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# Tensor envelopes of regular categories

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**Abstract**

We extend the calculus of relations to embed a regular category $\mathcal{A}$ into a family of pseudo-abelian tensor categories $\mathcal{T}(\mathcal{A},\delta)$ depending on a degree function $\delta$. Assume that all objects have only finitely many subobjects. Then our results are as follows:

1. Let $\mathcal{N}$ be the maximal proper tensor ideal of $\mathcal{T}(\mathcal{A},\delta)$. We show that $\mathcal{T}(\mathcal{A},\delta)/\mathcal{N}$ is semisimple provided that $\mathcal{A}$ is exact and Mal'cev. Thereby, we produce many new semisimple, hence abelian, tensor categories.
2. Using lattice theory, we give a simple numerical criterion for the vanishing of $\mathcal{N}$.
3. We determine all degree functions for which $\mathcal{T}(\mathcal{A},\delta)/\mathcal{N}$ is Tannakian. As a result, we are able to interpolate the representation categories of many series of profinite groups such as the symmetric groups $S_n$, the hyperoctahedral groups $S_n \ltimes \mathbb{Z}_2^n$, or the general linear groups $GL(n,\mathbb{F}_q)$ over a fixed finite field.

This paper generalizes work of Deligne, who first constructed the interpolating category for the symmetric groups $S_n$.


*Keywords:* Tensor categories; Semisimple categories; Regular categories; Mal'cev categories; Tannakian categories; Möbius function; Lattices; Profinite groups

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## 1. Introduction

A category $\mathcal{A}$ is called *regular* if it has all finite limits, has images, and where pull-backs preserve images. These are exactly the prerequisites for the *calculus of relations*. Recall that a relation (a.k.a. correspondence) between two objects $x$ and $y$ is a subobject $r$ of $x \times y$. Let $s \rightarrowtail y \times z$ be a second relation. Then the product $s \circ r$ of $r$ and $s$ is, by definition, the image of $r \times_y s$ in $x \times z$. The category of relations $\mathsf{Rel}(\mathcal{A})$ has the same objects as $\mathcal{A}$ but with relations as morphisms and the product of relations as composition.

In some applications, this procedure is too simplistic. For example, it does not conform to common practice in algebraic geometry[1]: let $X$, $Y$ and $Z$ be smooth complex projective varieties. Then the product of two cycles $C \subseteq X \times Y$ and $D \subseteq Y \times Z$ is not just the image $E$ of $C \times_Y D$ in $X \times Z$. It is rather a multiple of it (at least if $C \times_Y D$ is irreducible), the factor being the degree of the surjective morphism $e: C \times_Y D \twoheadrightarrow E$.

Guided by this example, we modify the relational product as follows. Fix a commutative field $K$ and a map $\delta$ which assigns to any surjective morphism $e$ of $\mathcal{A}$ an element $\delta(e)$ of $K$ (its "degree"). We define the product of $r \rightarrowtail x \times y$ and $s \rightarrowtail y \times z$ as

$$sr := \delta(e) s \circ r, \tag{1.1}$$

where $e$ is the surjective morphism $r \times_y s \twoheadrightarrow s \circ r$. Now, we define a new category $\mathcal{T}^0(\mathcal{A},\delta)$ as follows: it has the same objects as $\mathcal{A}$, the morphisms are formal $K$-linear combinations of relations, and the composition is given (on a basis) by (1.1). Of course, the degree function $\delta$ has to satisfy certain requirements for this to work. See Definition 3.1 for details.

The category $\mathcal{T}^0(\mathcal{A},\delta)$ is only of auxiliary nature. Since it is $K$-linear we can enlarge it by formally adjoining direct sums and images of idempotents (the pseudo-abelian closure). The result is our actual object of interest, the category $\mathcal{T}(\mathcal{A},\delta)$.

This category contains in the usual way $\mathcal{A}$ as a subcategory. But it has more structure: the direct product on $\mathcal{A}$ is converted into a tensor functor on $\mathcal{T}(\mathcal{A},\delta)$. It is not difficult to see that this way, $\mathcal{T}(\mathcal{A},\delta)$ is a rigid, symmetric, monoidal category (a tensor category, for short). Loosely

[1] This example is for motivation only. Our construction does not generalize cycle multiplication.

speaking, this means that the tensor product has a unit element, is associative and commutative, and that every object has a dual.

In the rest of the paper we investigate the structure of $\mathcal{T}(\mathcal{A}, \delta)$. Every $K$-linear tensor category has a maximal proper ideal (i.e., a certain class of morphisms) which is compatible with the tensor structure: the tensor radical $\mathcal{N}$. The quotient $\overline{\mathcal{T}}(\mathcal{A}, \delta) = \mathcal{T}(\mathcal{A}, \delta)/\mathcal{N}$ is again a $K$-linear, pseudo-abelian tensor category. Since its tensor radical vanishes, $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ has a chance to be a semisimple tensor category, i.e., one where every object is a direct sum of simple objects. This would entail, in particular, that $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is an abelian tensor category. In our first main theorem (Theorem 6.1), we show that $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is indeed semisimple for a large class of categories. Moreover, we are able to determine all simple objects. This way, we get a large number of new semisimple tensor categories. They are non-standard in the sense that they are not the representation category of a (pro-)reductive group. This was one of the main motivations of this paper.

The precise conditions for semisimplicity are that $\mathcal{A}$ is *subobject finite*, *exact* and *Mal'cev*. Here, "subobject finite" means that every object has only finitely many subobjects. This is required to make all morphism spaces finite dimensional. A regular category is exact if every equivalence relation has a quotient while Mal'cev essentially means that all relations are pullbacks. These last two conditions are quite technical and it is not clear whether they are required.

Nevertheless, the class of subobject finite, exact Mal'cev categories has many interesting examples: the categories of finite groups, finite rings (with or without unit), finite modules over a finite ring or, more generally, any subobject finite abelian category, or any finite algebraic structure containing a group operation. A particular interesting example is the category opposite to the category of finite sets. In that case, the construction of $\mathcal{T}(\mathcal{A}, \delta)$ is due to Deligne [8].

The construction of $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is quite implicit since it involves the (unknown) tensor radical $\mathcal{N}$. Therefore, it is a natural question when in fact $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is equal to $\mathcal{T}(\mathcal{A}, \delta)$, i.e., when $\mathcal{N}$ vanishes. We call degree functions with this property *non-singular*. Our second main result is a precise numerical criterion for non-singularity. The only assumption on $\mathcal{A}$ is subobject finiteness. For a surjective $\mathcal{A}$-morphism $e : x \twoheadrightarrow y$ we define the number

$$\omega_e := \sum_{\substack{w \subseteq x \\ e(w) = y}} \mu(w, x)\delta(w \twoheadrightarrow y) \in K \tag{1.2}$$

where $\mu$ is the Möbius function on the lattice of subobjects of $x$. A surjective morphism $e$ is *indecomposable* if $e$ is not an isomorphism and if any factorization $e = e'e''$ into surjective morphisms implies that one of $e'$ or $e''$ is an isomorphism. The criterion is that $\delta$ *is non-singular if and only if* $\omega_e \neq 0$ *for all indecomposable* $e$.

With this criterion it is very easy to compute the singular degree functions in many cases. For example, the degree functions of the category $\mathcal{A} = \mathsf{Set}^{\mathrm{op}}$ are parametrized by one number $t \in K$. The corresponding degree function is singular precisely when $t \in \mathbb{N}$, recovering a result of Deligne. Similarly, for $\mathcal{A} = \mathsf{Mod}_{\mathbb{F}_q}$, the category of finite $\mathbb{F}_q$-vector spaces, the singular parameters are precisely the powers $q^n$ with $n \in \mathbb{N}$. On the more abstract side, we can show that there always *exists* a non-singular degree function provided that $\mathcal{A}$ is exact and protomodular. The latter condition on $\mathcal{A}$ is stronger than Mal'cev but holds for all the examples mentioned above.

The best known semisimple tensor categories are the representation categories of pro-reductive groups (so-called *Tannakian categories*[2]). Thus it is a natural problem to determine

[2] At least if $K$ is algebraically closed. Assume this from now on.

degree functions $\delta$ for which $\overline{\mathcal{T}}(\mathcal{A},\delta)$ is Tannakian. Our third main result answers this question roughly as follows: assume $\mathcal{A}$ is a subobject finite, regular category and that $K$ is algebraically closed of characteristic zero. Then $\overline{\mathcal{T}}(\mathcal{A},\delta)$ is Tannakian if and only if $\delta$ is adapted to a uniform functor $P:\mathcal{A}\to\mathsf{Set}$. In this case, $\overline{\mathcal{T}}(\mathcal{A},\delta)\cong\operatorname{Rep}(G,K)$ where $G$ is the profinite group of automorphisms of $P$ (see Definitions 9.2 and 9.3 concerning "uniform" and "adapted").

We do not know of a construction of uniform functors in general but, in examples, it is not difficult to come up with many of them. More precisely, for certain categories $\mathcal{A}$ we are able to construct sufficiently many uniform functors $P_i$, $i\in I$, such that the corresponding adapted degree function $\delta_i$ are Zariski-dense in the space of all degree functions. Let $G_i:=\operatorname{Aut}(P_i)$ be the associated group. Since $\operatorname{Rep}(G_i,K)$ is a quotient of $\mathcal{T}(\mathcal{A},\delta_i)$ we say that $\mathcal{T}(\mathcal{A},\delta)$ *interpolates the categories* $\operatorname{Rep}(G_i,K)$, $i\in I$.

Let for example $\mathcal{A}=\mathsf{Set}^{\mathrm{op}}$. As already mentioned, it has a one-parameter family of degree functions $\delta_t$. It turns out that $\overline{\mathcal{T}}(\mathcal{A},\delta_t)\equiv\operatorname{Rep}(S_n,K)$ when $t=n\in\mathbb{N}$ (coincidentally(?) precisely the parameters for which $\delta_t$ is singular). Thus $\mathcal{T}(\mathcal{A},\delta_t)$ interpolates the representation categories of the symmetric groups $S_n$, $n\in\mathbb{N}$ (that was Deligne's starting point). Similarly, we find a category $\mathcal{T}(\mathcal{A},\delta_t)$ which interpolates the representation categories of $GL(n,\mathbb{F}_q)$, $n\in\mathbb{N}$, $q$ fixed. Other examples include the family of wreath products $S_n\wr G$, for $G$ a fixed finite group, or even the infinite wreath product $S_{n_1}\wr S_{n_2}\wr S_{n_3}\ldots$ and many more. We hope that our construction gives rise to a simultaneous treatment of the representations of the $G_i$, in the same way as the representations of the symmetric groups are best studied simultaneously.

The paper concludes with two appendices. In the first one, we give a very brief introduction to protomodular and Mal'cev categories. As already mentioned, we need "Mal'cev" for proving semisimplicity and "protomodular" for the existence of a non-singular degree function. In the second appendix, we use the Mal'cev property to compute degree functions.

We have tried to enhance our theory by including a fair number of examples. In addition to some isolated ones, the paper contains five more extensive blocks of examples. They cover regular categories (Section 2), degree functions (Section 3), singular degree functions (Section 8), interpolation of Tannakian categories (Section 9), and protomodular/Mal'cev categories (Appendix A).

The present work owes its existence to the paper [8] of Deligne where he constructs $\mathcal{T}(\mathcal{A},\delta)$ in the case $\mathcal{A}=\mathsf{Set}^{\mathrm{op}}$. His construction is carried out using different building blocks but the result is the same. Also the backbone of the proofs of our three main results is taken from Deligne's paper. We just added some more flesh to it. The main novelty of the present paper is probably the identification of the Mal'cev condition as being the key for the semisimplicity proof and the numerical non-singularity criterion in terms of Möbius functions.

Finally, it should be mentioned that this paper has a predecessor, [12], where the theory is carried out in the special case of abelian categories. One of my motivations for the present paper was to bring Deligne's case $\mathsf{Set}^{\mathrm{op}}$ and the case of abelian categories under a common roof.

## 2. Regular categories

Regular categories have been introduced by Barr, [3], but the extent limits are supposed to exist in their definition vary from author to author. In this section we make our notion of regularity precise and set up some other terminology.

Let $\mathcal{A}$ be a category. Monomorphisms in $\mathcal{A}$ will henceforth be called *injective* and will be indicated by the arrow "$\rightarrowtail$". Two injective morphisms $f:u\rightarrowtail x$ and $f':u'\rightarrowtail x$ with the same target are *equivalent*, $f\approx f'$, if there exists an isomorphism $g:u\xrightarrow{\sim}v$ with $f=f'g$. A *subob-*

*ject* of $x$ is an equivalence class of injective morphisms. We denote the class of subobjects of $x$ by $\operatorname{sub}(x)$. In most of this paper, we are going to assume that $\operatorname{sub}(x)$ is a set (*well-powered*) or even finite (*subobject finite*) for all $x$. The set $\operatorname{sub}(x)$ has the structure of a poset: in the notation above, we say $f \leqslant f'$ (or just $u \subseteq u'$) if there is a morphism $g$ with $f = f'g$. The morphism $g$ is injective and unique. Hence $f \leqslant f'$ and $f' \leqslant f$ imply $f \approx f'$.

The image, $\operatorname{image}(f)$, of any morphism $f : x \to y$ is the (absolutely) smallest subobject of $y$ through which $f$ factorizes. Clearly, the image may or may not exist. The morphism $f$ will be called *surjective* (or, more traditionally, an *extremal epimorphism*) if $\operatorname{image}(f) = y$. A surjective morphism will be indicated by the arrow "$\twoheadrightarrow$".

**2.1. Definition.** A category $\mathcal{A}$ is *complete and regular* if

**R0** $\mathcal{A}$ is well powered, i.e., $\operatorname{sub}(x)$ is a set for every object $x$.
**R1** $\mathcal{A}$ has all finite limits. In particular, it has a terminal object denoted by $\mathbf{1}$.
**R2** Every morphism has an image.
**R3** The pull-back of a surjective morphism along any morphism is surjective.

**Remarks. 1.** The first axiom, **R0**, is non-standard and is only thrown in for convenience.

**2.** Axiom **R2** (together with **R1**) implies that every morphism can be factorized as $f = me$ where $m$ is injective and $e$ is surjective. This factorization is essentially unique. Moreover, the classes of surjective and injective morphisms are closed under composition and their intersection consists of the isomorphisms.

**3.** Usually, only regular epimorphisms are called surjective. In particular, **R3** is stated only for regular epimorphisms. One can show (see, e.g., [4, §2.2]) that, in the presence of **R1**–**R3**, the concepts "extremal epimorphism", "strong epimorphism", and "regular epimorphism" are all the same.

For any category $\mathcal{A}$ let $\mathcal{A}^\emptyset$ be the category obtained by formally adjoining a (new) absolutely initial object $\emptyset$. More precisely, an object of $\mathcal{A}^\emptyset$ is either an object of $\mathcal{A}$ or equal to $\emptyset$. The morphisms between objects of $\mathcal{A}$ stay the same, for every object $x$ of $\mathcal{A}^\emptyset$ there a unique morphism from $\emptyset$ to $x$ and no morphism from $x$ to $\emptyset$ unless $x = \emptyset$.

**2.2. Definition.** A non-empty category $\mathcal{A}$ is *regular* if $\mathcal{A}^\emptyset$ is complete and regular.

In down to earth terms, this means:

**2.3. Proposition.** *A category $\mathcal{A}$ is regular if it satisfies* **R0**, **R2**, **R3** *above and if* **R1** *is replaced by*:

**R1.1** *$\mathcal{A}$ has a terminal object* $\mathbf{1}$.
**R1.2** *For every commutative diagram*

$$\begin{array}{ccc} y & \longrightarrow & v \\ \downarrow & & \downarrow \\ u & \longrightarrow & x \end{array} \tag{2.1}$$

*the pull-back $u \times_x v$ exists.*
**R1.3** *The pull-back of a surjective morphism by an arbitrary morphism exists.*

**Proof.** First observe that the inclusion of $\mathcal{A}$ in $\mathcal{A}^{\emptyset}$ preserves and reflects limits, injective morphism, and equality of subobjects.The same holds then for images and surjective morphisms. Then one checks easily:

$$\mathbf{R0}^{\emptyset} \quad \Leftrightarrow \quad \mathbf{R0}, \tag{2.2}$$

$$\mathbf{R1}^{\emptyset} \quad \Leftrightarrow \quad \mathbf{R1.1} \text{ and } \mathbf{R1.2}, \tag{2.3}$$

$$\mathbf{R2}^{\emptyset} \quad \Leftrightarrow \quad \mathbf{R2}, \tag{2.4}$$

$$\mathbf{R3}^{\emptyset} \quad \Leftrightarrow \quad \mathbf{R1.3} \text{ and } \mathbf{R3} \tag{2.5}$$

where $\mathbf{R}i^{\emptyset}$ means axiom $\mathbf{R}i$ for $\mathcal{A}^{\emptyset}$. □

**Remarks. 1.** Recall that a *cone* of a diagram $D:\mathcal{D} \to \mathcal{A}$ is an object $x$ together with morphisms $f_d : x \to D(d)$ which satisfy the obvious commutation relations. Call a diagram *bounded* if it has a cone. Then **R1.1** and **R1.2** are equivalent to the following completeness statement: *every bounded finite diagram has a limit.* This implies in particular that every regular category with an initial object is complete.

**2.** Many authors define regular categories to be complete. We opted for our present terminology mainly for two reasons. First, it accommodates some (for me) important examples, namely the category of (non-empty) affine spaces and the category of free actions of a group. Secondly, it has also conceptual advantages. See, e.g., the decomposition Theorem 3.6 below.

In the following we use freely the embedding of $\mathcal{A}$ into $\mathcal{A}^{\emptyset}$ in the way that ∅ stands for all non-existent limits.

**Examples.** Regular categories, even complete ones are abundant. The category of models of any equational theory is complete and regular. This includes the categories of sets, lattices, groups, rings, etc. The category of compact Hausdorff spaces is complete regular as is every abelian category. Also the category *opposite* to the category of sets is regular.

One reason for the abundance is that the concept of regular categories enjoys many permanence properties. The list below is not exhaustive. In the following let $\mathcal{A}$ be a regular category.

**1.** Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$ which contains the terminal object and is closed under products and subobjects. Then $\mathcal{B}$ is regular. This applies in particular to the category of finite models of an equational theory: finite sets, finite lattices, finite groups, etc.

**2.** Let $\mathcal{D}$ be a small category. Then the category of all functors $\mathcal{D}^{\mathrm{op}} \to \mathcal{A}$ is regular (a *diagram category*). Examples are the categories of all arrows $x \to y$ in $\mathcal{A}$ or the category of all objects equipped with a $G$-action where $G$ is a fixed group.

**3.** Fix an object $s$ of $\mathcal{A}$. Then the category $\mathcal{A}/s$ of all "$s$-objects", i.e., all arrows $x \to s$, a so-called *slice category*, is regular. This is one of the main mechanisms to obtain regular categories which are not pointed, i.e., do not possess a zero object.

**4.** For a fixed object $s$ the category $\mathcal{A}/\!\!/s$ of all *surjective* morphisms $x \twoheadrightarrow s$ is regular. A prime example is the category of (non-empty) affine spaces for a fixed field $k$. This category is equivalent to the category of finite dimensional $k$-vector spaces equipped with a *non-zero* linear form (the equivalence is given by taking the dual of the space of affine functions). As opposed to the previous constructions the categories produced by this one are usually not complete even if $\mathcal{A}$ is. In fact, this is one of our main motivations for our notion of regularity.

**5.** Again, fix an object $p$ of $\mathcal{A}$. Then the category $p\backslash\mathcal{A}$ of all arrows $p \to x$ (a *coslice category*) is regular. If $p = \mathbf{1}$ then $\mathcal{B}$ is the category of all pointed objects.

**6.** Fix an object $p$ and consider the full subcategory $p \to \mathcal{A}$ of objects $x$ for which there *exists* a morphism $p \to x$. It is regular if $p$ is projective in $p \to \mathcal{A}$, i.e., if for any surjective morphism $u \twoheadrightarrow v$ such that there is a morphism $p \to u$ every morphism $p \to v$ can be lifted to $p \to u$. Take, e.g., for $\mathcal{A}$ the category *opposite* to the category of $G$-sets ($G$ a fixed group). Let $p = G$ with left regular action. Then $p \to \mathcal{A}$ is the opposite category of the category of $G$-sets with *free* $G$-action. Again, this example is only regular and not complete.

**7.** A combination of slice and coslice category is the category of $s$-points $\mathrm{Pt}_s\,\mathcal{A} = s\backslash(\mathcal{A}/s) = (s\backslash\mathcal{A})/s$. It is the category of all triples $(x, e, d)$ where $e : x \to s$ is a morphism and $d : s \to x$ is a section of $e$. Its main virtue is that it is a pointed category.

In every regular category there is also the notion of a *quotient object of* $x$. It is an equivalence class of surjective morphisms with domain $x$. The *kernel pair* of an quotient object $x \twoheadrightarrow y$ is the double arrow $x \times_y x \rightrightarrows x$. It determines the quotient object uniquely since $y$ is the coequalizer of its kernel pair. In other words, every quotient object is encoded by the subobject $x \times_y x$ of $x \times x$.

The kernel pair is an example of an equivalence relation. In general, an *equivalence relation on* $x$ is a subobject $r$ of $x \times x$ which is reflexive (i.e., contains the diagonal), symmetric (i.e., is invariant under exchanging the two factors of $x \times x$, and transitive (i.e., the morphism $r \times_x r \to x \times x$ factorizes through $r$). In general, not every equivalence relation is the kernel pair $x \times_y x \rightarrowtail x \times x$ of a quotient object. If it is then it is called *effective*. Thus, there is a bijection between quotient objects of $x$ and effective equivalence relations on $x$.

**2.4. Definition.** A category is *exact* if it is regular and if every equivalence relation is effective.

An object $y$ is a *subquotient* of an object $x$ if $y$ is a quotient of a subobject of $x$, i.e., if there is a diagram $x \leftarrowtail u \twoheadrightarrow y$. In that case, we write $y \preccurlyeq x$. If we can find such a diagram such that at least one of the two arrows is not an isomorphism then this is denoted by $y \prec x$. In that case we call $y$ a *proper* subquotient of $x$.

**2.5. Lemma.** *The relations "$\preccurlyeq$" and "$\prec$" are transitive.*

**Proof.** Assume $z \preccurlyeq y \preccurlyeq x$. Then, we get a diagram

$$\begin{array}{ccccc} x'' & \rightarrowtail & x' & \rightarrowtail & x \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\ y' & \rightarrowtail & y & & \\ \downarrow\!\!\!\downarrow & & & & \\ z & & & & \end{array} \tag{2.6}$$

where the square is a pull-back showing $z \preccurlyeq x$. If both $x'' \to x$ and $x'' \to z$ were isomorphisms then all morphisms in diagram (2.6) were isomorphisms showing that "$\prec$" is transitive, as well. $\square$

**2.6. Lemma.** *Let $x$ be an object such that* $\operatorname{sub}(x)$ *satisfies the descending chain condition and* $\operatorname{sub}(x \times x)$ *satisfies the ascending chain condition. Then there is no infinite chain*

$$x \succ x_1 \succ x_2 \succ x_3 \succ \cdots. \tag{2.7}$$

*In particular, we have $x \not\prec x$.*

**Proof.** Since the quotient object $x \twoheadrightarrow y$ is determined by the subobject $x \times_y x \rightarrowtail x \times x$ the ascending chain condition for $\operatorname{sub}(x \times x)$ implies the descending chain condition for quotients of $x$. Let $z \rightarrowtail x$ be any subobject. Since $\operatorname{sub}(z \times z) \subseteq \operatorname{sub}(x \times x)$ we see that every subobject of $x$ satisfies the descending chain condition on quotient objects.

The chain (2.7) gives rise to the diagram

$$\begin{array}{ccccccc}
\cdots & z_{14} & \rightarrowtail & z_{13} & \rightarrowtail & z_{12} & \rightarrowtail & x_1 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & z_{24} & \rightarrowtail & z_{23} & \rightarrowtail & x_2 & & \\
 & \downarrow & & \downarrow & & & & \\
\cdots & z_{34} & \rightarrowtail & x_3 & & & & \\
 & \downarrow & & & & & & \\
\cdots & x_4 & & & & & & \\
 & & & & & & & \\
\iddots & & & & & & &
\end{array} \tag{2.8}$$

where all squares are pull-backs. By the descending chain condition for $\operatorname{sub}(x)$ there is a bound $N > 0$ such that $z_{1n+1} \xrightarrow{\sim} z_{1n}$ for all $n \geqslant N$. Then all morphisms in the second row $z_{2n+1} \rightarrowtail z_{2n}$ are also surjective, hence isomorphisms, for $n \geqslant N$. We conclude that all horizontal arrows $z_{in+1} \rightarrowtail z_{in}$ are isomorphisms for $n \geqslant N$ and $1 \leqslant i \leqslant n$. But then we get an infinite chain of quotients

$$z_{1N} \twoheadrightarrow x_N \twoheadrightarrow x_{N+1} \twoheadrightarrow \cdots$$

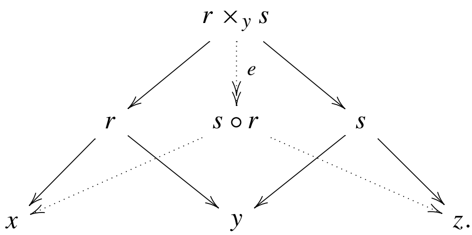
showing that $x_{n+1}$ cannot be a proper subquotient of $x_n$ for $n \gg 0$. □

## 3. The construction of the tensor envelope $\mathcal{T}(\mathcal{A}, \delta)$

First, we recall the classical calculus of relations. Let $\mathcal{A}$ be a complete regular category. A *relation* (a.k.a. *correspondence*) between $x$ and $y$ is by definition a subobject $r$ of $x \times y$. Let $s$ be a relation between $y$ and $z$. Then the *product* of $r$ and $s$ is defined as

$$s \circ r := \operatorname{image}(r \times_y s \to x \times z) \tag{3.1}$$

or, as a diagram




(3.2)

The regularity of $\mathcal{A}$ (or, more precisely, axiom **R3**) ensures that this product is associative. This way, one can define a new category $\mathsf{Rel}(\mathcal{A})$ with the same objects as $\mathcal{A}$ but with relations as morphisms.

The construction of $\mathsf{Rel}(\mathcal{A})$ completely ignores the structure of the surjective morphism $e$ in diagram (3.2). Our main construction can be roughly described as replacing $e$ be a numerical factor, its "degree" or "multiplicity". To carry this out we need to consider linear combinations of relations which actually enlarges the scope of the construction: since there is now a zero morphism not all pull-backs have to exist. They are just set to zero. Here are the precise definitions:

**3.1. Definition.** Let $\mathfrak{E}(\mathcal{A})$ be its class of surjective morphisms of a (just) regular category $\mathcal{A}$ and let $K$ be a commutative ring. Then a map $\delta : \mathfrak{E}(\mathcal{A}) \to K$ is called a *degree function* if

**D1** $\delta(1_x) = 1$ for all $x$.
**D2** $\delta(\bar{e}) = \delta(e)$ whenever $\bar{e}$ is a pull-back of $e$.
**D3** $\delta(e\,\bar{e}) = \delta(e)\delta(\bar{e})$ whenever $e$ can be composed with $\bar{e}$.

**Examples.** The degree functions in the following examples can be determined by simple ad-hoc arguments. Observe however that in Appendix B we have proved some general statements on the computation of degree functions which cover most of the examples below.

**1.** The morphism $\emptyset \to \emptyset$ of $\mathcal{A}^{\emptyset}$ is a pull-back of *every* morphism. Thus, the only degree function on $\mathcal{A}^{\emptyset}$ is the trivial one: $\delta \equiv 1$. Hence, it is *not* possible to reduce to the complete case by simply replacing $\mathcal{A}$ with $\mathcal{A}^{\emptyset}$.

**2.** [Deligne's case] By the same reason, all degree functions on the category of (finite) sets are trivial. On the other hand, if $\mathcal{A}$ is the category *opposite* to the category of finite sets then the surjective morphisms of $\mathcal{A}$ are the injective maps in $\mathsf{Set}$. In that case all degree functions are of the form

$$\delta(e : A \rightarrowtail B) = t^{|B \setminus e(A)|} \tag{3.3}$$

where $t \in K$ is arbitrary.

**3.** Let $\mathcal{A}$ be the category of finite dimensional $k$-vector spaces where $k$ is some field. Then all degree functions are of the form

$$\delta(e : U \twoheadrightarrow V) = t^{\dim \ker e} \tag{3.4}$$

where $t \in K$ is arbitrary.

**4.** More generally, let $\mathcal{A}$ be an abelian category in which every object is of finite length. Let $S$ be the class of simple objects. For an object $x$ let $\ell_s(x)$ be the multiplicity of $s \in S$ in $x$. Then all degree functions are of the form

$$\delta(e : x \twoheadrightarrow y) = \prod_{s \in S} t_s^{\ell_s(\ker e)} \tag{3.5}$$

where the parameters $t_s \in K$ are arbitrary.

Now we define a $K$-linear category as follows:

**3.2. Definition.** Let $\mathcal{A}$ be a regular category, $K$ a commutative ring, and $\delta$ a $K$-valued degree function on $\mathcal{A}$. Then the category $\mathcal{T}^0(\mathcal{A}, \delta)$ is defined as follows:

- The *objects* of $\mathcal{T}^0(\mathcal{A}, \delta)$ are those of $\mathcal{A}$. If an object $x$ of $\mathcal{A}$ is regarded as an object of $\mathcal{T}^0$ then we will denote it by $[x]$.
- The *morphisms* from $[x]$ to $[y]$ are the formal $K$-linear combinations of relations between $x$ and $y$. If $x \times y$ does not exist then $\mathrm{Hom}_{\mathcal{T}^0}([x], [y]) = 0$.
- The *composition* of $\mathcal{T}^0$-morphisms is defined on a basis as follows: let $r \rightarrowtail x \times y$ and $s \rightarrowtail y \times z$ be relations. Then their composition is (in the notation of (3.2))

$$sr := \begin{cases} \delta(e)\, s \circ r & \text{if } r \times_y s \text{ exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

**Remark.** If $\mathcal{A}$ is complete regular and $\delta \equiv 1$ then $\mathcal{T}^0(\mathcal{A}, \delta)$ is just the $K$-linear hull of $\mathsf{Rel}(\mathcal{A})$.

To facilitate further computations, we reformulate and extend the product formula (3.6). First, we adopt the following notation: if $x$ and $y$ are objects of $\mathcal{A}$ and $f : r \to x \times y$ is any $\mathcal{A}^{\emptyset}$-morphisms (i.e., $f$ may not be injective and $r$ may be $\emptyset$) with image $\bar{r}$ then we define the $\mathcal{T}^0$-morphism

$$\langle f \rangle : [x] \to [y]$$

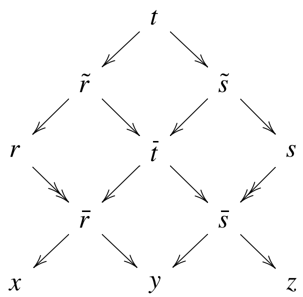
as

$$\langle f \rangle := \begin{cases} \delta(r \twoheadrightarrow \bar{r})\, \bar{r} & \text{if } r \neq \emptyset, \\ 0 & \text{if } r = \emptyset. \end{cases} \tag{3.7}$$

**3.3. Lemma.** *Let $x$, $y$, $z$ be objects of $\mathcal{A}$ and $r \to x \times y$ and $s \to y \times z$ be $\mathcal{A}^{\emptyset}$-morphisms. Then*

$$\langle s \to y \times z \rangle \langle r \to x \times y \rangle = \langle r \times_y s \to x \times z \rangle. \tag{3.8}$$

**Proof.** If one of $r$ or $s$ equals $\emptyset$ then $r \times_y s = \emptyset$, and both sides of (3.8) are zero. So assume $r, s \neq \emptyset$. Let $\bar{r}$ be the image of $r \to x \times y$ and $\bar{s}$ the image of $s \to y \times z$. Then we obtain the following diagram (in $\mathcal{A}^\emptyset$)




(3.9)

where all squares are pull-backs. Axiom **R3** implies that the two morphisms $t \to \tilde{s}$ and $\tilde{s} \to \bar{t}$ are surjective. Thus $t = \emptyset$ implies $\bar{t} = \emptyset$, in which case both sides of (3.8) are zero. So, assume $t \neq \emptyset$. Then we get

$$\begin{aligned} \langle s \to y \times z\rangle\langle r \to x \times y\rangle &= \delta(r \to \bar{r})\,\delta(s \to \bar{s})\langle \bar{t} \to x \times z\rangle \\ &\overset{\mathbf{D2}}{=} \delta(t \to \tilde{s})\delta(\tilde{s} \to \bar{t})\langle \bar{t} \to x \times z\rangle \overset{\mathbf{D3}}{=} \langle t \to x \times z\rangle. \qquad \square \end{aligned} \tag{3.10}$$

Now we can prove:

**3.4. Theorem.** *Let $\mathcal{A}$ be a regular category, $K$ a commutative ring, and $\delta : \mathfrak{E}(\mathcal{A}) \to K$ a degree function. Then $\mathcal{T}^0(\mathcal{A}, \delta)$ is a category.*

**Proof.** Condition **D1** makes sure that the diagonal relation $x \to x \times x$ is an identity morphism in $\mathcal{T}^0$. It remains to show that composition is associative. Let $F$, $G$, and $H$ be the $\mathcal{T}^0$-morphisms corresponding to relations $r \rightarrowtail x \times y$, $s \rightarrowtail y \times z$, and $t \rightarrowtail z \times u$, respectively. Then

$$\begin{aligned} (HG)F &= \langle s \times_z t \to y \times u\rangle \cdot \langle r \to x \times y\rangle \\ &\overset{(3.8)}{=} \big\langle r \times_y (s \times_z t) \to x \times u\big\rangle = \big\langle (r \times_y s) \times_z t \to x \times u\big\rangle \\ &\overset{(3.8)}{=} \langle t \to z \times u\rangle \cdot \langle r \times_y s \to x \times z\rangle = H(GF). \qquad \square \end{aligned} \tag{3.11}$$

The category $\mathcal{T}^0$ is only of auxiliary nature, our main interest being its *pseudo-abelian closure* $\mathcal{T}(\mathcal{A}, \delta)$. Recall that a category is pseudo-abelian (or also Karoubian) if it is additive and every idempotent has an image. We give a brief description of how to construct $\mathcal{T}$. For details, see, e.g., [1, §1].

The pseudo-abelian closure of $\mathcal{T}^0$ is constructed in two steps. First one forms the additive closure $\mathcal{T}'$ of $\mathcal{T}^0$. Its objects are formal direct sums $\bigoplus_{i=1}^n [x_i]$. Morphisms are matrices of $\mathcal{T}^0$-morphisms. Observe that the empty direct sum ($n = 0$) is allowed and provides a zero object.

The category $\mathcal{T}(\mathcal{A}, \delta)$ is now the idempotent closure of $\mathcal{T}'$: the objects of $\mathcal{T}$ are pairs $(X, p)$ where $X$ is an object of $\mathcal{T}'$ and $p \in \operatorname{End}(X)$ is idempotent. The morphism space between $(X, p)$

and $(X', p')$ is $p' \operatorname{Hom}_{\mathcal{T}'}(X, X')p$. This construction shows, in particular, that $\mathcal{T}^0$ is a full subcategory of $\mathcal{T}$.

The category $\mathcal{A}$ is a subcategory of $\mathcal{T}^0$. In fact, for an $\mathcal{A}$-morphism $f : x \to y$ let $[f] \rightarrowtail x \times y$ be its graph. Then one checks easily that $x \mapsto [x]$ and $f \mapsto [f]$ defines an embedding $\mathcal{A} \to \mathcal{T}^0$. Since $\mathcal{T}(\mathcal{A}, \delta)$ has a zero object, this embedding can be extended to a functor $\mathcal{A}^{\emptyset} \to \mathcal{T}(\mathcal{A}, \delta)$ by defining $[\emptyset] = 0$.

The direct product turns $\mathcal{A}^{\emptyset}$ into a symmetric monoidal category. This induces a $K$-linear tensor product on $\mathcal{T}(\mathcal{A}, \delta)$ by defining

$$[x] \otimes [y] := [x \times y]. \tag{3.12}$$

This tensor product is functorial: for relations $\langle r \rightarrowtail x \times x' \rangle$, $\langle s \rightarrowtail y \times y' \rangle$ let

$$\langle r \rangle \otimes \langle s \rangle : [x] \otimes [y] \to [x'] \otimes [y'] \tag{3.13}$$

be the relation

$$r \times s \to (x \times x') \times (y \times y') \xrightarrow{\sim} (x \times y) \times (x' \times y'). \tag{3.14}$$

The unit object is $\mathbb{1} = [\mathbf{1}]$. We claim that the tensor product is rigid, i.e., every object $X$ has a dual $X^\vee$. It suffices to prove this for objects of the form $X = [x]$. But then $X$ is even selfdual with evaluation morphism $\operatorname{ev} : [x] \otimes [x] \to \mathbb{1}$ and coevaluation morphism $\operatorname{ev}^\vee : \mathbb{1} \to [x] \otimes [x]$ represented by

$$x \times x \xleftarrow{\text{diag}} x \rightarrow \mathbf{1}, \qquad \mathbf{1} \leftarrow x \xrightarrow{\text{diag}} x \times x. \tag{3.15}$$

In a rigid monoidal category every morphism $F : X \to Y$ has a *transpose*

$$F^\vee : Y^\vee \to Y^\vee \otimes X \otimes X^\vee \xrightarrow{1 \otimes f \otimes 1} Y^\vee \otimes Y \otimes X^\vee \to X^\vee. \tag{3.16}$$

Concretely, if $F : [x] \to [y]$ is represented by $r \to x \times y$ then $F^\vee : [y] = [y]^\vee \to [x]^\vee = [x]$ is represented by the transposed relation $r \to x \times y \xrightarrow{\sim} y \times x$.

Every tensor category is linear over the endomorphism ring of $\mathbb{1}$. Therefore, the following statement is evident but crucial.

**3.5. Proposition.** $\operatorname{End}_{\mathcal{T}(\mathcal{A},\delta)}(\mathbb{1})$ *is the free $K$-module with basis* $\operatorname{sub}(\mathbf{1})$. *Multiplication is given by intersection* (*with the convention that $u \cdot v = 0$ if $u \cap v$ does not exist*).

For any $u \in \operatorname{sub}(\mathbf{1})$ let $\mathcal{A}_u \subseteq \mathcal{A}$ be the full subcategory whose objects are those $x$ such that $\operatorname{image}(x \to \mathbf{1}) = u$. This is again a regular category with terminal object $u$. The degree function $\delta$ on $\mathcal{A}$ restricts to a degree function $\delta_u$ on $\mathcal{A}_u$.

**3.6. Theorem.** *Assume that* $\operatorname{sub}(\mathbf{1})$ *is finite. Then $\mathcal{T}(\mathcal{A}, \delta)$ is tensor equivalent to the product of the categories $\mathcal{T}(\mathcal{A}_u, \delta_u)$ where $u$ runs through* $\operatorname{sub}(\mathbf{1})$.

**Proof.** Let $A := \mathrm{End}_{\mathcal{T}}(\mathbb{1})$. Then Proposition 3.5 implies that $A$ has a basis of orthogonal idempotents $p_u$ such that

$$p_u v = \begin{cases} p_u & \text{if } u \leqslant v, \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

(see Section 7 below for details). Let $p_u\mathcal{T}$ be the category with the same objects as $\mathcal{T}$ but with

$$\mathrm{Hom}_{p_u\mathcal{T}}(X, Y) = p_u \,\mathrm{Hom}_{\mathcal{T}}(X, Y). \tag{3.18}$$

It is easy to check that this is again a pseudo-abelian. Moreover the functor $X \mapsto (X)_u$, $F \mapsto (p_u F)_u$ is an equivalence of tensor categories $\mathcal{T} \xrightarrow{\sim} \prod_u p_u\mathcal{T}$. It remains to show that $p_u\mathcal{T}$ is equivalent to $\mathcal{T}(\mathcal{A}_u, \delta_u)$. Since $p_u\mathcal{T}$ is the pseudo-abelian closure of $p_u\mathcal{T}^0$ is suffices to prove that $p_u\mathcal{T}^0$ is equivalent to $\mathcal{T}^0(\mathcal{A}_u, \delta_u)$.

We claim that we can define a functor $\Phi : \mathcal{T}^0(\mathcal{A}_u, \delta_u) \to p_u\mathcal{T}^0$ by sending the object $[x]$ to itself and a morphism $F$ to $p_u F$. The only problem is for relations $r$ and $s$ in $\mathcal{A}_u$ such that $s \circ r$ is not in $\mathcal{A}_u$. In that case let $v := \mathrm{image}(s \circ r \to \mathbb{1}) \subset u$. Now, according to formula (3.13), the action of $u \in \mathrm{End}_{\mathcal{T}}(\mathbb{1})$ on a relation $r$ is given by

$$u\langle r\rangle = \langle u \times r\rangle. \tag{3.19}$$

Thus $p_u\langle s \circ r\rangle = p_u\langle v \times s \circ r\rangle = p_u v\langle s \circ r\rangle = 0$ by (3.17) proving the claim.

Now we show that $\Phi$ is a tensor equivalence. Let $x$ be any object of $\mathcal{A}$. We claim that $i : u \times x \rightarrowtail x$ induces an isomorphism $[u \times x] \xrightarrow{\sim} [x]$ in $p_u\mathcal{T}$. Indeed, $i^\vee i = 1_{[u\times x]}$ even in $\mathcal{T}$. Conversely, $p_u 1_{[x]} = p_u\langle x\rangle = p_u u\langle x\rangle = p_u\langle u \times x\rangle = p_u i i^\vee$ which proves the claim.

Put $v := \mathrm{image}(x \to \mathbf{1})$. If $u \subseteq v$ then $u \times x$ is an object of $\mathcal{A}_u$. If $u \not\subseteq v$ then $p_u 1_{[x]} = p_u v 1_{[x]} = 0$, hence $[x] = 0$ in $p_u\mathcal{T}^0$. This shows that every object of $p_u\mathcal{T}^0$ is isomorphic to an object in the image of $\Phi$.

Let now $x$ and $y$ be two objects of $\mathcal{A}_u$. Then

$$\mathrm{Hom}_{\mathcal{T}}\big([x], [y]\big) = \mathrm{Hom}_{\mathcal{T}^0(\mathcal{A}_u, \delta_u)}\big([x], [y]\big) \oplus C \tag{3.20}$$

where $C$ is spanned by all relations $r$ with $\mathrm{image}(r \to \mathbf{1}) \subset u$. This shows that

$$\mathrm{Hom}_{\mathcal{T}^0(\mathcal{A}_u, \delta_u)}\big([x], [y]\big) \xrightarrow{p_u\cdot} \mathrm{Hom}_{p_u\mathcal{T}}\big([x], [y]\big) \tag{3.21}$$

is an isomorphism, completing the proof that $\Phi$ is an equivalence of categories. Finally it is a tensor equivalence since $[x \times^{\mathcal{A}_u} y] \cong [x \times^{\mathcal{A}} y]$ in $p_u\mathcal{T}^0$ for all objects $x, y$ of $\mathcal{A}_u$. □

For our purposes, the preceding theorem allows us to assume without loss of generality that $\mathbf{1}$ has no proper subobject or, equivalently, that the endomorphism ring of $\mathbb{1}$ is $K$. This is one of the main reasons for our definition of regular categories.

## 4. The radical of a tensor category

In this section we review some general facts about tensor categories. Details can be found, e.g., in [1]. Let $\mathcal{T}$ be an arbitrary pseudo-abelian tensor category and denote the commutative ring $\mathrm{End}_{\mathcal{T}}(\mathbb{1})$ by $K$.

Let $\mathcal{I}$ be a map which assigns to any two objects $X$ and $Y$ of $\mathcal{T}$ a subspace $\mathcal{I}(X,Y)$ of $\mathrm{Hom}_{\mathcal{T}}(X,Y)$. Then $\mathcal{I}$ is called a *tensor ideal* if

(a) it is closed under arbitrary left and right multiplication, i.e., for all diagrams $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ holds: if $g \in \mathcal{I}(X,Y)$ then $hgf \in \mathcal{I}(W,Z)$, and
(b) it is closed under tensor products, i.e., for all morphisms $f : X \to Y$ and objects $Z$ holds: if $f \in \mathcal{I}(X,Y)$ then $f \otimes 1_Z \in \mathcal{I}(X \otimes Z, Y \otimes Z)$.

Given a tensor ideal $\mathcal{I}$, it is possible to define a tensor category $\mathcal{T}/\mathcal{I}$. Its objects are the same as those of $\mathcal{T}$ but the morphisms are:

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{I}}(X,Y) := \mathrm{Hom}_{\mathcal{T}}(X,Y)/\mathcal{I}(X,Y). \tag{4.1}$$

In fact, property (a) makes sure that composition of morphisms can be pushed down to $\mathcal{T}/\mathcal{I}$. Property (b) does the same for morphisms between tensor products. The category $\mathcal{T}/\mathcal{I}$ is clearly additive. For pseudo-abelian we need a further condition.

**4.1. Lemma.** *Assume $K$ is an Artinian ring and that all* $\mathrm{Hom}_{\mathcal{T}}$*-spaces are finitely generated $K$-modules. Then $\mathcal{T}/\mathcal{I}$ is also pseudo-abelian.*

**Proof.** Follows from the following well-known fact: let $A \twoheadrightarrow B$ be a surjective homomorphism between Artinian rings. Then every idempotent of $B$ can be lifted to an idempotent of $A$. □

Using the isomorphism

$$\iota_{XY} : \mathrm{Hom}_{\mathcal{T}}\big(\mathbb{1}, X^\vee \otimes Y\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(X,Y). \tag{4.2}$$

we have [1, 6.1.5]

$$\mathcal{I}(X,Y) = \iota_{XY}\big(\mathcal{I}\big(\mathbb{1}, X^\vee \otimes Y\big)\big). \tag{4.3}$$

This implies, in particular, that $\mathcal{I} = 0$ if and only if $\mathcal{I}(\mathbb{1}, Y) = 0$ for all $Y$.

Let $f : X \to X$ be an endomorphism in $\mathcal{T}$. The trace, $\mathrm{tr}\, f$, of $f$ is the composition

$$\mathbb{1} \xrightarrow{\delta} X \otimes X^\vee \xrightarrow{f \otimes 1_{X^\vee}} X \otimes X^\vee \xrightarrow{\sim} X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}. \tag{4.4}$$

The trace is an element of $K = \mathrm{End}_{\mathcal{T}}(\mathbb{1})$. Now we define the *tensor radical* $\mathcal{N}$ of $\mathcal{T}$ as

$$\mathcal{N}(X,Y) := \{f : X \to Y \mid \mathrm{tr}\, fg = 0 \text{ for all } g : Y \to X\}. \tag{4.5}$$

One can show that $\mathcal{N}$ is a tensor ideal [1, 7.1.1]. If $K$ is a field then $\mathcal{N}$ is the maximal proper tensor ideal of $\mathcal{T}$ [1, 7.1.4].

**4.2. Definition.** An object $X$ of $\mathcal{T}$ is called *$\varepsilon$-semisimple* (or *$\varepsilon$-simple*) if $\mathrm{End}_{\mathcal{T}}(X)$ is a semisimple ring (or a division ring).

**4.3. Lemma.** *Let $S$ and $X$ be objects of $\mathcal{T}$. Assume that $S$ is $\varepsilon$-simple.*

(i) *If* $\mathcal{N}(S, X) = 0$ *then every non-zero morphism* $S \to X$ *admits a retraction.*
(ii) *If* $\mathcal{N}(X, S) = 0$ *then every non-zero morphism* $X \to S$ *admits a section.*

**Proof.** We prove (i). The proof for (ii) is analogous. Let $f : S \to X$ be a non-zero morphism. Since $f \notin \mathcal{N}(S, X) = 0$ there is a morphism $g : X \to S$ with $\operatorname{tr}(gf) \neq 0$. This implies that $gf$ is a non-zero, hence invertible endomorphism of $S$. Then $\tilde{g} := (gf)^{-1}g$ is a retraction of $f$. □

This implies the following Schur type lemma:

**4.4. Lemma.** *Let* $S_1$ *and* $S_2$ *be two* $\varepsilon$*-simple objects of* $\mathcal{T}$. *Assume moreover that* $\mathcal{N}(S_1, S_2) = 0$. *Then every morphism* $S_1 \to S_2$ *is either zero or an isomorphism.*

**Proof.** Let $f : S_1 \to S_2$ be non-zero. By Lemma 4.3 there is a morphism $g : S_2 \to S_1$, with $gf = 1_{S_1}$. On the other hand, $fg$ is a non-zero idempotent, hence equal to $1_{S_2}$. □

$\varepsilon$-simple and $\varepsilon$-semisimple objects are related in the following way:

**4.5. Proposition.** *Let* $\mathcal{T}$ *be a pseudo-abelian tensor category with* $\mathcal{N} = 0$. *Let* $X$ *be an object of* $\mathcal{T}$. *Then the following are equivalent*:

(i) $X$ *is* $\varepsilon$*-semisimple.*
(ii) $X$ *is a direct sum of* $\varepsilon$*-simple objects.*

**Proof.** (i) ⇒ (ii). This direction works even without the assumption $\mathcal{N} = 0$. By the structure theory of semisimple rings we have

$$B := \operatorname{End}_{\mathcal{T}}(X) \cong M_{d_1}(K_1) \times \cdots \times M_{d_r}(K_r) \tag{4.6}$$

where the $K_i$ are division rings. The canonical set of minimal orthogonal idempotents of $B$ (their number is $\sum d_i$) splits $X$ as

$$X \cong X_1^{d_1} \oplus \cdots \oplus X_r^{d_r} \tag{4.7}$$

with

$$\operatorname{Hom}_{\mathcal{T}}(X_i, X_j) = \begin{cases} K_i & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.8}$$

The $X_i$ are, in particular, $\varepsilon$-simple.

(ii) ⇒ (i). Assume there is a decomposition (4.7) such that $K_i := \operatorname{End}_{\mathcal{T}}(X_i)$ is a division ring and such that $X_i \ncong X_j$ for $i \neq j$. Lemma 4.4 implies that $\operatorname{Hom}_{\mathcal{T}}(X_i, X_j) = 0$ for $i \neq j$. This implies (4.6). □

**4.6. Lemma.** *Let* $\mathcal{T}$ *be a pseudo-abelian tensor category with* $\mathcal{N} = 0$. *Let* $X_1$ *and* $X_2$ *be two objects. Then* $X_1 \oplus X_2$ *is* $\varepsilon$*-semisimple if and only if both* $X_1$ *and* $X_2$ *are* $\varepsilon$*-semisimple.*

**Proof.** If $X_1$ and $X_2$ are $\varepsilon$-semisimple then $X := X_1 \oplus X_2$ is a direct sum of $\varepsilon$-simple objects. Thus, $X$ is $\varepsilon$-semisimple. Assume conversely that $X$ is $\varepsilon$-semisimple. Then the decomposition $X = X_1 \oplus X_2$ corresponds to orthogonal idempotents $p_1$, $p_2$ of the semisimple ring $B = \mathrm{End}_{\mathcal{T}}(X)$. It is well known that $\mathrm{End}_{\mathcal{T}}(X_i) = p_i B p_i$ is again a semisimple ring. □

Here is our main criterion for semisimplicity:

**4.7. Corollary.** *Let $\mathcal{T}$ be a pseudo-abelian tensor category with $\mathcal{N} = 0$. Let $\mathcal{T}'$ be a full subcategory which generates $\mathcal{T}$ as a pseudo-abelian category. Then $\mathcal{T}$ is semisimple if and only if every object of $\mathcal{T}'$ is $\varepsilon$-semisimple.*

The decompositions (4.6) and (4.7) are related in a more canonical fashion which we recall now in a more general form. Let $B$ be a semisimple ring and let $\{M_\pi \mid \pi \in \widehat{B}\}$ be a set containing each simple $B$-module up to isomorphism exactly once. Then $K_\pi := (\mathrm{End}_B M_\pi)^{\mathrm{op}}$ is a division ring and $M_\pi$ is a $B$–$K_\pi$-bimodule. Moreover, $M_\pi^* := \mathrm{Hom}_{K_\pi}(M_\pi, K_\pi)$ is a $K_\pi$–$B$-bimodule. With this notation, the decomposition (4.6) corresponds to

$$B = \bigoplus_{\pi\in\widehat{B}} \mathrm{End}_{K_\pi} M_\pi = \bigoplus_{\pi\in\widehat{B}} M_\pi \otimes_{K_\pi} M_\pi^*. \tag{4.9}$$

Now assume an object $X$ of $\mathcal{T}$ is endowed with a homomorphism $B \to \mathrm{End}_{\mathcal{T}}(X)$. Then for any $\pi \in \widehat{B}$ put

$$X^\pi := M_\pi^* \otimes_B X. \tag{4.10}$$

Here $V \otimes_B X$ is the object representing the functor $Y \mapsto \mathrm{Hom}_B(V, \mathrm{Hom}_{\mathcal{T}}(X, Y))$ (see, e.g., [8, formula (3.7.1)]). Then $X^\pi$ is a left $K_\pi$-object of $\mathcal{T}$. The decomposition (4.7) becomes

$$X = \bigoplus_{\pi\in\widehat{B}} M_\pi \otimes_{K_\pi} X^\pi. \tag{4.11}$$

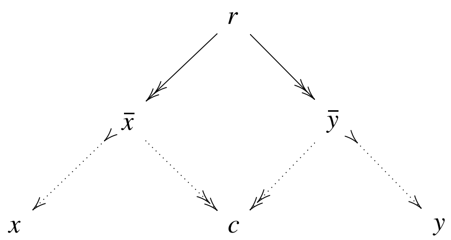
Moreover, if $B = \mathrm{End}_{\mathcal{T}}(X)$ then

$$\mathrm{Hom}_{\mathcal{T}}\big(X^\pi, X^{\pi'}\big) = \begin{cases} K_\pi & \text{if } \pi = \pi', \\ 0 & \text{if } \pi \neq \pi'. \end{cases} \tag{4.12}$$

## 5. The core of a relation

For general regular categories, it is difficult to control all subobjects of a product $x \times y$. Therefore, in this and the next section, we are going to restrict our attention to exact Mal'cev categories (see Definition A1.1) because there all subobjects of a product are basically pullbacks.

More precisely, let $r \rightarrowtail x \times y$ be a relation. To get hold of $r$ we first consider the images $\bar{x}$ and $\bar{y}$ of $r$ in $x$ and $y$, respectively. Then we form the push-out of $r \twoheadrightarrow \bar{x}$ along $r \twoheadrightarrow \bar{y}$ (possible by Proposition A1.2). Thus, we arrive at the following diagram




(5.1)

where the square is a push-out. The point is now, that in an exact Mal'cev category $r$ can be recovered from the dotted part of the diagram. In fact, Proposition A1.2 implies that the square is also a pull-back diagram. Thus, we obtain a bijection between subobjects of $x \times y$ and isomorphisms between subquotients of $x$ and $y$ up to some obvious equivalence. For the category of groups, this observation is due to Goursat [10, p. 47–48].

**5.1. Definition.** Let $r \rightarrowtail x \times y$ be a relation. Then the object $c$ of diagram (5.1) is called the *core* of $r$.

The significance of this definition is summarized in the following lemma.

**5.2. Lemma.** *Let $\mathcal{A}$ be an exact Mal'cev category, let $r \rightarrowtail x \times y$ be a relation, and let $c$ be its core.*

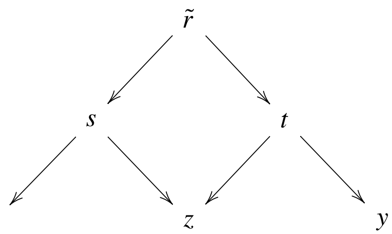
- (i) *The morphism $\langle r\rangle$ factorizes in $\mathcal{T}(\mathcal{A}, \delta)$ through* $[c]$.
- (ii) *Assume that $\lambda\langle r\rangle$, with $\lambda \neq 0$ and $r \neq \emptyset$, factorizes in $\mathcal{T}(\mathcal{A}, \delta)$ through an object* $[z]$. *Then $c \preccurlyeq z$.*

**Proof.** (i) is obvious from diagram (5.1).

(ii) Assume $\lambda\langle r\rangle$ is equal to the composition

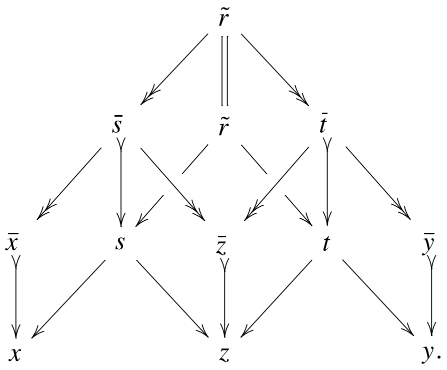
$$[x] \xrightarrow{G} [z] \xrightarrow{H} [y].$$

Then $G$ and $H$ "contain" relations $s \rightarrowtail x \times z$ and $t \rightarrowtail z \times y$ such that $r = t \circ s$. In other words, there is a diagram



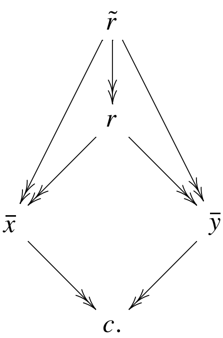

(5.2)

where the square is a pull-back such that $r$ is the image of $\tilde{r}$ in $x \times y$. Let $\bar{z}$ be the image of $\tilde{r}$ in $z$. Then we get the following diagram




(5.3)

Here, the upper square is the pull-back of the lower one by $\bar{z} \rightarrowtail z$ while $\bar{x}$ and $\bar{y}$ are the images of $\bar{s} \to x$ and $\bar{t} \to y$, respectively. The morphisms $\bar{s} \to \bar{z}$ and $\bar{t} \to \bar{z}$ are surjective since $\tilde{r} \to \bar{z}$ is. The upper square is also a pull-back diagram. Hence, the two morphisms from $\tilde{r}$ to $\bar{s}$ and $\bar{t}$ are surjective, as well. This implies that $\bar{x}$ and $\bar{y}$ are the images of $\tilde{r}$ in $x$ and $y$, respectively. By definition of $c$ there is a diagram




(5.4)

Since the upper square of (5.3) is also a push-out (Proposition A1.2), we obtain a morphism $\bar{z} \to c$ which is surjective since $\tilde{r} \to c$ is. This yields the desired subquotient diagram $z \leftarrowtail \bar{z} \twoheadrightarrow c$. □

Here is the linearized version of the preceding theorem:

**5.3. Corollary.** *Let $x$ and $y$ be objects of an exact Mal'cev category $\mathcal{A}$.*

(i) *Every $\mathcal{T}(\mathcal{A}, \delta)$-morphism $[x] \to [y]$ factorizes through an object of the form $[z_1] \oplus \cdots \oplus [z_n]$ with $z_i \preccurlyeq x$ and $z_i \preccurlyeq y$ for all $i$.*
(ii) *Assume $x \not\prec x$* (*see Lemma* 2.6). *Then there is a decomposition*

$$\mathrm{End}_{\mathcal{T}}\big([x]\big) = K\big[\mathrm{Aut}_{\mathcal{A}}(x)\big] \oplus \mathrm{End}_{\mathcal{T}}^{\prec}\big([x]\big). \tag{5.5}$$

*Here,* $\mathrm{End}^{\prec}_{\mathcal{T}}([x])$ *is the two-sided ideal of all endomorphisms which factorize through an object of the form* $[z_1] \oplus \cdots \oplus [z_n]$ *with* $z_i \prec x$ *for all* $i$.

**Proof.** (i) Follows directly from Lemma 5.2(i).

(ii) Assume the core $c$ of a relation $r \rightarrowtail x \times x$ is not a proper subquotient of $x$. Then the dotted arrows of diagram (5.1) (with $y = x$) are all isomorphisms. Thus also the two solid arrows are isomorphism which means that $r$ is the graph of an automorphism of $x$. This and Lemma 5.2(i) imply that

$$\mathrm{End}_{\mathcal{T}}\big([x]\big) = K\big[\mathrm{Aut}_{\mathcal{A}}(x)\big] + \mathrm{End}^{\prec}_{\mathcal{T}}\big([x]\big). \tag{5.6}$$

To show that the sum is direct assume that the linear combination $F = \sum_j \lambda_j [f_j]$ factorizes through $[z_1] \oplus \cdots \oplus [z_n]$ with $z_i \prec x$ and with pairwise different $f_j \in \mathrm{Aut}_{\mathcal{A}}(x)$. Suppose $\lambda_j \neq 0$. Then there are relations $r \rightarrowtail x \times z_i$ and $s \rightarrowtail z_i \times x$ such that the $\mathcal{T}$-composition $\langle s\rangle\langle r\rangle$ is a non-zero multiple of $[f_j]$. Lemma 5.2(ii) implies that $x = \mathrm{core}(f_j) \preccurlyeq z_i$ in contradiction to $z_i \prec x$. □

## 6. The semisimplicity of $\overline{\mathcal{T}}(\mathcal{A}, \delta)$

We return to our pseudo-abelian tensor category $\mathcal{T}(\mathcal{A}, \delta)$ attached to a regular category $\mathcal{A}$ and a $K$-valued degree function $\delta$. In this section we address the problem whether

$$\overline{\mathcal{T}}(\mathcal{A}, \delta) := \mathcal{T}(\mathcal{A}, \delta)/\mathcal{N} \tag{6.1}$$

is a semisimple, hence abelian, tensor category. Except for very degenerate cases (e.g., $\delta = 1$), semisimplicity cannot be expected unless all Hom-spaces are finite dimensional over $K$. Therefore, we are going to assume that $\mathcal{A}$ is *subobject finite*, i.e., that every object has only finitely many subobjects.

But even then there is a problem: Deligne [8, Mise en garde 5.8] has constructed a pseudo-abelian tensor category over $\mathbb{C}$ with finite dimensional Hom-spaces and $\mathcal{N} = 0$ which is not semisimple.

To state our main criterion, let $\widehat{\mathcal{T}}$ be the class of isomorphism classes of pairs $(x, \pi)$ where $x$ is an object of $\mathcal{A}$ and $\pi$ is an irreducible $K$-representation of $\mathrm{Aut}_{\mathcal{A}}(x)$.

**6.1. Theorem.** *Let* $\mathcal{A}$ *be a subobject finite, exact Mal'cev category and let* $\delta$ *be a* $K$*-valued degree function on* $\mathcal{A}$ *where* $K$ *is a field of characteristic zero. Then*:

- (i) $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ *is a semisimple* (*hence abelian*) *tensor category.*
- (ii) *For every* $(x, \pi) \in \widehat{\mathcal{T}}$ *there is, up to isomorphism, at most one simple object* $M$ *of* $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ *with*
  - *–* $M$ *occurs in the isotypic component* $[x]^{\pi}$ (*notation of* (4.10)).
  - *–* $M$ *does not occur in* $[y]$ *for any* $y \prec x$.
- (iii) *Let* $\widehat{\mathcal{T}}_\delta$ *be the set of* $(x, \pi) \in \widehat{\mathcal{T}}$ *such that* $M$ *as in* (ii) *exists and denote, in that case,* $M$ *by* $M^{(x,\pi)}$. *Then* $(x, \pi) \mapsto M^{(x,\pi)}$ *is a bijection between* $\widehat{\mathcal{T}}_\delta$ *and isomorphism classes of simple objects.*
- (iv) *If* $\mathcal{N} = 0$ *then* $\widehat{\mathcal{T}}_\delta = \widehat{\mathcal{T}}$.

**Proof.** In view of Theorem 3.6 we may assume that **1** has no proper subobject. This means, in particular, that $K = \mathrm{End}_{\mathcal{T}}(\mathbb{1})$.

By Corollary 4.7, we have to show that every object of the form $[x]$ is $\varepsilon$-semisimple, i.e., has a semisimple endomorphism ring. By Lemma 2.6 it suffices to prove the following statement: *let $x$ be an object of $\mathcal{A}$ such that $[y]$ is $\varepsilon$-semisimple for all $y \prec x$. Then $x$ is $\varepsilon$-semisimple.*

Let $\mathfrak{S}$ be the (finite) set of all $\varepsilon$-simple summands occurring in some $[y]$ with $y \prec x$. If we apply Lemma 4.3 successively to $[x]$ and all elements of $\mathfrak{S}$, we obtain a decomposition

$$[x] = [x]_0 \oplus [x]_1 \tag{6.2}$$

such that $[x]_1$ is a direct sum of elements of $\mathfrak{S}$ and

$$\mathrm{Hom}_{\mathcal{T}}\big([x]_0, [y]\big) = \mathrm{Hom}_{\mathcal{T}}\big([y], [x]_0\big) = 0 \tag{6.3}$$

for all $y \prec x$. The decomposition (5.3) implies

$$\mathrm{End}_{\overline{\mathcal{T}}}\big([x]\big) = K\big[\mathrm{Aut}_{\mathcal{A}}(x)\big] + \mathrm{End}_{\overline{\mathcal{T}}}^{\preceq}\big([x]\big) \tag{6.4}$$

but the sum may no longer be direct. It is clear that $\mathrm{End}^{\prec}$ kills $[x]_0$. Thus, we obtain a surjective homomorphism

$$K\big[\mathrm{Aut}_{\mathcal{A}}(x)\big] \to B := \mathrm{End}_{\mathcal{T}}\big([x]_0\big). \tag{6.5}$$

Since $K$ is of characteristic zero and $\mathrm{Aut}_{\mathcal{A}}(x)$ is a finite group we conclude that $B$ is a semisimple ring. Thus $[x]_0$ and therefore $[x]$ is $\varepsilon$-semisimple (Lemma 4.6), showing (i).

Let

$$[x]_0 = \bigoplus_{\pi \in \widehat{B}} M_\pi \otimes M^{(x,\pi)} \tag{6.6}$$

be the $B$-isotypic decomposition (see (4.11)). Then $M^{(x,\pi)}$ is $\varepsilon$-simple (see (4.12)), hence simple. Since $K[\mathrm{Aut}_{\mathcal{A}}(x)] \to B$ is surjective, we can think of $(x, \pi)$ as being an element of $\widehat{\mathcal{T}}$.

If $M$ is any simple object as in (ii) then $M$ cannot appear in $[x]_1$. Thus $M \cong M^{(x,\pi)}$ proving (ii).

The decomposition (6.2) shows, by induction, that $[x]$ is a direct sum of objects $M^{(y,\pi)}$ with $y \preccurlyeq x$. In particular, every simple object of $\mathcal{T}$ is of the form $M^{(x,\pi)}$. Now assume that there is an isomorphism $f : M^{(x,\pi)} \xrightarrow{\sim} M^{(x',\pi')}$. This isomorphism extends to a morphism $[x] \to [x']$. Corollary 5.3(i) implies that $M^{(x,\pi)}$ occurs already in an object $[y]$ where $y$ is a subquotient of both $x$ and $x'$. By definition of $M^{(x,\pi)}$ and $M^{(x',\pi')}$, this subquotient cannot be proper. Thus we obtain $x \xrightarrow{\sim} x'$. We conclude $\pi = \pi'$ (see (4.12)), showing (iii).

Finally (iv) follows from the fact that (6.5) is an isomorphism if $\mathcal{N} = 0$. □

**Example** *(Deligne's case).* Let $\mathcal{A} = \mathsf{Set}^{\mathrm{op}}$, the opposite category of the category of finite sets. Then $\widehat{\mathcal{T}}$ is the union over $n \geqslant 0$ of $\widehat{S}_n$. Therefore, $\widehat{\mathcal{T}}$ is parametrized by Young diagrams of arbitrary size.

**Remark.** I see no inherent reason why $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ should only be semisimple for exact Mal'cev categories. It is just the proof which requires that condition. On the other hand, the description of simple objects is probably only valid in the exact Mal'cev case.

## 7. The Möbius algebra of a semilattice

In preparation for the next section, we review and refine in this section some results from (semi)lattice theory. Recall that a *semilattice* is a set $L$ equipped with an associative and commutative product $\wedge$ which is idempotent, i.e., with $u \wedge u = u$ for all $u \in L$. Any semilattice is partially ordered by

$$u \leqslant v \quad \Leftrightarrow \quad u \wedge v = u. \tag{7.1}$$

Conversely, one can recover the product from the partial order because $u \wedge v$ is the largest lower bound of $\{u, v\}$.

If $L$ is any partially ordered set let $L^{\emptyset}$ be $L$ with a new minimum $\emptyset$, i.e., $L^{\emptyset} := L \,\dot{\cup}\, \{\emptyset\}$ with $\emptyset < u$ for all $u \in L$. We call $L$ a *partial semilattice* if $L^{\emptyset}$ is a semilattice. In analogy to (and, in fact, a special case of) Proposition 2.3, a poset $L$ is a partial semilattice if and only if any two element set $\{u, v\} \subseteq L$ either has no lower bound at all or has an infimum.

**Examples. 1.** Let $x$ be an object of a category $\mathcal{A}$. If $\mathcal{A}$ is complete regular then $\operatorname{sub}(x)$, the partially ordered set of subobjects of $x$, is a semilattice. If $\mathcal{A}$ is just regular then $\operatorname{sub}(x)$ is a partial semilattice. That is why we are interested in them.

**2.** If $L$ is a partial semilattice then every upper subset $U$ (i.e., one with $u \in U, u \leqslant v \Rightarrow v \in U$) is a partial semilattice.

For a finite partial semilattice $L$ let $P$ be the $L \times L$-matrix with $P_{uv} = 1$ if $u \leqslant v$ and $P_{uv} = 0$ otherwise. This matrix is unitriangular and therefore has an inverse $M$. The entries $\mu(u, v) := M_{uv}$ are the values of the *Möbius function* $\mu$ of $L$. It is $\mathbb{Z}$-valued with $\mu(u, v) = 0$ unless $u \leqslant v$.

The Möbius function has a natural interpretation in terms of the *Möbius algebra* $A(L)$. If $L$ is a semilattice then $A(L)$ is the free abelian group with basis $L$ and multiplication induced by $\wedge$. For a partial semilattice, we put $A(L) = A(L^{\emptyset})/\mathbb{Z}\emptyset$. Thus $A(L)$ is the free abelian group over $L$ and multiplication is induced by $\wedge$ with the proviso that $u \wedge v = 0$ if $u \wedge v$ does not exist in $L$.

Define the elements

$$p_v = \sum_{u \leqslant v} \mu(u, v) u \in A(L). \tag{7.2}$$

Then one can show (see, e.g., [11]) that the $p_v$ form a basis of $A(L)$ with

$$p_u \wedge p_v = \delta_{u,v} p_v \tag{7.3}$$

and

$$p_u \wedge v = \begin{cases} p_u & \text{if } u \leqslant v, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

Now we generalize a formula of Lindström [15] and Wilf [20] to partial semilattices.

**7.1. Lemma.** *Let $\varphi : L \to K$ be a function on a finite partial semilattice $L$. Then*

$$\det\big(\varphi(u \wedge v)\big)_{u,v\in L} = \prod_{w\in L} \varphi(p_w), \tag{7.5}$$

*where, on the left-hand side $\varphi(\emptyset) := 0$ while, on the right-hand side, $\varphi$ is extended linearly to a map $A(L) \to K$.*

**Proof.** The map $u \mapsto p_u$ is a unitriangular base change of $A(L)$. Thus

$$\det\big(\varphi(u \wedge v)\big)_{u,v\in L} = \det\big(\varphi(p_u \wedge p_v)\big)_{u,v\in L} = \prod_{w\in L} \varphi(p_w). \qquad \square \tag{7.6}$$

Next, we need a generalization of a formula of Greene [11, Thm. 5] which compares the minimal idempotents $p_u$ of two Möbius algebras. For that we write $p_u^L$ and $\mu_L(u, v)$ to make the dependence on $L$ explicit. Recall that a pair of maps $e^* : M \to L$, $e_* : L \to M$ between posets is a *Galois connection* if

$$l \leqslant e^*(m) \quad \Leftrightarrow \quad e_*(l) \leqslant m \tag{7.7}$$

for all $l \in L$ and $m \in M$. In that case, it is known (see, e.g., [9, Prop. 3]) that both maps are order preserving and that $e^*$ preserves infima. In particular, if $L$ and $M$ are partial semilattices then $e^*$ is multiplicative (with $e^*(\emptyset) := \emptyset$).

**7.2. Lemma.** *Let $e^* : M \to L$ and $e_* : L \to M$ be a Galois connection between finite partial semilattices. For any $l \in L$ put $m := e_*(l) \in M$ and*

$$p^L_{l\to m} := \sum_{\substack{l' \leqslant l \\ e_*(l')=m}} \mu_L(l', l)\, l'. \tag{7.8}$$

*Then*

$$p_l^L = e^*\big(p_m^M\big) \wedge p^L_{l\to m}. \tag{7.9}$$

**Proof.** Let $x := e^*(p_m^M) \wedge p_l^L$. Then $x$ is a linear combination of elements of the form $\tilde{l} := e^*(m') \wedge l'$ with $m' \leqslant m$ and $l' \leqslant l$. Clearly $\tilde{l} \leqslant l$ and $\tilde{l} = l$ if $m' = m$ and $l' = l$. Conversely, suppose $\tilde{l} = l$. Then $l' \geqslant l$ and therefore $l' = l$. Moreover $e^*(m') \geqslant l$ implies $m' \geqslant e_*(l) = m$, hence $m' = m$. This shows that $x = l +$ lower order terms. On the other hand, $x \in A(L)p_l^L = \mathbb{Z}p_l^L$ and therefore $x = p_l^L$. Thus we have

$$p_l^L = e^*\big(p_m^M\big) \wedge \sum_{l' \leqslant l} \mu_L(l', l)\, l'. \tag{7.10}$$

We are done if we show that $l' \leqslant l$ and $e^*(p_m^M) \wedge l' \neq 0$ implies $e_*(l') = m$. Put $m' := e_*(l')$. Then $e_*(l') \leqslant m'$ implies $l' \leqslant e^*(m')$, hence $l' = l' \wedge e^*(m')$. Thus also $e^*(p_m^M) \wedge e^*(m') \neq 0$. Hence $p_m^M \wedge m' \neq 0$ which implies $m \leqslant m'$. On the other hand, $l' \leqslant l$ implies $m' \leqslant m$ and therefore $m' = m$, as claimed. $\square$

## 8. The tensor radical of $\mathcal{T}(\mathcal{A}, \delta)$

As opposed to $\overline{\mathcal{T}}(\mathcal{A}, \delta)$, the categories $\mathcal{T}(\mathcal{A}, \delta)$ form a nice family in dependence of $\delta$. In fact, assume for the moment that $\mathcal{A}$ is essentially small (i.e., equivalent to a small category). Then we can define $K(\mathcal{A})$ as the commutative ring generated by symbols $\langle e\rangle$ (with $e$ surjective) and relations

(i) $\langle 1_x\rangle = 1$ for all objects $x$,
(ii) $\langle \bar{e}\rangle = \langle e\rangle$ if $\bar{e}$ is a pull-back of $e$, and
(iii) $\langle e\bar{e}\rangle = \langle e\rangle\langle \bar{e}\rangle$ for all $e$, $\bar{e}$ which can be composed.

It is clear that

$$\Delta : \mathfrak{E}(\mathcal{A}) \to K(\mathcal{A}) : e \mapsto \langle e\rangle \tag{8.1}$$

is a universal degree function on $\mathcal{A}$, i.e., every degree function factorizes uniquely through $K(\mathcal{A})$. Moreover, the category $\mathcal{T}(\mathcal{A}, \Delta)$ is a universal family of tensor categories in the sense that

$$\mathcal{T}(\mathcal{A}, \delta) = \text{idempotent closure of } \mathcal{T}(\mathcal{A}, \Delta) \otimes_{K(\mathcal{A})} K. \tag{8.2}$$

Observe that all Hom-spaces of the universal category $\mathcal{T}(\mathcal{A}, \Delta)$ are projective $K(\mathcal{A})$-modules (of finite type, in case $\mathcal{A}$ is subobject finite).

Thus, since $\mathcal{T}(\mathcal{A}, \delta)$ is a "fiber" of the family $\mathcal{T}(\mathcal{A}, \Delta)$ it is of interest when $\mathcal{T}(\mathcal{A}, \delta)$ itself is semisimple, or at least when $\mathcal{T}(\mathcal{A}, \delta) = \overline{\mathcal{T}}(\mathcal{A}, \delta)$, i.e., when its tensor radical vanishes. The purpose of this section is to give a simple numerical criterion.

**8.1. Definition.** Let $K$ be a field. A $K$-valued degree function $\delta$ is *non-singular* if the tensor radical $\mathcal{N}$ of $\mathcal{T}(\mathcal{A}, \delta)$ is 0.

Assume in this section that $\mathcal{A}$ is a subobject finite regular category and that $K$ is a field (of any characteristic). Let $X$ be an object of $\mathcal{T}(\mathcal{A}, \delta)$. If $\operatorname{sub}(\mathbf{1}) = \{\mathbf{1}\}$ then $\operatorname{End}_{\mathcal{T}}(\mathbb{1}) = K$ and there is a pairing

$$\beta_X : \operatorname{Hom}_{\mathcal{T}}(\mathbb{1}, X) \times \operatorname{Hom}_{\mathcal{T}}(X, \mathbb{1}) \to K : (G, F) \mapsto FG. \tag{8.3}$$

Basically by definition, this pairing is non-degenerate if and only if $\mathcal{N}(\mathbb{1}, X) = \mathcal{N}(X, \mathbb{1}) = 0$.

**8.2. Lemma.** *Assume* $\operatorname{sub}(\mathbf{1}) = \{\mathbf{1}\}$. *Then the pairing* $\beta_{[x]}$ *is non-degenerate if and only if*

$$\Omega_x := \det\big(\delta(u \cap v \to \mathbf{1})\big)_{u,v \subseteq x} \neq 0 \tag{8.4}$$

*where we put* $\delta(\emptyset \to \mathbf{1}) = 0$.

**Proof.** Every subobject $u$ of $x$ induces relations $\delta_u : u \rightarrowtail \mathbf{1} \times x$ and $\varepsilon_u : u \rightarrowtail x \times \mathbf{1}$. These relations form a $K$-basis in their respective $\operatorname{Hom}_{\mathcal{T}}$-space. Moreover $\beta_{[x]}(\delta_u, \varepsilon_v) = \delta(u \cap v \to \mathbf{1})$. Thus $\Omega_x$ is just the determinant of $\beta_{[x]}$ with respect to these bases. $\square$

We proceed with the calculation of $\Omega_x$. Observe that $\operatorname{sub}(x)$ is a finite partial semilattice (see Section 7 for the definition) with respect to intersection: $u \wedge v := u \cap v := u \times_x v$. Thus, for any surjective morphism $e : x \twoheadrightarrow y$ we may define

$$\omega_e := \sum_{\substack{w \in \operatorname{sub}(x) \\ e(w)=y}} \mu(w,x)\delta(w \twoheadrightarrow y) \in K \tag{8.5}$$

where $\mu$ is the Möbius function of $\operatorname{sub}(x)$. If $\operatorname{sub}(\mathbf{1}) = \{\mathbf{1}\}$ and $y = \mathbf{1}$ then this specializes to

$$\omega_{x \twoheadrightarrow \mathbf{1}} = \sum_{w \in \operatorname{sub}(x)} \mu(w,x)\delta(w \twoheadrightarrow \mathbf{1}). \tag{8.6}$$

Now we have the following factorization:

**8.3. Lemma.** *Assume* $\operatorname{sub}(\mathbf{1}) = \{\mathbf{1}\}$. *Then*

$$\Omega_x = \prod_{u \in \operatorname{sub}(x)} \omega_{u \twoheadrightarrow \mathbf{1}}. \tag{8.7}$$

**Proof.** Apply Lemma 7.1 to $L = \operatorname{sub}(x)$ and $\varphi(u) := \delta(u \to \mathbf{1})$. □

The elements $\omega_{x \twoheadrightarrow \mathbf{1}}$ factorize further. This is similar to Stanley's factorization, [19], of the characteristic polynomial of a lattice.

**8.4. Lemma.** *The element $\omega_e$ is multiplicative in e, i.e., if $x \overset{\bar{e}}{\twoheadrightarrow} y \overset{e}{\twoheadrightarrow} z$ are surjective morphisms then $\omega_{e\bar{e}} = \omega_e \omega_{\bar{e}}$.*

**Proof.** Every surjective morphism $e : x \twoheadrightarrow y$ induces two maps

$$e_* : \operatorname{sub}(x) \to \operatorname{sub}(y) : u \mapsto e(u) = \operatorname{im}(u \rightarrowtail x \overset{e}{\twoheadrightarrow} y), \tag{8.8}$$

$$e^* : \operatorname{sub}(y) \to \operatorname{sub}(x) : v \mapsto x \times_y v. \tag{8.9}$$

They form a Galois connection since both $e_*(u) \leqslant v$ and $u \leqslant e^*(v)$ are equivalent to $u \to y$ factorizing through $v$.

Now we apply Lemma 7.2 to $L = \operatorname{sub}(x)$, $M = \operatorname{sub}(y)$ and $l = x$. Then $m = y$ since $e$ is surjective. Thus we get

$$\sum_{u \in \operatorname{sub}(x)} \mu(u,x)u = \left(\sum_{v \in \operatorname{sub}(y)} \mu(v,y)e^*(v)\right) \cap \left(\sum_{\substack{w \in \operatorname{sub}(x) \\ e(w)=y}} \mu(w,x)z\right). \tag{8.10}$$

Let $v \subseteq y$ and $w \subseteq x$ with $e(w) = y$. Put $u := e^*(v) \cap w$. Then the double pull-back diagram

$$\begin{array}{ccccccc} u & \rightarrowtail & e^*(v) & \twoheadrightarrow & v & & \\ \downarrow & & \downarrow & & \downarrow & & \\ w & \rightarrowtail & x & \overset{e}{\twoheadrightarrow} & y & \overset{\bar{e}}{\twoheadrightarrow} & z \end{array} \qquad (8.11)$$

yields $e\bar{e}(u) = z$ if and only if $\bar{e}(v) = z$ and in this case

$$\delta(u \twoheadrightarrow z) = \delta(v \twoheadrightarrow z) \cdot \delta(w \twoheadrightarrow y). \qquad (8.12)$$

Thus, if we apply the function

$$\varphi(u) := \begin{cases} \delta(u \to z) & \text{if } u \to z \text{ is surjective,} \\ 0 & \text{otherwise} \end{cases} \qquad (8.13)$$

to both sides of (8.10) then we get $\omega_{e\bar{e}} = \omega_e \omega_{\bar{e}}$. □

Let us call a surjective morphism $e$ *indecomposable* if in any factorization $e = e'e''$ with $e'$, $e''$ surjective precisely one of the factors is an isomorphism. Combining the above yields:

**8.5. Corollary.** *Let $\mathcal{A}$ be a subobject finite, regular category with* $\mathrm{sub}(\mathbf{1}) = \{\mathbf{1}\}$. *Then the pairing $\beta_{[x]}$ is non-degenerate if and only if $\omega_e \neq 0$ for all indecomposable surjective morphisms $e : u \twoheadrightarrow v$ with $u \preccurlyeq x$.*

**Proof.** The set $\mathrm{sub}(u)$ being finite for any object $u$ implies that $u$ has only finitely many quotient objects. This, in turn, entails that every surjective map is the composition of finitely many indecomposable ones. We conclude with Lemmas 8.2, 8.4, and 8.3. □

From this we get our main vanishing theorem:

**8.6. Theorem.** *Let $\mathcal{A}$ be a subobject finite, regular category and $K$ a field. Then a $K$-valued degree function $\delta$ is non-singular if and only if $\omega_e \neq 0$ for all indecomposable surjective morphisms $e$.*

**Proof.** Theorem 3.6 reduces the assertion to the case $\mathrm{sub}(\mathbf{1}) = \{\mathbf{1}\}$. If $\omega_{x \twoheadrightarrow y} = 0$ then $\Omega_x = 0$ and therefore $\mathcal{N}(\mathbb{1}, [x]) \neq 0$. Conversely, the non-vanishing of $\omega_e$ for all indecomposable $e$ implies $\Omega_x \neq 0$ and therefore $\mathcal{N}(\mathbb{1}, [x]) = 0$ for all $x$. From $\mathcal{N}(\mathbb{1}, X \oplus Y) = \mathcal{N}(\mathbb{1}, X) \oplus \mathcal{N}(\mathbb{1}, Y)$ we conclude $\mathcal{N}(\mathbb{1}, X) = 0$ for all objects $X$ of $\mathcal{T}$. This implies $\mathcal{N} = 0$ by (4.3). □

**8.7. Corollary.** *Let $\mathcal{A}$ be an essentially small, subobject finite, complete, exact, protomodular category. Then $\mathcal{A}$ has non-singular degree functions.*

**Proof.** Consider the universal degree function $\Delta : \mathfrak{E}(\mathcal{A}) \to K(\mathcal{A})$. Theorem B1.4(ii) implies that $K(\mathcal{A})$ is a polynomial ring over $\mathbb{Z}$. For any surjective morphism $e : x \twoheadrightarrow y$ consider $\omega_e$ computed with respect to $\Delta$. Then $\omega_e$ is a polynomial. Theorem B1.5 asserts that the monomial $\Delta(e)$ occurs only once in $\omega_e$ which implies $\omega_e \neq 0$. Now we can take for $K$ the field of fractions of $K(\mathcal{A})$ (or any bigger field) and for $\delta$ the composition $\mathfrak{E}(\mathcal{A}) \xrightarrow{\Delta} K(\mathcal{A}) \hookrightarrow K$. □

**Remark.** We show in the last example below that there are exact Mal'cev categories without non-singular degree functions. So the condition of protomodularity cannot be dropped.

**Examples. 1.** [Deligne's case] Let $\mathcal{A} = \mathsf{Set}^{\mathrm{op}}$ where $\mathsf{Set}$ is the category of finite sets. Then surjective morphisms in $\mathcal{A}$ are injective maps in $\mathsf{Set}$. Let $t \in K$. The degree functions are parametrized by $t \in K$ (see (3.3)). An injective map $e : A \rightarrowtail B$ is indecomposable if $B \setminus e(A) = \{b\}$ is a one-point set. To compute $\omega_e$ we have to consider diagrams

$$\begin{array}{ccc} A & \rightarrowtail & B = A \cup \{b\} \\ & \searrow & \downarrow \\ & & Q. \end{array} \tag{8.14}$$

There are two cases: either $Q = B$ or $Q = A$. In the first case, $\mu(Q, B) = 1$ and $\delta(A \rightarrowtail Q) = t$. The second case depends on the image of $b$, so there are $|A|$ possibilities. Moreover, $\mu(Q, B) = -1$ and $\delta(A \rightarrowtail Q) = 1$. This implies $\omega_e = t - |A|$. Since $|A|$ is an arbitrary natural number we conclude: *$\delta$ is non-singular if and only if $t \notin \mathbb{N}$.*

**2.** Let $G$ be a finite group and let $\mathcal{A}$ be the opposite category of the category of finite sets with a *free* $G$-action. This category is regular but not complete. Let $\ell_0(A) := |A/G|$, the number of $G$-orbits. Then all degree functions are of the form

$$\delta(e : A \rightarrowtail B) = t^{\ell_0(B \setminus e(A))}. \tag{8.15}$$

Then as before one deduces that $e : A \rightarrowtail B$ is indecomposable if $B \setminus e(A)$ is just one orbit and in that case $\omega_e = t - |A|$. Thus *$\delta$ is non-singular if and only if $t \notin \mathbb{N}\,|G|$.*

**3.** Let $\mathcal{A}$ be the category of non-empty finite sets. Then there is only one degree function with $\delta(e) = 1$ for all surjective maps $e$. The map $e : A \twoheadrightarrow B$ is indecomposable if it identifies exactly one pair $\{a_1, a_2\}$ of points to one point. There are exactly three subsets $A_0 \subseteq A$ such that $A_0 \to B$ is still surjective. Hence $\omega_e = 1 - 1 - 1 = -1 \neq 0$. This shows that $\mathcal{N} = 0$ for any field (even in positive characteristic). Because $\mathcal{A}$ is not Mal'cev we cannot apply Theorem 6.1. Thus, it is not clear whether $\mathcal{T}(\mathcal{A}, \delta)$ is semisimple.

**4.** Let $\mathcal{A} = \mathsf{Mod}_{\mathbb{F}_q}$ be the category of finite dimensional $\mathbb{F}_q$-vector spaces. The degree functions are given by formula (3.4). The homomorphism $e$ is indecomposable if $\dim \ker e = 1$. To compute $\omega_e$ we have to consider diagrams

$$\begin{array}{ccc} S & & \\ \downarrow & \searrow & \\ V \oplus \mathbb{F}_q \cong U & \twoheadrightarrow & V \end{array} \tag{8.16}$$

up to automorphisms of $S$. Again, there are two possibilities: $S = U$ or $S = V$. In the first case $\mu(S, U) = 1$ and $\delta(S \twoheadrightarrow V) = t$. In the second case, $S$ is a section of $e$, hence there are $|V|$ possibilities. Since $\mu(S, U) = -1$ and $\delta(S \twoheadrightarrow V) = 1$ we get $\omega_e = t - |V|$. We conclude: *$\delta$ is non-singular if and only if $t \notin q^{\mathbb{N}}$.* Observe that, in particular, $t = 0$ is also non-singular.

**5.** Let, more generally, $\mathcal{A}$ be a subobject finite abelian category. The degree functions are given by formula (3.5). A surjective morphism $e : x \twoheadrightarrow y$ is indecomposable if and only if $s = \ker e$ is simple. A calculation as above shows that $\omega_e = t_s - \alpha$ where $\alpha$ is the number of sections of $e$. This number can be zero unless $s$ is injective. Otherwise, $\alpha$ is a power of $q_s := |\mathrm{End}_{\mathcal{A}}(s)|$. We conclude: *$\delta$ is non-singular if and only if $t_s \notin q_s^{\mathbb{N}}$ for all $s \in S$ and $t_s \neq 0$ for all non-injective $s \in S$.*

**6.** Let $\mathcal{A}$ be the category of homomorphisms $f : U \to V$ between finite dimensional $\mathbb{F}_q$-vector spaces. This is the category of $\mathbb{F}_q$-representation of the quiver $\bullet \to \bullet$ and therefore abelian. The simple objects are $s_1 = (\mathbb{F}_q \to 0)$ and $s_2 = (0 \to \mathbb{F}_q)$. Only $s_1$ is injective. Let $t_i := t_{s_i}$. Then *$\delta$ is non-singular if and only if $t_1 \notin q^{\mathbb{N}}$ and $t_2 \notin q^{\mathbb{N}} \cup \{0\}$.*

**7.** Let $\mathcal{A}$ be the category of (non-empty) affine spaces over $\mathbb{F}_q$. The degree functions are given by

$$\delta(e : X \twoheadrightarrow Y) = t^{\dim X - \dim Y}. \tag{8.17}$$

Moreover, $e$ is indecomposable if $X \cong Y \times \mathbf{A}^1$. In that case, $\omega_e = t - \alpha$ where $\alpha$ is the number of section of $e$. Since $\alpha = q\,|Y|$ we get *$\delta$ is non-singular if and only if $t \notin q^{\mathbb{N}^*}$.*

**8.** Let $\mathcal{A}$ be the category of finite solvable groups. For a prime $p$ let $v_p$ be the corresponding valuation of $\mathbb{Z}$. Then all degree functions on $\mathcal{A}$ are given by

$$\deg(e : G \twoheadrightarrow H) = \prod_p t_p^{v_p(|\ker e|)} \tag{8.18}$$

with infinitely many parameters $t_2, t_3, t_5, \ldots \in K$. The map $e : G \twoheadrightarrow H$ is indecomposable if and only if $K = \ker e$ is a minimal non-trivial normal subgroup of $G$. Then $K$ is an elementary abelian group of order $p^n$, say. Let $L \subseteq G$ be a subgroup with $e(L) = H$, i.e., $G = KL$. The intersection $L \cap K$ is a subgroup of $G$ which is normalized by $K$ (since $K$ is abelian) and $L$ hence by $G = KL$. Minimality of $K$ implies either $K \subseteq L$ or $K \cap L = 1$. In the first case, we have $L = G$, $\mu(L, G) = 1$ and $\deg(L \twoheadrightarrow H) = t_p^n$. In the second case holds $G = L \ltimes K$. Thus $\mu(L, G) = -1$ and $\deg(L \twoheadrightarrow H) = 1$. The number of complements can be zero in which case $\omega_e = t_p^n$. Otherwise, the conjugacy classes of complements are parametrized by $H^1(H, K)$ and each conjugacy class has $|K/K^H|$ elements. Thus, the number of complements is a power $p^N$ with $N \in \mathbb{N}$. We conclude that *$\delta$ is non-singular if and only if $t_p \neq 0$ and $t_p \neq p^r$ for all primes $p$ and all $r \in \mathbb{Q}_{\geqslant 0}$.*

**9.** We present an example for which *every* degree function is degenerate. Let $\mathcal{A}$ be the category of finite pointed Mal'cev algebras. Objects of this category are finite sets $A$ equipped with a base point $0 \in A$ and a ternary operation $m : A^3 \to A$ satisfying the identities $m(a, a, c) = c$ and $m(a, c, c) = a$ for all $a, c \in A$. Now we define ternary operations $m_3$ and $m_2$ on $A := \{0, 1, 2\}$ and $B := \{0, 1\}$, respectively by

$$m_3(a_1, a_2, a_3) = \begin{cases} a_3 & \text{if } a_1 = a_2, \\ a_1 & \text{if } a_2 = a_3, \\ 0 & \text{if exactly one of } a_1, a_2, a_3 \text{ equals } 0, \\ 1 & \text{otherwise} \end{cases} \tag{8.19}$$

and

$$m_2(b_1, b_2, b_3) = b_1 - b_2 + b_3 \mod 2. \tag{8.20}$$

Then one verifies easily:

- $m_3$ and $m_2$ are Mal'cev operations.
- The only (pointed) subalgebras of $A$ are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, and $A$ (the crucial point is $m_3(0, 2, 0) = 1$). The only subalgebras of $B$ are $B_0 = \{0\}$ and $B$.
- The map $e : A \twoheadrightarrow B$ with $e(0) = 0$ and $e(1) = e(2) = 1$ is an $\mathcal{A}$-morphism. For this notice that in the "otherwise" case of (8.19) either none or exactly two of $a_1, a_2, a_3$ are equal to 0.

Since $A_0 = \varphi^{-1}(B_0)$, the pull-back of $e$ by $B_0 \to B$ is an isomorphism. Hence $\delta(e) = 1$ and

$$\omega_e = \mu(A, A)\delta(A \twoheadrightarrow B) + \mu(A_1, A)\delta(A_1 \twoheadrightarrow B) = 1 \cdot 1 + (-1) \cdot 1 = 0. \tag{8.21}$$

Thus, $\delta$ *is always degenerate*. Observe that $\mathcal{A}$ is a pointed, exact Mal'cev category.

If one traces through the proof of Theorem 6.1 and analyzes exactly which $\mathcal{N}(X, Y)$ have to be zero, one obtains the following semisimplicity statement:

**8.8. Theorem.** *Let $\mathcal{A}$ be a subobject finite, exact Mal'cev category, $K$ a field of characteristic zero and $\delta$ a $K$-valued degree function. Assume $\omega_f \neq 0$ for every indecomposable $f : u \twoheadrightarrow v$ with $u \preccurlyeq x \times y$ for some $y \prec x$. Then* $\mathrm{End}_{\mathcal{T}}([x])$ *is a semisimple $K$-algebra.*

**Example.** If $\mathcal{A} = \mathsf{Set}^{\mathrm{op}}$ then $\mathrm{End}_{\mathcal{T}}([x])$ is known as *partition algebra* (Martin [16]). If $x = A$ is a set with $n$ elements then $y$ and $x \times y$ have at most $n - 1$ and $2n - 1$ elements, respectively. Thus $|v| \leqslant 2n - 2$. We conclude that $\mathrm{End}_{\mathcal{T}}([x])$ is semisimple for $t \neq 0, \ldots, 2n - 2$ which was first proved by Martin–Saleur [17]. Similarly, the $\mathbb{F}_q$-analog of the partition algebra is semisimple unless $t = 1, q, \ldots, q^{2n-2}$.

## 9. Tannakian degree functions

In this section, we investigate tensor functors from $\mathcal{T}(\mathcal{A}, \delta)$ to the category of $K$-vector spaces. This will answer in particular when $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is Tannakian, i.e., equivalent to $\mathrm{Rep}(G, K)$ for some pro-algebraic group over $K$, at least if $K$ is algebraically closed of characteristic zero.

Let $\mathsf{Set}$ be the category of *finite* sets and let $\mathsf{Mod}_K$ be the tensor category of finite dimensional $K$-vector spaces. There is a functor $\mathsf{Set} \to \mathsf{Mod}_K$ which maps a set $A$ to $K[A]$, the vector space with basis $A$. Let $\pi : A \to B$ be a map. Then $K[\pi]$, also denoted by $\pi$, is the homomorphism

$$\pi : K[A] \to K[B] : a \mapsto \pi(a). \tag{9.1}$$

Its transpose is the homomorphism

$$\pi^\vee : K[B] \to K[A] : b \mapsto \sum_{a \in \pi^{-1}(b)} a. \tag{9.2}$$

**9.1. Lemma.** *Consider the commutative diagram of sets*

$$\begin{array}{ccc} A & \xrightarrow{\pi_1} & B \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\ C & \xrightarrow[\pi_2]{} & D. \end{array} \tag{9.3}$$

*If* (9.3) *is a pull-back then*

$$\tau_2^\vee \pi_2 = \pi_1 \tau_1^\vee : K[C] \to K[B]. \tag{9.4}$$

*The converse is true if* $\operatorname{char} K = 0$.

**Proof.** For $c \in C$ let $A_c := \tau_1^{-1}(c)$, $d := \pi_2(c)$, and $B_d := \tau_2^{-1}(d)$. Diagram (9.3) is a pull-back if and only if the map $\pi_c : A_c \to B_d$ induced by $\pi_1$ is an isomorphism for all $c \in C$. Now the assertion follows from

$$\tau_2^\vee \pi_2(c) = \sum_{b \in B_d} b,$$
$$\pi_1 \tau_1^\vee(c) = \sum_{b \in B_d} \left|\pi_c^{-1}(b)\right| b. \qquad \square \tag{9.5}$$

A map $\pi$ between two finite sets is called *uniform* if all of its fibers have the same cardinality. If $\pi : A \to B$ is uniform and $B \neq \emptyset$ then we call $\deg \pi := |A|/|B|$ the *degree of* $\pi$. In other words, $\deg \pi$ is the cardinality of the fibers of $\pi$. Observe that $\emptyset \to \emptyset$ has no degree.

Let $\mathcal{A}$ be a regular category.

**9.2. Definition.** Let $P : \mathcal{A} \to \mathsf{Set}$ be a functor and extend it to a functor $P^\emptyset : \mathcal{A}^\emptyset \to \mathsf{Set}$ by setting $P^\emptyset(\emptyset) = \emptyset$. Then $P$ is called *uniform* if

– $P^\emptyset$ preserves finite limits (i.e., is *left exact*), and
– $P$ maps surjective morphisms to uniform maps.

**Remark.** In terms of $P$ alone, the first condition says that $P$ preserves finite limits and that $P(u) \times_{P(y)} P(v) = \emptyset$ whenever $u \times_y v$ does not exist. In particular, for a complete regular category the first condition could be replaced by "$P$ left exact".

**9.3. Definition.** A degree function $\delta : \mathfrak{E}(\mathcal{A}) \to K$ is *adapted* to a uniform functor $P : \mathcal{A} \to \mathsf{Set}$ if

$$\delta(x \twoheadrightarrow y) = \deg\bigl(P(x) \to P(y)\bigr) \quad \text{whenever } P(y) \neq \emptyset. \tag{9.6}$$

**Remark.** Call a uniform functor $P$ *non-degenerate* if $P(x) \neq \emptyset$ for all $x$. In that case, it is easy to check that (9.6) defines a degree function on $\mathcal{A}$. Thus, for a non-degenerate uniform functor $P$ there is precisely one degree function $\delta_P$ adapted to it. Observe that if $\mathcal{A}$ is pointed then all uniform functors are non-degenerate. Indeed, the left-exactness of $P$ implies that $P(\mathbf{1})$ is a terminal object of $\mathsf{Set}$, i.e., a one-point set. Since $\mathcal{A}$ is pointed, the unique map $\mathbf{1} \to x$ induces $P(\mathbf{1}) \to P(x)$ which implies $P(x) \neq \emptyset$.

**Example** *(Deligne's case).* At this point, we give only one example. There will be more after Corollary 9.9. If $\mathcal{A} = \mathsf{Set}^{\mathrm{op}}$ and $X$ is any finite set then $P(A) := \operatorname{Hom}_{\mathsf{Set}}(A, X)$ is uniform functor. In fact, it is clearly left exact. Moreover, let $e : A \rightarrowtail B$ be injective then any map $f : A \to X$ can be extended to $B$ by freely choosing the extension on $B \setminus e(A)$. Thus, the number of extensions is $|X|^{|B \setminus e(A)|}$, independent of $f$. There is exactly one degree function adapted to $P$, namely the one with $t = |X|$ (notation of (3.3)).

**9.4. Theorem.** *Let $\mathcal{A}$ be a regular category and $K$ a field.*

(i) *Let $P : \mathcal{A} \to \mathsf{Set}$ be a uniform functor and $\delta$ a $K$-valued degree function. Then there is a tensor functor $T_P$ such that the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{P} & \mathsf{Set} \\ {\scriptstyle [*]}\big\downarrow & & \big\downarrow{\scriptstyle K[*]} \\ \mathcal{T}(\mathcal{A}, \delta) & \xrightarrow{T_P} & \mathsf{Mod}_K \end{array} \tag{9.7}$$

*commutes if and only if $\delta$ is adapted to $P$. Moreover, $T_P$ is unique.*

(ii) *Assume that $K$ is algebraically closed of characteristic zero. Let $T : \mathcal{T}(\mathcal{A}, \delta) \to \mathsf{Mod}_K$ be a tensor functor. Then there is uniform functor $P : \mathcal{A} \to \mathsf{Set}$ (unique up to equivalence) such that $\delta$ is adapted to $P$ and $T$ is equivalent to $T_P$.*

**Proof.** (i) Assume $\delta$ is adapted to $P$. Because $\mathcal{T}(\mathcal{A}, \delta)$ is the universal pseudo-abelian extension of $\mathcal{T}^0(\mathcal{A}, \delta)$, it suffices to construct $T_P$ on $\mathcal{T}^0$. The commutativity of (9.7) forces us to define on objects

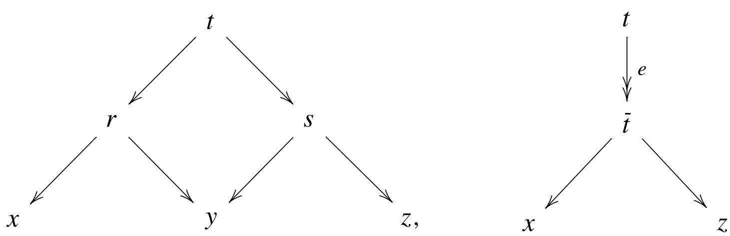
$$T_P\big([x]\big) = K\big[P(x)\big]. \tag{9.8}$$

If $r \to x \times y$ is a relation then, as a $\mathcal{T}$-morphism, $\langle r \rangle = [r \to y][r \to x]^\vee$. Thus, we have to define on morphisms

$$T_P\big(\langle r \rangle\big) = P(r \to y)P(r \to x)^\vee : K\big[P(x)\big] \to K\big[P(y)\big]. \tag{9.9}$$

This shows uniqueness.

Next, we show that (9.8) and (9.9) define indeed a tensor functor. For that, consider the diagrams




(9.10)

where the square is a pull-back (in $\mathcal{A}^\emptyset$) and where $\bar{t}$ is the image of $t$ in $x \times z$. Apply $P$ to (9.10). Then the left-exactness of $P^\emptyset$ and (9.4) imply

$$\begin{aligned} T_P\big(\langle s \rangle\big)T_P\big(\langle r \rangle\big) &= P(s \to z)P(s \to y)^\vee P(r \to y)P(r \to x)^\vee \\ &= P(s \to z)P(t \to s)P(t \to r)^\vee P(r \to x)^\vee \\ &= P(t \to z)P(t \to x)^\vee = P(\bar{t} \to z)P(e)P(e)^\vee P(\bar{t} \to x)^\vee. \end{aligned} \tag{9.11}$$

This is zero if $P(\bar{t}) = \emptyset$. Otherwise the uniformity of $P(e)$ implies

$$P(e)P(e)^\vee = \deg P(e) \cdot 1_{P(\bar{t})} \tag{9.12}$$

and therefore

$$T_P\big(\langle s\rangle\big)T_P\big(\langle r\rangle\big) = \deg P(e) \cdot T_P\big(\langle \bar{t}\rangle\big). \tag{9.13}$$

On the other hand,

$$T_P\big(\langle s\rangle\langle r\rangle\big) = T_P\big(\delta(e)\langle \bar{t}\rangle\big) = \delta(e) \cdot T_P\big(\langle \bar{t}\rangle\big). \tag{9.14}$$

Thus, $T_P$ is a functor if and only if $\delta$ is adapted to $P$. The fact that $T_P$ is a tensor functor follows from

$$\begin{aligned} T_P\big([x] \otimes [y]\big) &= T_P\big([x \times y]\big) = K\big[P^\emptyset(x \times y)\big] = K\big[P(x) \times P(y)\big] \\ &= K\big[P(x)\big] \otimes K\big[P(y)\big] = T_P(x) \otimes T_P(y). \end{aligned} \tag{9.15}$$

Now we prove part (ii) of the theorem. The map $x \mapsto (x) := [x]^\vee$ defines a *contravariant* functor from $\mathcal{A}$ to $\mathcal{T}(\mathcal{A}, \delta)$. It still has the property $(x \times y) = (x) \otimes (y)$. Thus, the unique morphism $x \to \mathbf{1}$ and the diagonal morphism $x \rightarrowtail x \times x$ define $\mathcal{T}$-morphisms

$$(x) \to \mathbb{1} \quad \text{and} \quad (x) \otimes (x) \to (x) \tag{9.16}$$

which equip $(x)$ with the structure of a unital commutative ring object. Every multiplication $m : X \otimes X \to X$ induces a trace on $X$,

$$\mathrm{tr} : X = X \otimes \mathbb{1} \xrightarrow{1_X \otimes \mathrm{ev}^\vee} X \otimes X \otimes X^\vee \xrightarrow{m \otimes 1_{X^\vee}} X \otimes X^\vee \xrightarrow{\sim} X^\vee \otimes X \xrightarrow{\mathrm{ev}} \mathbb{1}, \tag{9.17}$$

and therefore a trace form

$$X \otimes X \xrightarrow{m} X \xrightarrow{\mathrm{tr}} \mathbb{1}. \tag{9.18}$$

An easy calculation shows that the trace on $(x)$ is induced by the relation $x \xrightarrow{\sim} \mathbf{1} \times x$. Therefore, the trace form comes from the relation

$$\begin{array}{ccc} & x & \\ \swarrow & & \searrow^{\mathrm{diag}} \\ \mathbf{1} & & x \times x. \end{array} \tag{9.19}$$

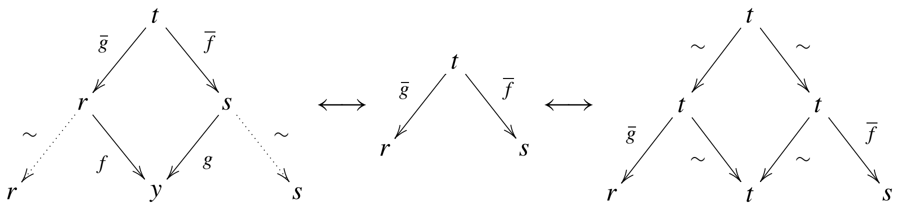
This relation is just the evaluation morphism of the selfduality $(x)^\vee = (x)$ (see (3.15)).

Let now $T : \mathcal{T}(\mathcal{A}, \delta) \to \mathsf{Mod}_K$ be a tensor functor. Then $\mathcal{O}(x) := T((x)) = T([x]^\vee)$ inherits all the properties above from $(x)$: it is a finite dimensional unital commutative $K$-algebra such that the trace form is non-degenerate. Since $K$ is algebraically closed, this implies that $\mathcal{O}(x)$ is isomorphic to $K \times \cdots \times K$ as a $K$-algebra. More canonically, put

$$P(x) := \mathrm{Spec}\, \mathcal{O}(x) = \mathrm{AlgHom}_K\big(\mathcal{O}(x), K\big). \tag{9.20}$$

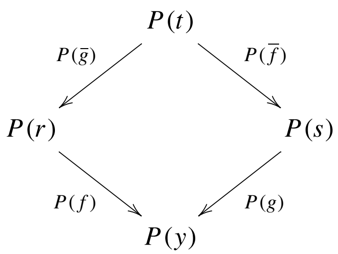
Then $P$ is a covariant functor $\mathcal{A} \to \mathsf{Set}$ such that the functors $T_P(x) = K[P(x)]$ and $T([x])$ from $\mathcal{A}$ to $\mathsf{Mod}_K$ are equivalent.

The proof of uniqueness above did not use any properties of $P$. This implies that also $T_P$ and $T$ are equivalent to each other. It remains to show that $P$ is uniform and that $\delta$ is adapted to $P$. For that, consider the following sequence of diagrams in $\mathcal{A}$



(9.21)

where the left square is an arbitrary pull-back. It implies the relation $[g]^\vee[f] = [\overline{f}][\bar{g}]^\vee$. Thus we also have $P(g)^\vee P(f) = P(\overline{f})P(\bar{g})^\vee$. From Lemma 9.1 we infer that also



(9.22)

is a pull-back. Since $P(\emptyset)$ and $P(\mathbf{1})$ have to be the empty and one-point set, respectively, we see that $P$ is left exact.

Finally, let $e : x \twoheadrightarrow y$ be surjective, inducing an algebra morphism $(e) : (y) \to (x)$. The restriction of $\mathrm{tr}_{(x)} : (x) \to \mathbb{1}$ to $(y)$ is given by the relation $x \to \mathbf{1} \times y$ which implies $\mathrm{tr}_{(x)} \circ (e) = \delta(e)\,\mathrm{tr}_{(y)}$. Thus also for $\mathcal{O}(y) \to \mathcal{O}(x)$ holds that the restriction of $\mathrm{tr}_{\mathcal{O}(x)}$ to $\mathcal{O}(y)$ is $\delta(e)\,\mathrm{tr}_{\mathcal{O}(y)}$. But that means that the map on spectra $P(x) \to P(y)$ is uniform of degree $\delta(e)$ (if $P(y) \neq \emptyset$). This shows that $\delta$ is adapted to $P$. $\square$

Finally, we investigate kernel and image of a tensor functor $T : \mathcal{T}(\mathcal{A}, \delta) \to \mathsf{Mod}_K$. Using Tannaka theory (see, e.g., [18]) this would be easy if $\mathcal{T}(\mathcal{A}, \delta)$ were an abelian category or, more generally, if $T$ would factorize through $\overline{\mathcal{T}}(\mathcal{A}, \delta)$. This is actually true but will be shown only a posteriori.

For any profinite group $G$ let $\mathsf{Set}(G)$ be the category of finite sets equipped with a continuous $G$-action. Let $\mathrm{Rep}(G, K)$ be the category of continuous *finite dimensional* representations of $G$ over $K$. Then $A \mapsto K[A]$ provides also a functor $\mathsf{Set}(G) \to \mathrm{Rep}(G, K)$.

Let $P : \mathcal{A} \to \mathsf{Set}$ be a uniform functor. If $\mathcal{A}$ is essentially small then $\mathrm{Aut}\, P$ is an example of a profinite group. Indeed, $\mathrm{Aut}\, P$ is the closed subgroup of $\prod_x S_{P(x)}$ defined by the equations $P(f)\pi_x = \pi_y P(f)$. Here, $S_{P(x)}$ is the symmetric group on a set $P(x)$. Moreover, $x$ and

$f : x \to y$ run through all objects and morphisms of a skeleton of $\mathcal{A}$. Clearly $P$ will factorize canonically through $\mathsf{Set}(\operatorname{Aut} P)$ and diagram (9.7) can be refined to

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{P} & \mathsf{Set}(\operatorname{Aut} P) \\ \downarrow & & \downarrow \\ \mathcal{T}(\mathcal{A}, \delta) & \xrightarrow{\mathcal{T}_P} & \operatorname{Rep}(\operatorname{Aut} P, K) \end{array} \tag{9.23}$$

(provided $\delta$ is adapted to $P$).

For every injective morphism $y \rightarrowtail x$ the map $P(y) \to P(x)$ is injective, as well. Thus we may define

$$P^*(x) := P(x) \setminus \bigcup_{y \subsetneq x} P(y). \tag{9.24}$$

Note that $P^*$ is functorial for surjective morphisms only. Since $P^{\emptyset}$ preserves intersections, for every $a \in P(x)$ there is a unique minimal subobject $z$ with $a \in P(y)$. This means that $P(x)$ is the disjoint union of the sets $P^*(y)$ with $y \in \operatorname{sub}(x)$.

From now on we assume that $\mathcal{A}$ is an essentially small, subobject finite, regular category and that $P : \mathcal{A} \to \mathsf{Set}$ is a uniform functor.

**9.5. Lemma.** *Let $e : x \twoheadrightarrow y$ be a surjective morphism. Then $P^*(x) \to P^*(y)$ is uniform.*

**Proof.** For $a \in P^*(y)$ let $P(x)_a \subseteq P(x)$ and $P^*(x)_a \subseteq P^*(x)$ be the fibers over $a$. Since $P(x)_a$ is the disjoint union of $P^*(z)_a$ with $z \subseteq x$, Möbius inversion yields

$$\left|P^*(x)_a\right| = \sum_{z \subseteq x} \mu(z, x)\left|P(z)_a\right|. \tag{9.25}$$

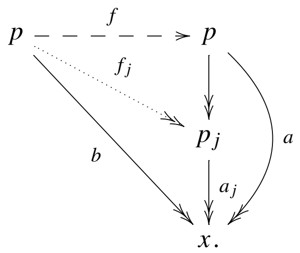
Because $a \in P^*(y)$, it suffices to sum over those $z$ with $z \to y$ surjective. In that case, uniformity of $P$ implies that $|P(z)_a|$ is independent of $a$. Hence the same holds for $|P^*(x)_a|$. □

The next assertion shows that $\operatorname{Aut} P$ is sufficiently big.

**9.6. Lemma.** *Let $x$ be an object of $\mathcal{A}$. Then* $\operatorname{Aut} P$ *acts transitively on $P^*(x)$.*

**Proof.** According to [2, Cor. App. 2.8], every left exact functor from $\mathcal{A}$ to the category of (not necessarily finite) sets is pro-representable. This means that there is a small filtering category $\mathcal{C}$ and a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{A} : i \mapsto p_i$ such that $P(u) = \varinjlim \operatorname{Hom}_{\mathcal{A}}(p_i, u)$ for all objects $u$ of $\mathcal{A}$. We denote the corresponding pro-object by $p$. Then $P^*(x)$ is the set of surjective morphisms $p \twoheadrightarrow x$.

So, let $a, b : p \twoheadrightarrow x$ be two surjective morphisms. We have to show that there is an automorphism $f$ of $p$ such that $a = bf$. Our plan is to construct $f$ by lifting $b : p \to x$ to a morphism $p \to p$ as in the following diagram




(9.26)

For that observe that $a : p \twoheadrightarrow x$ is represented by a surjective morphism $p_i \to x$ for some object $i$ of $\mathcal{C}$. By replacing $\mathcal{C}$ with a cofinal subcategory, we may assume that $i$ is an initial object of $\mathcal{C}$. Moreover, the descending chain condition for subobjects implies that we may assume that all morphisms $p_j \to p_k$ are surjective (simply replace $p_j$ by the intersection of all images of the $p_i \to p_j$).

Now a surjective morphism $f_j : p \twoheadrightarrow p_j$ corresponds to an element of $P^*(p_j)$. By Lemma 9.6, the $P^*(p_j)$ form a projective system of finite sets such that all structure maps $P^*(p_k) \to P^*(p_j)$ are uniform. The canonical projection $p \to p_j$ furnishes an element of $P^*(p_j)$ which shows that this set is not empty. We conclude that all maps $P^*(p_k) \to P^*(p_j)$ are surjective. Now let $P^*(p_j)_b$ be the fiber over $b \in P^*(x)$. Then also the $P^*(p_j)_b$ form a projective system of finite sets such that all structure maps are surjective. It follows that $\varprojlim P^*(p_j)_b$ is non-empty. Any element of that limit corresponds to an endomorphism $f$ of $p$ such that $b = af$.

This $f$ induces an endomorphism $\varphi$ of the functor $P$ with $\varphi(a) = b$. It remains to show that $\varphi$ is invertible. By construction, $f$ is represented by surjective morphisms $p \twoheadrightarrow p_j$. This implies that for any object $y$ of $\mathcal{A}$ the self-map $\varphi_y : P(y) \to P(y)$ is injective. But then it is invertible since $P(y)$ is finite. □

**9.7. Lemma.** *Assume $\delta$ is adapted to $P$. Then the functor*

$$T_P : \mathcal{T}(\mathcal{A}, \delta) \to \mathrm{Rep}(\mathrm{Aut}\, P, K) \tag{9.27}$$

*is full, i.e., surjective on* Hom*-spaces.*

**Proof.** Let $X$, $Y$ be objects of $\mathcal{T}(\mathcal{A}, \delta)$. Because of $\mathrm{Hom}_{\mathcal{T}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(\mathbb{1}, X^\vee \otimes Y)$ and because $T_P$ is a tensor functor we may assume $X = \mathbb{1}$. Moreover, since $T_P$ commutes with direct sums, is suffices to consider $Y = [x]$. Thus we have to show that

$$\mathrm{Hom}_{\mathcal{T}}\big(\mathbb{1}, [x]\big) \to P(x)^G \tag{9.28}$$

is surjective for all objects $x$ of $\mathcal{A}$ (with $G := \mathrm{Aut}\, P$). By Lemma 9.6, the $G$-orbits in $P(x)$ are precisely the non-empty sets among the sets $P^*(y)$ with $y \in \mathrm{sub}(x)$. For a subset $A$ of $P(x)$ let $\sum U = \sum_{\alpha \in A} \alpha$. Thus the right-hand space of (9.28) is spanned by all $\sum P^*(y)$ with $y \in \mathrm{sub}(x)$

(it is not a basis since some of these sums may be zero). By triangularity, $P(x)^G$ is also spanned by the elements $\sum P(y)$ with $y \in \operatorname{sub}(x)$. But $\sum P(y) = T_P(F)$ where $F$ is the relation

$$\begin{array}{ccc} & y & \\ \swarrow & & \searrow \\ \mathbf{1} & & x. \end{array} \qquad \square \tag{9.29}$$

Now we can prove the specialization theorem:

**9.8. Theorem.** *Let $\mathcal{A}$ be an essentially small, subobject finite, regular category, let $P : \mathcal{A} \to \mathsf{Set}$ be a uniform functor, and let $\delta$ be $K$-valued degree function adapted to $P$ where $K$ is a field of characteristic zero. Then $T_P$ induces an equivalence of tensor categories*

$$\overline{\mathcal{T}}(\mathcal{A}, \delta) \xrightarrow{\sim} \operatorname{Rep}(\operatorname{Aut} P, K). \tag{9.30}$$

**Proof.** Since $T_P$ is full and preserves traces, a $\mathcal{T}$-morphism is in $\mathcal{N}$ if and only if its image is. Because $K$ is of characteristic zero, the category $\operatorname{Rep}(\operatorname{Aut} P, K)$ is semisimple. This shows that $T_P$ factorizes through $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ and that the functor (9.30) fully faithful. The action of $G$ on the entirely of all sets $P([x])$ is effective. Therefore, $\operatorname{Rep}(\operatorname{Aut} P, K)$ is generated, as a pseudo-abelian tensor category, by the representations of the form $T_P([x])$. This implies that (9.30) is an equivalence. $\square$

**9.9. Corollary.** *Let $K$ be an algebraically closed field of characteristic zero. Then $\overline{\mathcal{T}}(\mathcal{A}, \delta)$ is Tannakian if and only if $\delta$ is adapted to some uniform functor $P : \mathcal{A} \to \mathsf{Set}$.*

Before we go on with examples we introduce the following language:

**9.10. Definition.** Let $\mathcal{A}$ be subobject finite, regular category, $A$ a commutative domain with field of fractions $F$, and $\Delta$ a $A$-valued degree function on $\mathcal{A}$. Let $\mathfrak{T} := \{\mathcal{T}_i \mid i \in I\}$ be a family of semisimple tensor categories such that $K_i := \operatorname{End}_{\mathcal{T}_i}(\mathbb{1})$ is a field for all $i \in I$. Then we say that $\mathcal{T}(\mathcal{A}, \Delta)$ *interpolates the family* $\mathfrak{T}$ if:

(i) The $F$-valued degree function of $\mathcal{A}$ induced by $\Delta$ is non-singular.
(ii) There is a family of homomorphism $\varphi_i : A \to K_i$ such that
  (a) the product homomorphism $\prod_i \varphi_i : A \to \prod_i K_i$ is injective, and
  (b) the category $\mathcal{T}_i$ is tensor equivalent to $\overline{\mathcal{T}}(\mathcal{A}, \varphi_i \circ \Delta)$ for all $i \in I$.

Observe that (i) is equivalent to $\omega_e \neq 0$ for all indecomposable surjective morphisms $e$ where we consider $\omega_e$ as an element of $A$. By Corollary 8.7 this is automatically true for protomodular categories.

Every object of $\mathcal{T}(\mathcal{A}, \Delta)$ can be considered as an object of $\overline{\mathcal{T}}(\mathcal{A}, \varphi_i \circ \Delta)$ and, via the equivalence in (b), as an object of $\mathcal{T}_i$. For any two objects $X$ and $Y$ we get maps

$$\operatorname{Hom}_{\mathcal{T}(\mathcal{A},\Delta)}(X, Y) \otimes_A K_i \overset{\Phi}{\twoheadrightarrow} \operatorname{Hom}_{\overline{\mathcal{T}}(\mathcal{A},\varphi_i\circ\Delta)}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}_i}(X, Y). \tag{9.31}$$

The homomorphism $\Phi$ is always surjective and its kernel is it the tensor radical of $\mathcal{T}(\mathcal{A}, \varphi \circ \Delta)$. Its vanishing is equivalent to the non-vanishing (in $K_i$) of certain $\omega_e$ which are finite in number. Condition (a) says that the $K_i$-valued points $\varphi_i$ of Spec $A$ are Zariski dense. Together this shows that also $\Phi$ is an isomorphism for infinitely many $i \in I$. If, for example, $\mathrm{Hom}_{\mathcal{T}(\mathcal{A},\Delta)}(X, Y)$ were a free $A$-module then we would get a basis of all infinitely many of the spaces $\mathrm{Hom}_{\mathcal{T}_i}(X, Y)$. Moreover, the structure constants of the composition

$$\mathrm{Hom}_{\mathcal{T}_i}(X, Y) \times \mathrm{Hom}_{\mathcal{T}_i}(Y, Z) \to \mathrm{Hom}_{\mathcal{T}_i}(X, Z) \tag{9.32}$$

were all specializations from $A$. This holds even true if we consider any finite family of pairs $(X, Y)$ simultaneously.

In the following examples, we take for $\Delta$ the universal degree function with $A = K(\mathcal{A})$ and put $\mathcal{T}(\mathcal{A}) := \mathcal{T}(\mathcal{A}, \Delta)$. We choose a field $K$ of characteristic zero.

**Examples. 1.** [Deligne's case] Let $\mathcal{A} = \mathcal{B}^{\mathrm{op}}$ where $\mathcal{B} = \mathsf{Set}$ is the category of finite sets. Then $K(\mathcal{A}) = \mathbb{Z}[t]$. The uniform functors are of the form $P(A) := \mathrm{Hom}_{\mathsf{Set}}(A, X)$ where $X$ be a finite set. The adapted degree function corresponds to $t = n := |X|$. Since $\mathrm{Aut}_{\mathsf{Set}}(X) \cong S_n$ we see that $\mathcal{T}(\mathcal{A})$ *interpolates the categories* $\mathrm{Rep}(S_n, K)$, $n \geqslant 0$.

**2.** More generally, let $G$ be a finite group and let $\mathcal{A} = \mathcal{B}^{\mathrm{op}}$ where $\mathcal{B}$ is the category of finite sets with *free* $G$-action. Then $K(\mathcal{A}) = \mathbb{Z}[t]$. For any object $X$ of $\mathcal{A}$ we get a uniform functor $P(A) = \mathrm{Hom}_{\mathcal{B}}(A, X)$ whose adapted degree function has parameter $t = |X| = |G|n$, where $n = |X/G|$. We have $\mathrm{Aut}_{\mathcal{A}}(X) \cong S_n \wr G = S_n \ltimes G^n$. Thus $\mathcal{T}(\mathcal{A})$ *interpolates* $\mathrm{Rep}(S_n \wr G, K)$, $n \geqslant 0$. In particular, for $G = \mathbb{Z}_2$, we get the representation categories of the hyperoctahedral groups, i.e., the Weyl groups of type $\mathsf{BC}_n$.

**3.** Let $\mathcal{A} = \mathcal{B}^{\mathrm{op}}$ where $\mathcal{B}$ is the category of chains

$$A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow \cdots \tag{9.33}$$

of finite sets with $A_i = \emptyset$ for $i \gg 0$. This category is also equivalent to the category of finite rooted trees with graph maps which preserve the distance to the root. Then $K(\mathcal{A}) = \mathbb{Z}[t_1, t_2, \ldots]$. The uniform pro-objects are chains of finite sets

$$X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow X_4 \leftarrow \cdots \tag{9.34}$$

(possibly non-empty for all $i$) such that all connecting maps are uniform. The corresponding parameters are $t_i = |X_i| =: n_i$. Thus $\mathcal{T}(\mathcal{A})$ *interpolates* $\mathrm{Rep}(S_{n_1} \wr (S_{n_2} \wr (S_{n_3} \wr (S_{n_4} \wr \ldots))), K)$, *with* $n_i \geqslant 1$.

**4.** Let $\mathcal{A}$ be the category of finite solvable groups. Then $K(\mathcal{A}) = \mathbb{Z}[t_p \mid p \text{ prime}]$. Let $FS_n$ be the free pro-solvable group on $n$ letters (i.e., the completion of the free group $F_n$ on $n$ letters with respect to the topology defined by all normal subgroups $N$ such that $F_n/N$ is solvable). This is a uniform pro-object. Its adapted degree function has $t_p = p^n$. A moments thought shows that these functions are Zariski-dense in Spec $K(\mathcal{A})$. Thus $\mathcal{T}(\mathcal{A})$ *interpolates* $\mathrm{Rep}(\mathrm{Aut}\, FS_n, K)$, $n \geqslant 0$.

**5.** Let $\mathcal{A} = \mathsf{Mod}_{\mathbb{F}_q}$, the category of finite dimensional $\mathbb{F}_q$-vector spaces. Then $K(\mathcal{A}) = \mathbb{Z}[t]$. Every object $X$ of $\mathcal{A}$ is uniform with adapted degree function corresponding to the parameter $t = |X| = q^n$. Thus $\mathcal{T}(\mathcal{A})$ *interpolates* $\mathrm{Rep}(GL_n(\mathbb{F}_q), K)$, $n \geqslant 0$, *q fixed.*

**6.** Let $\mathcal{A}$ be the category of chains of homomorphisms of finite dimensional $\mathbb{F}_q$-vector spaces

$$\cdots \to V_{-2} \to V_{-1} \to V_0 \to V_2 \to V_{\to} V_4 \to \cdots \tag{9.35}$$

with $V_i = 0$ for $|i| \gg 0$. Then $K(\mathcal{A}) = \mathbb{Z}[\ldots, t_{-1}, t_0, t_1, \ldots]$. Moreover, a uniform pro-object is also a chain but all homomorphisms have to be injective and the vanishing condition on the $V_i$ has to be replaced by the condition that the inverse limit is zero. In other words, a uniform object is an $\mathbb{F}_q$-vector space $X$ with a $\mathbb{Z}$-filtration $X_i$ such that $X = \bigcup_i X_i$ and $0 = \bigcap_i X_i$. This shows that $\mathcal{A}(\mathcal{A})$ *interpolates* $\operatorname{Rep}(P_{\ldots n_{-1} n_0 n_1 \ldots}, K)$. Here $P_{\ldots n_{-1} n_0 n_1 \ldots}$ is group of invertible block matrices

$$\begin{pmatrix} \ddots & & & & \\ & B_{-1-1} & B_{-10} & B_{-11} & \\ & 0 & B_{00} & B_{01} & \\ & 0 & 0 & B_{11} & \\ & & & & \ddots \end{pmatrix} \tag{9.36}$$

where $B_{ij}$ is an $n_i \times n_j$-matrix with entries in $\mathbb{F}_q$.

**7.** Let $\mathcal{A}$ be the category of (non-empty) affine spaces. Then $K(\mathcal{A}) = \mathbb{Z}[t]$ and every object is uniform. Thus $\mathcal{T}(\mathcal{A})$ *interpolates* $\operatorname{Rep}(A_n, K)$, $n \geqslant 0$, where $A_n = GL(n, \mathbb{F}^q) \ltimes \mathbb{F}_q^n$ is the affine group.

## Appendix A. Protomodular and Mal'cev categories

In this appendix, we recall two classes of categories, namely protomodular and Mal'cev ones, which generalize various aspects of the category of groups and abelian categories. Mal'cev categories are more general than protomodular ones.

In a nutshell, the short five-lemma holds for protomodular categories while some version of Jordan–Hölder theorem can be proved for Mal'cev categories (see Appendix B). Moreover, Mal'cev categories have a nice theory of relations. That is why they are of particular interest to us. It should be pointed out, though, that most, if not all, natural examples are already protomodular.

**A1.1. Definition.** A regular category is a *Mal'cev category* if every reflexive relation is an equivalence relation.

This definition is due to Carboni, Lambek, and Pedicchio [6]. It is the following property of exact Mal'cev categories which we are actually going to use.

**A1.2. Proposition.** *(See [7, Thm. 5.7].) Let $\mathcal{A}$ be an exact Mal'cev category. Then every pair of surjective morphisms with the same domain has a push-out. Moreover for any commutative diagram*

$$\begin{array}{ccc} u & \twoheadrightarrow & y \\ \downarrow & & \downarrow \\ x & \twoheadrightarrow & z \end{array} \tag{A1.1}$$

*of surjective morphisms the following are equivalent*:

(i) *Diagram* (A1.1) *is a pull-back.*
(ii) *Diagram* (A1.1) *is a push-out and* $u \to x \times y$ *is injective.*

**Remark.** The implication (i) $\Rightarrow$ (ii) is a general property of regular categories. It is the characterization of pull-backs in term of push-outs which is of particular interest.

Now, we define protomodular categories. They will be used in Corollary 8.7 stating the non-singularity of the generic degree function. First, recall the category $\mathrm{Pt}_s\,\mathcal{A}$ of triples $(x, e, d)$ where $e : x \to s$ and $d : s \to x$ are morphisms with $ed = 1_s$. It is a pointed category with zero object $(s, 1_s, 1_s)$.

**A1.3. Definition.** A regular category $\mathcal{A}$ is *protomodular* if for any morphism $\bar{s} \to s$ the pull-back functor $\mathrm{Pt}_s\,\mathcal{A} \to \mathrm{Pt}_{\bar{s}}\,\mathcal{A}$ reflects isomorphisms.

In protomodular categories we have the following form of the short five-lemma:

**A1.4. Lemma.** *Let* $\mathcal{A}$ *be a pointed protomodular category and consider the following commutative diagram*:

$$\begin{array}{ccccc} x & \overset{i}{\rightarrowtail} & y & \overset{e}{\twoheadrightarrow} & z \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ x' & \overset{i'}{\rightarrowtail} & y' & \overset{e'}{\twoheadrightarrow} & z' \end{array} \tag{A1.2}$$

*where* $i, i'$ *is the kernel of* $e, e'$*, respectively. Assume that* $a$ *and* $c$ *are isomorphisms. Then* $b$ *is an isomorphism, as well.*

**Remark. 1.** For further reading we recommend the book [5] by Borceux and Bourn.

**2.** Our notion of a Mal'cev/protomodular category differs slightly from the one in the literature (like, e.g., [5]) since these categories are usually not required to be regular. Instead, a certain amount of limits is required to exist. For us, it does not really matter since we are only interested in regular categories anyway. A sufficient amount of limits is already build into them.

**Examples. 1.** Every abelian category is protomodular since in that case the pull-back functor $\mathrm{Pt}_s\,\mathcal{A} \to \mathrm{Pt}_{\bar{s}}\,\mathcal{A}$ is even an equivalence of categories.

**2.** Let $\mathcal{A}$ be the category of models of an equational theory. Then $\mathcal{A}$ is always exact. If the theory contains a group operation then $\mathcal{A}$ is protomodular. Therefore, the categories of groups, rings (with or without unity), Boolean algebras, Lie algebras, or any other type of algebras is protomodular. The full story is as follows (see [5, Thm. 3.1.6]): $\mathcal{A}$ is protomodular if and only if there is an $n \in \mathbb{N}$ such that the theory contains $n$ constants $e_1, \ldots, e_n$, $n$ binary operations $a_1(x, y), \ldots, a_n(x, y)$, and one $(n+1)$-ary operation $b(x_0, \ldots, x_n)$ such that

$$a_1(x, x) = e_1, \ldots, a_n(x, x) = e_n, \qquad b\big(x, a_1(x, y), \ldots, a_n(x, y)\big) = y. \tag{A1.3}$$

For example, for groups one has $n = 1$, $e_1 = e$, $a_1(x, y) = x^{-1}y$ and $b(x, y) = xy$.

**3.** The opposite of the category of sets or, more generally, of an elementary topos is exact protomodular [5, Ex. 3.1.17].

**4.** Every protomodular category is Mal'cev (see, e.g., [5, Prop. 3.1.19]).

**5.** Let $\mathcal{A}$ be the category of models of an equational theory. Then $\mathcal{A}$ is Mal'cev if and only if the theory contains a ternary operation $m(x, y, z)$ with $m(x, x, z) = z$ and $m(x, z, z) = x$ for all $x, z$ [5, Thm. 2.2.2]. Any group structure gives rise to such an operation, namely $m(x, y, z) = xy^{-1}z$. If $\mathcal{A}$ is the category of sets equipped with such a ternary operation then $\mathcal{A}$ is Mal'cev but not protomodular. This is shown in Section 8, Example 9.

**6.** The categories of (finite) sets, (finite) monoids, (finite) posets, and (finite) lattices are not Mal'cev.

Let $\mathfrak{P}$ be one of the properties "protomodular" or "Mal'cev". Then the class of $\mathfrak{P}$-categories enjoys many permanence properties. In the following, let $\mathcal{A}$ be any $\mathfrak{P}$-category.

**7.** Any full subcategory of $\mathcal{A}$ which is closed under products, subobjects and quotients is again $\mathfrak{P}$. Thus the category of finite models for an equational $\mathfrak{P}$-theory is again $\mathfrak{P}$. Examples are the categories of finite groups, finite rings, and finite Boolean algebras. The latter example is, by the way, equivalent to $\mathsf{Set}^{\mathrm{op}}$, the opposite category of finite sets.

**8.** Let $\mathcal{D}$ be a small category. Then the diagram category $[\mathcal{D}^{\mathrm{op}}, \mathcal{A}]$ is $\mathfrak{P}$. This applies, e.g., to the category of arrows $x \to y$ in $\mathcal{A}$ or the category of objects with $G$-action where $G$ a fixed group (or monoid).

**9.** Fix an object $s$ of $\mathcal{A}$. Then the slice category $\mathcal{A}/s$ of $s$-objects $x \to s$ is $\mathfrak{P}$. The same holds for the coslice category $s\backslash\mathcal{A}$ of arrows $s \to x$ and the category of points $\mathrm{Pt}_s\,\mathcal{A} = (s\backslash\mathcal{A})/(s \to s)$.

**10.** The category $\mathcal{A}/\!\!/s$ of "dominant" $s$-objects $x \twoheadrightarrow s$ is $\mathfrak{P}$. The same holds for the full subcategory category $s \to \mathcal{A}$ of objects such that there exists an arrow $s \to x$ provided $s$ is projective in it.

## Appendix B. Degree functions on Mal'cev categories

In this appendix, we determine the degree functions on certain Mal'cev categories. We also state a result to the effect that degree functions separate certain morphisms. For simplicity, we restrict to categories $\mathcal{A}$ which are essentially small. This has the effect that $\mathcal{A}$ has a universal degree function $\Delta : \mathfrak{E}(\mathcal{A}) \to K(\mathcal{A})$ (see (8.1)).

### *B1. Results*

In this section, we are only stating the results. Proofs are given in Section B3.

First, we need the following finiteness condition.

**B1.1. Definition.** A regular category is of *finite type* if $\mathrm{sub}(x)$ satisfies for every object $x$ the ascending and the descending chain condition.

This condition implies, in particular, that also the set of quotient objects of any $x$ satisfies the ascending and descending chain condition.

In determining degree functions, we first consider pointed categories. This means that there is an object $\mathbf{0}$ which is both initial and terminal. An object $x \not\cong \mathbf{0}$ is *simple* if, up to isomorphism, $\mathbf{0}$ and $x$ are its only quotients.

**B1.2. Theorem.** *Let $\mathcal{A}$ be an essentially small, pointed, exact Mal'cev category of finite type. Let $S(\mathcal{A})$ be its set of isomorphism classes of simple objects. Then the map $s \mapsto \langle s \twoheadrightarrow \mathbf{0}\rangle$ induces an isomorphism $\mathbb{Z}[S(\mathcal{A})] \xrightarrow{\sim} K(\mathcal{A})$.*

The case of non-pointed categories can be often reduced to the pointed case. Let $s$ be an object of $\mathcal{A}$. Recall that $\mathrm{Pt}_s\,\mathcal{A}$ is the category of triples $(x, e, d)$ where $e : x \to s$ and $d : s \to x$ are morphisms with $ed = 1_s$. This is a pointed category with zero object $(s, 1_s, 1_s)$.

**B1.3. Theorem.** *Let $\mathcal{A}$ be an essentially small, exact Mal'cev category of finite type. Assume that $\mathcal{A}$ has an initial object $\mathbf{0}$. Then the forgetful functor $\mathrm{Pt}_{\mathbf{0}}\,\mathcal{A} \to \mathcal{A}$ induces an isomorphism $K(\mathrm{Pt}_{\mathbf{0}}\,\mathcal{A}) \xrightarrow{\sim} K(\mathcal{A})$.*[3]

If $\mathcal{A}$ does not have an initial element then we look at *minimal* objects, i.e., objects which have no proper subobjects. Equivalently, an object $m$ is minimal if every morphism $x \to m$ is surjective. If $\mathcal{A}$ is of finite type then every object has a minimal subobject. If, moreover, $\mathcal{A}$ has all finite limits (hence intersections) then this minimal subobject $x_{\min}$ is even unique. Another consequence is that for any two minimal objects $m$ and $\overline{m}$ there is at most one morphism $\overline{m} \to m$. In fact, the graph of this morphism would have to be $(\overline{m} \times m)_{\min}$. The set $M(\mathcal{A})$ of isomorphism classes of minimal objects is a join-semilattice with $\overline{m} \geqslant m$ if there is a morphism $\overline{m} \to m$ and $m \vee \overline{m} = (m \times \overline{m})_{\min}$. If there is a morphism $\overline{m} \to m$ then there is a pull-back functor $\Phi_{\overline{m}\to m} : \mathrm{Pt}_m\,\mathcal{A} \to \mathrm{Pt}_{\overline{m}}\,\mathcal{A}$.

**B1.4. Theorem.** *Let $\mathcal{A}$ be a essentially small, complete, exact Mal'cev category of finite type.*

(i) *Let $m, \overline{m}$ be two minimal objects. Then $\Phi_{\overline{m}\to m}$ preserves simple objects. In particular we can define*

$$S(\mathcal{A}) := \lim_{m\in M(\mathcal{A})} S(\mathrm{Pt}_m\,\mathcal{A}). \tag{B1.1}$$

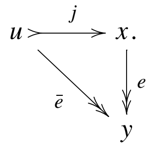
(ii) *The map $(s \rightleftarrows m) \mapsto \langle s \twoheadrightarrow m\rangle$ induces an isomorphism $\mathbb{Z}[S(\mathcal{A})] \xrightarrow{\sim} K(\mathcal{A})$.*

**Example.** Let $\mathcal{A}$ be the category of finite unital commutative rings. Then a ring $A$ is minimal if $A = \mathbb{Z}_n$ is cyclic. Via the correspondence, $I \mapsto I \oplus \mathbb{Z}_n$, the category $\mathrm{Pt}_{\mathbb{Z}_n}\,\mathcal{A}$ is equivalent to the category of finite non-unital rings $I$ with $nI = 0$. The pull-back functor $\Phi$ sends $I \oplus \mathbb{Z}_n$ to $I \oplus \mathbb{Z}_m$ (provided $n \mid m$). Thus, $S(\mathcal{A})$ is the set of isomorphism classes of finite simple non-unital commutative rings. They are easily classified: let $I$ be finite simple commutative. Because of simplicity we have $xI = 0$ or $xI = I$ for all $x \in I$. For the same reason, the set $J = \{x \in I \mid xI = 0\}$ is either $0$ or $I$. If $J = I$ then $II = 0$. Thus $I = N_p$ where $p$ is a prime and $N_p = \mathbb{Z}_p$ as an additive group but with zero multiplication. If $J = 0$ then multiplication by any $x \neq 0$ is surjective. Thus there is $e \in I$ with $xe = x$. But then $ze = z$ for all $z \in xI = I$, i.e., $e \in I$ is an identity element. This implies that $I = \mathbb{F}_q$ is a finite field. Thus $K(\mathcal{A})$ is a polynomial ring over two sets of variables $n_p$, $p$ prime and $f_q$, $q$ a prime power.

[3] Observe $\mathrm{Pt}_{\mathbf{0}}\,\mathcal{A} = \mathcal{A}/\mathbf{0}$ since $s : \mathbf{0} \to x$ is redundant.

An application of the theory above is the following statement on the values of the universal degree functions.

**B1.5. Theorem.** *Let $\mathcal{A}$ be an essentially small, complete, exact protomodular category of finite type with universal degree function $\Delta:\mathfrak{E}(\mathcal{A}) \to K(\mathcal{A})$. Consider the commutative diagram*



(B1.2)

*Then*

$$\Delta(e) = \Delta(\bar{e})$$

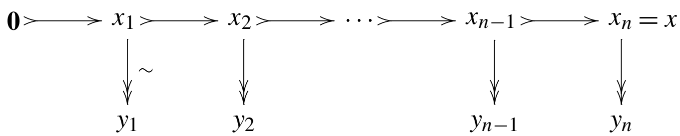
*if and only if $j$ is an isomorphism.*

**Remark.** Example 9 at the end of Section 8 shows that the theorem fails for Mal'cev categories.

*B2. Lambek's Jordan–Hölder theorem*

In this section (only), *the classical product of relations will be denoted by $rs$ instead of $r \circ s$.*

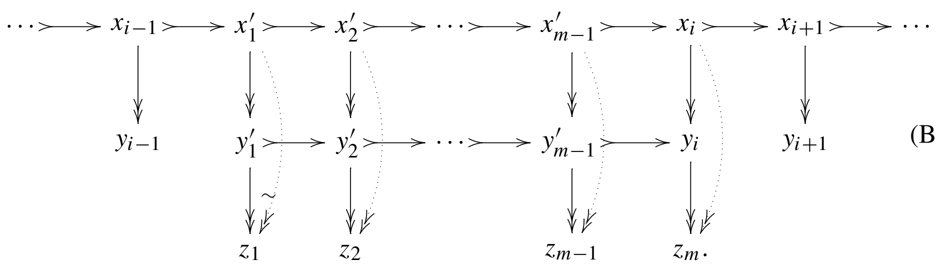
Let $\mathcal{A}$ be a *pointed* regular category. Then we can talk about the *kernel* $\ker f = x \times_y \mathbf{0}$ of a morphism $f: x \to y$. A *normal series* of an object $x$ is a diagram



(B2.1)

where each (horizontal) injective morphism is the kernel of the following (vertical) surjective morphism. The $y_i$ are called the *factors* of the normal series. Two series are *equivalent* if, after a suitable permutation, their factors are isomorphic.

Given a normal series of one of the factors $y_i$ one can *refine* the normal series of $x$:



(B2.2)

Here all squares are pull-backs. Of course one can refine all $y_i$ simultaneously giving the most general refinement. Now we have the following Schreier type theorem:

**B2.1. Theorem.** *Let $x$ be an object of a pointed, exact Mal'cev category $\mathcal{A}$. Then any two normal series of $x$ have equivalent refinements.*

The theorem is essentially due to Lambek [13,14], except that he proves it just for equational theories and that our notion of refinement seems to be stricter than his. Therefore, we repeat his proof and observe that it stays valid.

First, we reformulate the theorem in terms of relations and state a series of lemmas. Every subquotient diagram $x \hookleftarrow y \twoheadrightarrow z$ gives rise to an equivalence relation $r$ on $y$ which we may consider as a subobject of $x \times x$. Conversely, the subquotient is uniquely determined by $r$: for any subobject $u$ of $x$ we denote the image of the first projection $r \times_x u \to x$ by $ru$. Then $y = rx$ and $z = y/r$. Since $\mathcal{A}$ is exact, the relations which arise as $r$ are precisely the *subequivalence relations*, i.e., the relations which are symmetric and transitive but not necessarily reflexive.

A normal series of $x$ can be encoded as a sequence $r_1, r_2, \ldots, r_n$ of subequivalence relations where $r_i$ corresponds to the subquotient diagram $x \hookleftarrow x \twoheadrightarrow y_i$. In other words, $r_i = x_i \times_{y_i} x_i \rightarrowtail x \times x$. Since $x_i = r_i x$ and $\ker(x_i \twoheadrightarrow y_i) = x_i\mathbf{0}$, the kernel conditions are equivalent to

$$\mathbf{0} = r_1\mathbf{0},\ r_1 x = r_2\mathbf{0},\ \ldots,\ r_{n-1}x = r_n\mathbf{0},\ r_n x = x. \tag{B2.3}$$

Next, we need to encode refinements:

**B2.2. Definition.** A *refinement* of a subequivalence relation $r$ on $x$ is a sequence of subequivalence relations $s_1, \ldots, s_m$ with

$$rs_j r = s_j, \quad j = 1, \ldots, m, \quad \text{and} \tag{B2.4}$$

$$r\mathbf{0} = s_1\mathbf{0},\ s_1 x = s_2\mathbf{0},\ s_2 x = s_3\mathbf{0}, \ldots, s_{m-1}x = s_m\mathbf{0},\ s_m x = rx. \tag{B2.5}$$

**Remark.** It is condition (B2.4) which is missing in Lambek's notion of refinement.

**B2.3. Lemma.** *There is a natural one-to-one correspondence between refinements of $r$ and normal series of $rx/r$.*

**Proof.** Consider the quotient $rx \twoheadrightarrow r/x$ and let $u$ be a subobject of $x$. Then it is well known that $u$ is the pull-back of a subobject of $rx/r$ if and only if $ru = u$. If we apply this to the morphism $x \times x \twoheadrightarrow x/r \times x/r$ then we obtain that a relation $s$ on $x$ is the pull-back of a relation on $x/r$ if and only if $rsr = s$. Thus, (B2.4) makes asserts that the $s_j$ are pull-backs from subequivalence relations $s'_j$ on $y := rx/r$. The conditions (B2.5) imply that the $s'_j$ form a normal series of $y$. □

**B2.4. Lemma.** *Let $\mathcal{A}$ be a pointed Mal'cev category and $r$, $s$ two subequivalence relations on $x$. Then $rsr\mathbf{0} = rs\mathbf{0}$ and $rsrx = rsx$.*

**Proof.** Since $\mathcal{A}$ is Mal'cev, for any relation $p \subseteq x \times y$ holds

$$pp^{\vee}p = p. \tag{B2.6}$$

Applying this to $p = rs$ we get

$$rsrs = rs. \tag{B2.7}$$

Thus,

$$rs\mathbf{0} \subseteq rsr\mathbf{0} \subseteq rsrs\mathbf{0} = rs\mathbf{0}, \tag{B2.8}$$

$$rsx = rsrsx \subseteq rsrx \subseteq rsx. \qquad \square \tag{B2.9}$$

**Proof of Theorem B2.1.** Let $r_1, \ldots, r_n$ and $s_1, \ldots, s_m$ encode two normal series and put $r_{ij} := r_i s_j r_i$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. For a fixed $i$ we claim that $r_{i1}, \ldots, r_{im}$ refines $r_i$. First, $r_{ij}$ is clearly symmetric. Transitivity holds by (B2.7):

$$r_{ij} r_{ij} = r_i s_j r_i s_j r_i = r_i s_j r_i = r_{ij}. \tag{B2.10}$$

Hence $r_{ij}$ is a subequivalence relation. Condition (B2.4) holds trivially. Finally, we have, using Lemma B2.4,

$$r_{i1}\mathbf{0} = r_i s_1 r_i \mathbf{0} = r_i s_1 \mathbf{0} = r_i \mathbf{0}, \tag{B2.11}$$

$$r_{ij}\mathbf{0} = r_i s_j r_i \mathbf{0} = r_i s_j \mathbf{0} = r_i s_{j-1} x = r_i s_{j-1} r_i x = r_{ij-1} x, \quad j = 2, \ldots, m, \tag{B2.12}$$

$$r_{im} x = r_i s_m r_i x = r_i s_m x = r_i x. \tag{B2.13}$$

Symmetrically, put $s_{ji} := s_j r_i s_j$. Then $s_{j1}, \ldots, s_{jn}$ forms a refinement of $s_j$. The assertion follows now from Lambek's version of the Zassenhaus butterfly lemma ([13, Prop. 3, Thm. I], [6, Prop. 4.2]):

$$r_{ij} x / r_{ij} \cong s_{ji} x / s_{ji}. \qquad \square \tag{B2.14}$$

A normal series is called a *composition series* if all of its factors are simple. Now we have the following Jordan–Hölder type theorem:

**B2.5. Theorem.** *Let $\mathcal{A}$ be a pointed, exact Mal'cev category of finite type. Then every object has a composition series and any two composition series are equivalent.*

**Proof.** The additional condition (B2.4) makes sure that a composition series can be only refined in a trivial way. In fact, if $y_i$ is simple then the only possible refinements are

$$\begin{array}{ccccccccccccc} \mathbf{0} & \rightarrowtail & \cdots & \rightarrowtail & \mathbf{0} & \rightarrowtail & y_i & \overset{\sim}{\rightarrowtail} & y_i & \overset{\sim}{\rightarrowtail} & \cdots & \overset{\sim}{\rightarrowtail} & y_i \\ \downarrow & & & & \downarrow & & \downarrow\sim & & \downarrow & & & & \downarrow \\ \mathbf{0} & & \cdots & & \mathbf{0} & & y_i & & \mathbf{0} & & \cdots & & \mathbf{0}. \end{array} \tag{B2.15}$$

Thus all non-zero factors stay the same. Now the assertion follows from Theorem B2.1. $\square$

*B3. Proofs of Theorems B1.2–B1.5*

Let $\mathcal{A}$ be a pointed regular category and $\mathfrak{S}(\mathcal{A})$ its class of simple objects. A *rank function* on $\mathcal{A}$ is a function $\varrho : \operatorname{Ob}\mathcal{A} \to K$ such that

$$\varrho(x) = \varrho(\ker e) \cdot \varrho(y) \quad \text{for all surjective } e : x \twoheadrightarrow y. \tag{B3.1}$$

**B3.1. Proposition.** *Let $\mathcal{A}$ be a pointed, exact Mal'cev category of finite type. Then every function $\varrho_0 : S(\mathcal{A}) \to K$ extends uniquely to a rank function $\varrho : \operatorname{Ob}\mathcal{A} \to K$.*

**Proof.** For any object $x$ we choose a composition series with simple factors $y_1, \ldots, y_n$. Then we are forced to define

$$\varrho(x) = \prod_{i=1}^{n} \varrho_0(y_i),$$

showing the uniqueness of the extension. Moreover, Theorem B2.5 shows that $\varrho(x)$ is well defined. If $e : x \twoheadrightarrow y$ is surjective then the composition factors of $x$ are those of $y$ together with those of $\ker e$ which shows that $\varrho$ is a rank function. □

**Proof of Theorem B1.2.** In view of Proposition B3.1 it suffices to show that

$$\delta(e) := \varrho(\ker e), \qquad \varrho(x) := \delta(x \twoheadrightarrow \mathbf{0}) \tag{B3.2}$$

establishes a bijection between rank and degree functions on $\mathcal{A}$.

First we start with $\delta$. Then $\varrho$ as in (B3.2) is a rank function. Indeed, let $f : x \twoheadrightarrow y$ be surjective. Then

$$\begin{aligned} \varrho(x) &= \delta(x \twoheadrightarrow y \twoheadrightarrow \mathbf{0}) = \delta(x \twoheadrightarrow y)\delta(y \twoheadrightarrow \mathbf{0}) \\ &= \delta(\ker x \twoheadrightarrow \mathbf{0})\delta(y \twoheadrightarrow \mathbf{0}) = \varrho(\ker f)\varrho(y). \end{aligned} \tag{B3.3}$$

Conversely, given $\varrho$ then $\delta$ is a degree function. Indeed, the invariance under pull-backs is clear. Multiplicativity under composition follows from the following diagram where all squares are pull-backs.

$$\begin{array}{ccccc} \ker e' & \twoheadrightarrow & \mathbf{0} & & \\ \downarrow & & \downarrow & & \\ \ker ee' & \twoheadrightarrow & \ker e & \twoheadrightarrow & \mathbf{0} \\ \downarrow & & \downarrow & & \downarrow \\ x & \overset{e'}{\twoheadrightarrow} & y & \overset{e}{\twoheadrightarrow} & z \end{array} \tag{B3.4}$$

It is clear that both assignments are inverse to each other. □

**Proof of Theorem B1.3.** First observe that since $\mathbf{0}$ is initial, the arrow $\mathbf{0} \to x$ is redundant and the category $\operatorname{Pt}_{\mathbf{0}}\mathcal{A}$ is the same as the slice category $\mathcal{A}/\mathbf{0}$. Now we can define maps between $K(\mathcal{A})$ and $K(\operatorname{Pt}_0 \mathcal{A})$

$$K(\mathcal{A}) \to K(\mathrm{Pt}_{\mathbf{0}}\,\mathcal{A}) : \langle x \twoheadrightarrow y\rangle \mapsto \left\langle \begin{array}{ccc} x \times_y \mathbf{0} & \twoheadrightarrow & \mathbf{0} \\ & \searrow \quad \swarrow & \\ & \mathbf{0} & \end{array} \right\rangle,$$

$$K(\mathrm{Pt}_{\mathbf{0}}\,\mathcal{A}) \to K(\mathcal{A}) : \left\langle \begin{array}{ccc} x & \twoheadrightarrow & y \\ & \searrow \quad \swarrow & \\ & \mathbf{0} & \end{array} \right\rangle \mapsto \langle x \twoheadrightarrow y\rangle \tag{B3.5}$$

which are clearly inverse to each other. □

**Proof of Theorem B1.4.** Consider the following pull-back diagram

$$\begin{array}{ccc} \bar{s} & \xrightarrow{\bar{e}} & \overline{m} \\ \downarrow & & \downarrow \\ s & \xrightarrow{e} & m \end{array} \tag{B3.6}$$

where $s$ is simple in $\mathrm{Pt}_m\,\mathcal{A}$, i.e., $e$ is indecomposable in $\mathcal{A}$. We have to show that $\bar{e}$ is indecomposable, as well. The morphism $\bar{e}$ is not an isomorphism since diagram (B3.6) is also a push-out diagram (see remark after Proposition A1.2). Now assume that $\bar{e}$ factorizes. Then we get the diagram

$$\begin{array}{ccccc} \bar{s} & \twoheadrightarrow & u & \twoheadrightarrow & \overline{m} \\ \downarrow & & & & \downarrow \\ s & & \twoheadrightarrow & & m. \end{array} \tag{B3.7}$$

Since $\mathcal{A}$ is an exact Mal'cev category, the quotient objects of $x$ form a *modular* lattice [7, Prop. 3.3]. We have $s \vee \overline{m} = \bar{s}$ since (B3.6) is a pull-back. The indecomposability of $s \twoheadrightarrow m$ implies either $u \wedge s = s$ or $u \wedge s = m$. Thus

$$u = u \wedge \bar{s} = u \wedge (s \vee \overline{m}) = (u \wedge s) \vee \overline{m} = \begin{cases} s \vee \overline{m} = \bar{s} \text{ or} \\ m \vee \overline{m} = \overline{m}. \end{cases} \tag{B3.8}$$

Thus $\bar{s}$ is simple as well.

(ii) If $m$ is a minimal object of $\mathcal{A}$ let $m\backslash\mathcal{A}$ be the coslice category of all arrows $m \to x$. Since any morphism $m \to x$ is necessarily unique this is a full subcategory of $\mathcal{A}$ with initial element $m$. It is clear that $m\backslash\mathcal{A}$ is again an exact Mal'cev category. Moreover, $\mathcal{A}$ is the union of all subcategories $m\backslash\mathcal{A}$ where $m$ runs through all minimal objects.

For a morphism $\overline{m} \twoheadrightarrow m$ of minimal objects consider the commutative diagram

$$\begin{array}{ccc} K(\mathrm{Pt}_m\,\mathcal{A}) & \xrightarrow{\sim} & K(m\backslash\mathcal{A}) \\ {\scriptstyle \Phi_{\overline{m}\twoheadrightarrow m}} \downarrow & & \downarrow \\ K(\mathrm{Pt}_{\overline{m}}\,\mathcal{A}) & \xrightarrow{\sim} & K(\overline{m}\backslash\mathcal{A}). \end{array} \tag{B3.9}$$

The two horizontal arrows are isomorphisms by Theorem B1.2. The right vertical arrow comes from the inclusion $m\backslash\mathcal{A} \subseteq \overline{m}\backslash\mathcal{A}$. It is easy to check that the diagram commutes. Thus, we have to show that

$$\lim_{m\in M(\mathcal{A})} K(m\backslash\mathcal{A}) \xrightarrow{\sim} K(\mathcal{A}). \tag{B3.10}$$

Let $e: x \twoheadrightarrow y$ be surjective. Then $e$ is in the image of $K(m\backslash\mathcal{A})$ with $m = x_{\min}$ showing that (B3.9) is surjective. On the other hand, let $e_i \in \mathfrak{E}(m_i\backslash\mathcal{A})$ with $\langle e_1\rangle = \langle e_2\rangle$ in $K(\mathcal{A})$. To show this inequality only finitely many objects and morphisms of $\mathcal{A}$ are needed. Thus, this equality holds already in $K(m\backslash\mathcal{A})$ with $m \in M(\mathcal{A})$ big enough. This shows that (B3.9) is injective. □

**Proof of Theorem B1.5.** Pull-back by the surjective morphism $\bar{e}$ reflects isomorphisms and preserves $\Delta$. Thus we may assume that $\bar{e}$ has a splitting $s$. Thus the diagram takes place in $\mathrm{Pt}_y\,\mathcal{A}$. By (B3.10) there is a minimal object $m \geqslant y_{\min}$ such that $e$ and $\bar{e}$ have the same image in $m\backslash\mathcal{A}$. This means that also the pull-backs of $e, \bar{e}$ by $m \to y$ have the same image in $\mathrm{Pt}_m\,\mathcal{A}$. On the hand, since $\mathcal{A}$ is protomodular, the pull-back functor $\mathrm{Pt}_y\,\mathcal{A} \to \mathrm{Pt}_m\,\mathcal{A}$ reflects isomorphisms. Thus it suffices to prove the assertion for $\mathrm{Pt}_m\,\mathcal{A}$, i.e., we may assume from now on that $\mathcal{A}$ is pointed and $y = 0$.

Consider a composition series (B2.1) of $x$. Then the $u_i := x_i \cap u$ form a normal series of $u$ with factors $z_i := \mathrm{image}(x_i \cap u \to y_i)$. I claim that $z_i = y_i$ for all $i$. For that, we construct a directed graph. The vertices are the $i$ with $z_i \neq y_i$. We draw an arrow $i \to j$ if $y_i$ is isomorphic to a composition factor of $z_i$. In that case

$$y_i \preccurlyeq z_j \preccurlyeq y_j. \tag{B3.11}$$

The assumption $\Delta(u \twoheadrightarrow \mathbf{0}) = \Delta(x \twoheadrightarrow \mathbf{0})$ means that $u$ and $x$ have the same composition factors. This implies that each vertex has at least one outgoing edge. Therefore, the graph must contain a directed cycle. If $j$ is part of such a cycle then Lemma 2.6 and (B3.11) imply $z_j = y_j$, contradicting the choice of $j$. This proves the claim.

Finally, we show by induction on $n$, the number of composition factors, that $u = x$. For that consider the following diagram

$$\begin{array}{ccccc} u_{n-1} & \rightarrowtail & u_n & \twoheadrightarrow & z_n \\ \downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow{\scriptstyle\sim} \\ x_{n-1} & \rightarrowtail & x_n & \twoheadrightarrow & y_n. \end{array} \tag{B3.12}$$

Here the left horizontal arrows are the kernels of the right horizontal ones. The left vertical arrow is an isomorphism by induction. The right vertical arrow is an isomorphism by what we showed above. Lemma A1.4 implies $u = u_n \xrightarrow{\sim} x_n = x$. □

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