time along $\mathbf{v}$ from (x, t) to the level surface $h^{-1}(h(x_0, t))$ where x_0 is a base point in F. K_s is clearly a diffeomorphism for each fixed s since $\mathbf{v}$ is transversal to the fibers and g is bounded. It is also clear from the construction that K_s covers an isotopy on M^n. Thus the lemma is proved.

Proof of Theorem 2. Since both ω and $p^*(dt)$ are in non-zero cohomology classes we may apply Lemma 1.

Remark. Theorem 1 holds for compact manifolds with boundary if we assume that the closed l-form is non-vanishing when restricted to the boundary.

REFERENCES

1. H. ROSENBERG: Actions of R^n on manifolds, *Comment. math. helv.* **41** (1966), 170–178.
2. H. ROSENBERG: Foliations by planes, *Topology* **7** (1968), 131–138.
3. R. SACKSTEDER: Foliations and pseudogroups, *Am. J. Math.* **87** (1965), 79–102.
4. J. STALLINGS: On fibering certain 3-manifolds, *Topology of 3-Manifolds*, Prentice-Hall (1962).

University of Maryland