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A Strengthening of Brooks' Theorem

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We show that for sufficiently large Δ, any graph with maximum degree at most Δ and no cliques of size Δ has a $\Delta-1$ colouring. © 1999 Academic Press

It is easy to see that if the maximum degree of G is Δ, then the chromatic number $\chi(G)$ is at most $\Delta+1$. In fact there is a simple recursive greedy procedure for colouring a graph G with $\Delta(G)+1$ colours. Having coloured $G-x$ we simply colour x with a colour not appearing on the (at most) Δ neighbours of $G-x$. Of course, this colouring may not be optimal. In fact, in 1941, Brooks [5] proved that $\chi(G)=\Delta+1$ precisely if some component of G is a $\Delta+1$-clique or $\Delta=2$ and G is not bipartite.

In [15], Reed showed that this result is just the tip of the iceberg. In fact, as the maximum clique size, $\omega(G)$, of G moves further away from $\Delta(G)+1$, so does the chromatic number. Specifically, we have:

Theorem 1. *There is a positive constant ε such that if $\Delta \geqslant 3$ then $\chi(G) \leqslant (1-\varepsilon)(\Delta(G)+1)+\varepsilon\omega(G)$.*

Theorem 2. *For every b there is a Δ_b such that if $\Delta(G) \geqslant \Delta_b$ and $\omega(G) \leqslant (\Delta(G)+1)-2b$ then $\chi(G) \leqslant (\Delta(G)+1)-b$.*

As shown in [15], the second theorem is tight up to a $o(b)$ term. More strongly, the following conjecture, if true, would also be essentially best possible:

Conjecture 3. $\chi(G) \leqslant \lceil \frac{1}{2}(\Delta(G)+1)+\frac{1}{2}\omega(G)\rceil$.

In this paper, we consider graphs for which $\omega(G) \leqslant \Delta-1$. Borodin and Kostochka [4] conjectured that if $\Delta(G) \geqslant 9$ and $\omega(G) \leqslant \Delta(G)-1$ then

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$\chi(G) \leqslant \Delta - 1$ (this is problem 4.8 in [7], and we refer readers to [7] for more details). Beutelspacher and Hering [3] independently conjectured that this statement holds for sufficiently large Δ. We prove their conjecture, i.e., we show:

THEOREM 4. *There is a Δ_0 such that if $\Delta(G) \geqslant \Delta_0$ and $\omega(G) \leqslant \Delta(G) - 1$ then $\chi(G) \leqslant \Delta(G) - 1$. In fact, $\Delta_0 = 10^{14}$ will do.*

The crux of the proof is dealing with cliques which have about Δ vertices. For example, applying Theorem 2, we see that Theorem 4 holds for graphs with no $\Delta - 3$ cliques. In fact, if we impose a stronger condition, that each vertex neighbourhood contains at most $\frac{9}{10}\binom{\Delta}{2}$ edges, then, as shown in [9], there is a $\frac{9999}{10000}\Delta$ colouring of G.

Three interesting earlier papers also point out that it is these large cliques with which we need to concern ourselves. First, Borodin and Kostochka [4] showed that if $\omega(G) \leqslant \Delta(G)/2$ then $\chi(G) \leqslant \Delta(G) - 1$. Next, Kostochka [8] showed that if $\omega(G) \leqslant \Delta(G) - 29$ then $\chi(G) \leqslant \Delta(G) - 1$. Finally, in 1987, Mozhan in his Ph.D. thesis showed that if $\omega(G) \leqslant \Delta(G) - 4$ and $\Delta \geqslant 31$ then $\chi(G) \leqslant \Delta(G) - 1$.

Our proof of Theorem 4 relies on the fact that if G is a minimal counterexample to the theorem, then any reasonably dense set of about Δ vertices in G is either a clique or the intersection of two cliques. More precisely, we show:

LEMMA 5. *If G is a minimal counter-example to Theorem 4 and H is a subgraph of G with at most $7\Delta/6$ vertices such that every vertex of H has at least $3\Delta/4$ neighbours in H then H is either a clique or consists of a clique C_H with less than $\Delta - 1$ vertices and a vertex v_H.*

We mention one corollary of this lemma which we prove separately:

LEMMA 6. *If G is a minimal counter-example to Theorem 4 of maximum degree Δ, then its $\Delta - 1$ cliques are disjoint.*

The proof we give yields a similar result of some independent interest:

LEMMA 7. *If G is a minimal counter-example to the Borodin–Kostochka conjecture of maximum degree Δ, then its $\Delta - 1$ cliques are disjoint.*

We will also need a strengthening of Lemma 6.

LEMMA 8. *If G is a minimal counter-example to Theorem 4 of maximum degree Δ, and K is a $\Delta - 1$ clique of G then no vertex of $G - K$ sees more than four vertices of K. Furthermore, at most four vertices of K have degree $\Delta - 1$.*

These results will allow us to handle the dense sets. We defer any more precise explanation of how we do so, because our explanation requires machinery developed in the proofs of Lemmas 5–8, presented in the next section.

Our technique for dealing with vertices with sparse neighbourhoods is that presented in [9]. It involves examining a naive colouring procedure which consists of choosing a random colour for each vertex with each possibility equally likely, uncolouring those involved in a conflict, and then completing the colouring, essentially by greedily extending the partial colouring. We shall see that, surprisingly, analyzing a slight variant of this procedure using simple but powerful probabilistic tools allows us to obtain the theorem stated above. The proof technique is similar to that in [15]. We refer the reader to [10–14] for others results in the same vein. In particular, [13] surveys a number of results using this or related techniques.

2. THE DENSE SETS

Proof of Lemma 6. Let G be a minimal counterexample to Theorem 4 of maximum degree Δ. We first observe:

9. *No two intersecting* $\Delta-1$ *cliques intersect in fewer than* $\Delta-3$ *vertices.*

To see this note that if two $\Delta-1$ cliques intersect in $\Delta-k$ vertices, then a vertex in their intersection has $\Delta+k-3$ neighbours in their union.

We next observe:

10. *No two* $\Delta-1$ *cliques intersect in* $\Delta-3$ *vertices.*

Proof. Suppose that there are two $\Delta-1$ cliques which intersect in a $\Delta-3$ clique C. Let a, b, c, d be the other four vertices in the union of these two $\Delta-1$ cliques and note that each of these four vertices is adjacent to every element of C. Now, there is no triangle in the graph F induced by a, b, c, d as otherwise this triangle along with C forms a Δ clique, a contradiction. Thus, F is bipartite and has maximum degree two. It follows that we can partition a, b, c, d into two pairs of non-adjacent vertices. By relabelling, we can assume that neither ab nor cd is an edge of G.

By the minimality of G (and Brooks' Theorem), we know that $H=G-C-a-b-c-d$ has a $\Delta-1$ colouring. We shall fix some such colouring and extend it to a colouring of G. To begin, we note that any vertex in $G-H$ has at most 2 neighbours in H because it is in a $\Delta-1$ clique in $G-H$. Thus, we can choose some colour i which does not appear on any of the at most four vertices of H which are incident to at least one of a or b, and assign this colour to both a and b . (Recall that $\Delta \geqslant 9$ so there are at least 8 colours in total.) Similarly, we can choose a colour different from i to assign to both c and d thereby obtaining a $\Delta - 1$ colouring of $G - C$. We can greedily extend this to a $\Delta - 1$ colouring of G because each vertex of C sees all four of a, b, c, and d, and only two colours are used on these four vortices. ∎

Finally, we prove:

11. *No two $\Delta - 1$ cliques intersect in $\Delta - 2$ vertices.*

Suppose there are two $\Delta - 1$ cliques which intersect in a $\Delta - 2$ clique C. Let x and y be the two other vertices in these two intersecting cliques. Note that x and y are non-adjacent since G contains no Δ clique.

If no vertex of C sees any of $G - C - x - y$. As above, we can extend this to a colouring of $G - C$ in which x and y have the same colour. Now, since each vertex of C has at most $\Delta - 1$ neighbours and sees both x and y, we can greedily complete this colouring to a $\Delta - 1$ colouring of G.

So, we assume there is a vertex z of $G - C - x - y$ adjacent to a vertex in C. If z sees all of C, then since G has no Δ clique, it misses both x and y. In this case, we $\Delta - 1$ colour $H - z$ and extend this colouring to a $\Delta - 1$ colouring of $G - C$ by colouring all of x, y, z with the same colour, one which appears on none of the at most six vertices of H adjacent to an element of this triple. We can now greedily extend this colouring to a $\Delta - 1$ colouring of G because each vertex of C sees all of $\{x, y, z\}$.

Thus, we can assume that z misses some vertex w in C. Note that w sees all of $C + x + y - w$ and hence has at most one neighbour in H. If z sees three or more vertices of C, then we will extend a $\Delta - 1$ colouring of $H - z$ to a $\Delta - 1$ colouring of G. We first colour z and w with some colour i which appears neither on any of the at most $\Delta - 3$ neighbours of z in H nor on the neighbour of w in H if such an animal exists. We next colour both x and y with a colour different from i which appears on none of the at most four vertices of H adjacent to at least one vertex of this pair. Finally, we colour the vertices of $C - w$ saving some neighbour v of z to colour last. When we come to colour a vertex u of $C - v$, there will be one uncoloured neighbour of $u : v$ and there will be a pair of neighbours of u with the same colour: $\{x, y\}$. Thus, we can greedily extend our colouring of $G - C$ to a $\Delta - 1$ colouring of $G - v$. Finally, v sees all of w, x, y, z but we used only two colours on these four vertices so we can actually extend our colouring to a $\Delta - 1$ colouring of G, a contradiction.

Thus, we can assume that z sees at most two vertices of C. We can also assume that every non-neighbour w of z has a neighbour in H. Otherwise, we can $\Delta - 1$ colour H, colour w with the same colour as z and extend this colouring to a colouring of G as we did in the last paragraph.

Our next step is to show that there is some vertex w of C not adjacent to z such that the graph H_{zw} obtained from H by adding an edge from z to the neighbour of w in H contains no Δ clique. To this end, note that if H_{zw} contains a Δ clique then $N_H(z)$ contains a $\Delta-2$ clique D, and the neighbour u of w in H sees all of D. If $N_H(z)$ contains a $\Delta-2$ clique, then by considering the degrees of the vertices in this clique, we see that there are at most 2 vertices of $H-z-N_H(z)$ which see at least $\Delta-2$ vertices of $N_H(z)$. Furthermore, each such vertex has at most two neighbours in C. Thus, there are at most four w in C not adjacent to z for which H_{zw} is a Δ clique. Since there are at least seven vertices in C and at most two of them are adjacent to z, it follows that we can choose some non-neighbour w of z in C such that H_{zw} contains no Δ clique.

Now, we note that $\Delta(H_{zw}) \leqslant \Delta$ as z has a neighbour in C and w has only one neighbour in H. Thus, by the minimality of G, we can $\Delta-1$ colour H_{zw}. This yields a $\Delta-1$ colouring of $H+w$ in which z and w have the same colour. As above, we can extend this to a $\Delta-1$ colouring of G if we first colour x and y with the same colour and save a neighbour v of z to colour last. This contradiction completes the proof of Lemma 6. ▌

Exactly the same proof yields Lemma 7. Essentially the same proof yields Lemma 8.

Proof of Lemma 8. Suppose there is a $\Delta-1$ clique K and a vertex not in K which has at least five neighbours in K. Then, we let x be a vertex of $G-K$ which has the maximum number of neighbours in K, note that this is at most $\Delta-3$. Let y be a non-neighbour of x in K, let $C=K-y$ and let D be the neighbourhood of x in C.

If no vertex of D sees any of $G-C-x-y$, then each vertex in D has degree $\Delta-1$. In this case, by the minimality of G, we can $\Delta-1$ colour $H=G-C-x-y$. We can extend this to a colouring of $G-C$ in which x and y have the same colour. We can then complete this to a $\Delta-1$ colouring of $G-D$, since each vertex in $C-D$ has two neighbours in D. Now, since each vertex of D has at most $\Delta-1$ neighbours and sees both x and y, we can greedily complete this colouring to a $\Delta-1$ colouring of G. Thus, we can assume there is a vertex z in $G-C-x-y$ adjacent to a vertex in D.

If z sees four or more vertices of D, then we will extend a $\Delta-1$ colouring of $H-z$ to a $\Delta-1$ colouring of G. To do so, we choose some non-neighbour w of z in C other than y. We first colour z and w with some colour i which appears neither on any the at most $\Delta-4$ neighbours of z in H nor on the at most two neighbours of w in H. We next colour both x and y with a colour different from i which appears on none of the at most $\Delta-3$ vertices of H adjacent to at least one vertex of this pair. Next, we colour the vertices of $C-D$, all of which have two uncoloured neighbours in D. Finally, we colour the vertices of D saving some neighbour v of z to colour last. When we come to colour a vertex u of $D-v$, there will be one uncoloured neighbour of $u: v$ and there will be a pair of neighbours of u with the same colour: $\{x, y\}$. Thus, we can greedily extend our colouring of $G-D$ to a $\Delta-1$ colouring of $G-v$. Finally, v sees all of w, x, y, z but we used only two colours on these four vertices so we can actually extend our colouring to a $\Delta-1$ colouring of G, a contradiction.

Thus, we can assume that z sees at most three vertices of D, and hence misses a vertex w in D. Now, if w has a neighbour u in H, we set $H'=H+zu$. Otherwise, we set $H'=H$. In either case, by Lemma 6, we know H' contains no Δ clique. Further, $\Delta(H') \leqslant \Delta$ as z and u have neighbours in C. Thus, by the minimality of G, we can $\Delta-1$ colour H'. This yields a $\Delta-1$ colouring of $H+w$ in which z and w have the same colour. As above, we can extend this to a $\Delta-1$ colouring of G if we first colour x and y with the same colour and save a neighbour v of z in D to colour last. This contradiction completes the proof of the first statement of Lemma 8.

Now, suppose there is some $\Delta-1$ clique C in G containing a set S of five vertices each with degree $\Delta-1$. Let v be a vertex of S, and let z be the neighbour of v outside C. Since z has at most four neighbours in C, there is at least one non-neighbour w of z in S. We can mimic the proof above to find a colouring of $G-C+w$ in which z and w have the same colour. We can complete the colouring if we colour v last and some other vertex of S second last. ▮

We turn now to the last result on dense sets stated in the introductory section.

Proof of Lemma 5. Consider a minimal counterexample G to Theorem 4, and a set H with at most $7\Delta/6$ vertices which induces a graph of minimum degree $3\Delta/4$. By our degree condition, for each pair $\{x, y\}$ of vertices of H, there must be at least $\Delta/3$ vertices in the set $S_{x,y}=N(x) \cap N(y) \cap H$.

Thus, if H has 4 disjoint pairs of nonadjacent vertices $\{(x_1, y_1), ..., (x_4, y_4)\}$, then there must be at least $\Delta/48>6$ vertices which are in two of the S_{x_i, y_i}. In particular this implies that H contains two vertices a and b such that $N(a) \cap N(b) \cap H$ contains two disjoint pairs of nonadjacent vertices, call them (x, y) and (v, w). By the minimality of G, we can $\Delta-1$ colour $G-H$. We can extend this to a colouring of $G-H+x+y+w+v$ in which (x, y) and (w, v) are pairs of vertices with the same colour, because of our degree condition on H. Set $S=a+b+(N(a) \cap N(b) \cap H)-v-w-x-y$. We next extend our colouring to a $\Delta-1$ colouring of $G-S$ by first colouring those vertices of $G-S$ which have at most one neighbour in S and then colouring the set T of vertices of $G-S$ which have at least two neighbours in S. We note that our degree condition ensures that $|S| \geqslant \Delta/3-2$. If $|S| \leqslant \Delta/2-1$ then each vertex of S must see at least $\Delta/4+2$ vertices of $H-S$ and hence there are at least $\Delta^2/12$ edges from S to $H-S$. In this case, since $|S| \leqslant \Delta/2$, we obtain: $|T| \geqslant \Delta/6$. Thus, in either case, $|S \cup T| \geqslant \Delta/2 - 2$. So, each vertex of $H - S - T$ has at least two neighbours in $S + T$ and there is no problem colouring them. Each vertex of T has two neighbours in S, so we can indeed extend our $\Delta - 1$ colouring to $G - S$. Now, we can colour all of $S - a - b$, because each vertex in this set sees both a and b. Finally, we can colour a and b because both their neighbourhoods contain the four vertices v, w, x, y on which we used only two colours. This is a contradiction.

So, we can assume that H has no four disjoint pairs of non-adjacent vertices. Hence by considering the clique obtained by deleting a maximum family of pairs of disjoint pairs of non-adjacent vertices of H, we see that if we let C_H be a maximum clique of H then $H - C_H$ has at most six vertices. If $H - C_H$ has at least two vertices then by the maximality of C_H, there are either two pairs of nonadjacent vertices in H or there is a stable set of size three in H. That is, there are four vertices of H which permit a two colouring, and more strongly, we see we can choose two of these vertices in C_H. In either case, we can colour $G - H$ with $\Delta - 1$ colours by the minimality of G, extend this by colouring some set X of four vertices of H including two in C_H with only two colours, then colour $H - C_H$, and finally colour C_H, saving two of the at least $\Delta/2 - 8$ vertices which see all of X to colour last. This will yield a $\Delta - 1$ colouring of G, a contradiction. So, we see that H does indeed consist of a clique C_H and a vertex v_H. Furthermore, if H is not a clique then by Lemma 8, C_H has at most $\Delta - 2$ vertices. ∎

For the rest of the paper, G is a minimal counter example to Theorem 4 with maximum degree Δ and $\mathscr{C}$ is the set of maximal cliques of G which contain at least $3\Delta/4 + 1$ vertices. We note that the minimality of G implies

12. *Every vertex of G has degree Δ or $\Delta - 1$.*

Our results imply:

13. *If two elements C_1 and C_2 of $\mathscr{C}$ with $|C_1| \leqslant |C_2|$ intersect, then $|C_1 - C_2| \leqslant 1$.*

Proof. By considering a vertex in the intersection of the two cliques, we see that their union contains at most $\Delta + 1$ vertices. Hence, we can apply Lemma 5 to the graph obtained from their union. ∎

14. *No element of $\mathscr{C}$ intersects two other elements of $\mathscr{C}$.*

Proof. By (13), these three elements of S would have to have a common intersection. Further, since a vertex in this common intersection sees all of the rest of the vertices in the graph H induced by their union, we see that H has at most $\Delta+1$ vertices. Applying Lemma 5 to H yields a contradiction. ∎

This result implies that we can partition $V(G)$ up into sets $S_1, ..., S_l$, $L = V(G) - \bigcup \{C \mid C \in \mathscr{C}\}$ so that each S_i is either a clique C_i of $\mathscr{C}$ or a clique C_i of $\mathscr{C}$ and a vertex v_i of $V - C_i$ which sees at least $3\Delta/4$ vertices of C_i. In the second case, we say S_i is a *near clique*.

We shall need the following results concerning this partition:

Lemma 15. *If v is a vertex in some C_i such that $|C_i| = \Delta - p$ then there is a most one neighbour of v outside C_i which sees more than $p+3$ vertices of C_i. Furthermore, if v has degree $\Delta - 1$ there are no such neighbours.*

Proof. Suppose, for a contradiction that there are two such vertices x and y. By the minimality of G, we can $\Delta - 1$ colour $G - C_i - x - y$. By (13) and (14), there must exist distinct w and z in C_i such that w misses x and z misses y. We can extend our colouring so as to give w and x the same colour and y and z the same colour. We can now complete our colour by colouring the vertices of $C_i - w - z$, saving v to colour last, and some neighbour of x to colour second last.

If v has degree $\Delta - 1$, a similar argument applies if v has a neighbour x in $G - C_i$ with at least $p+3$ neighbours in C_i. We first colour $G - C_i - x$, then colour x and w the same colour for some non-neighbour w of x in C_i and finally colour C_i, saving v for last, and some neighbour of x for second last. ∎

Lemma 16. *For any C_i, if $|C_i| = \Delta - p$ then we can find at least $\Delta/15$ disjoint triples each of which consists of a vertex v of C_i and two neighbours of v outside of C_i both of which have at most $p+3$ neighbours in C_i.*

Proof. Take, a maximal such set of triples, suppose it has k elements. Let S be the set of $2k$ vertices outside C_i contained in one of these triples. Let T be the set of k vertices of C_i contained in some triple. Since G has minimum degree at least $\Delta - 1$, the maximality of our triple set and lemma 15 imply that each vertex of $C_i - T$ has at least $p-1$ neighbours in S. Thus there are at least $(p-1)(\Delta - k)$ edges between S and $C_i - T$. On the other hand there are at most $2k(p+3)$ such edges. So, if p is at least 2, then the desired result holds. If $p=1$, then the result holds by Lemma 8, because all but four of the vertices of C_i see at least two vertices of $G - C_i$ each of which has degree at most four in C_i. Hence we can actually find a disjoint set of $(\Delta - 4)/7$ disjoint triples. ∎

Lemma 17. *If v is a vertex whose neighbourhood contains fewer than $\Delta^2/32 - \Delta/4$ pairs of non-adjacent vertices then it is in some S_i.*

Proof. If v is not in any S_i then by Lemma 5, there is some vertex v_0 in $N(v)$ which sees less than $3\Delta/4-1$ vertices of $N(v)$. More strongly, there is a sequence $v_0, ..., v_{\lfloor \Delta/4 \rfloor}$ of vertices of G such that v_i sees less than $3\Delta/4$ vertices of $N(v)-\{v_j \mid j<i\}$. The result follows. ∎

Now that we have a better grasp of what the dense sets in G look like, we can prove Theorem 4.

3. A PROOF SKETCH

Crucial to the proof is the following straightforward result.

LEMMA 18. *Any partial $\Delta-1$ colouring of G satisfying the three following conditions can be extended to a $\Delta-1$ colouring of G.*

(i) *for every vertex $v \in L$ there are at least 2 colours appearing twice in the neighbourhood of v,*

(ii) *for each near clique S_i, there are two uncoloured neighbours of v_i in C_i, and*

(iii) *for every C_i, there are two uncoloured vertices w_i and x_i of C_i whose neighbourhoods contain two repeated colours.*

Proof. Simply colour the uncoloured vertices one at a time. If S_i is a near-clique, then we colour v_i before colouring any of C_i, (ii) ensures this is possible. For any S_i, we colour w_i and x_i after all of the rest of C_i. This ensures that we can colour $C_i - w_i - x_i$ because all the vertices in this set see both w_i and x_i. We can colour w_i and x_i by (iii). We can colour the vertices of L because of (i). ∎

Now, we will attempt to find a partial $\Delta-1$ colouring satisfying conditions (i), (ii), and (iii) by analyzing the probabilistic procedure described in the introduction. That is, by considering a partial colouring obtained by colouring each vertex with a uniformly independently chosen colour and then uncolouring vertices which are involved in conflicts.

It turns out that ensuring that (i) is satisfied is straightforward. To introduce our technique, we first prove this result. So, consider a uniformly random partial $\Delta-1$ colouring of the vertices of G. Let v be a vertex of L.

Our first step is to examine the number of repeated colours we expect in $N(v)$. We let U_v be the set of pairs of vertices of $N(v)$ which receive the same colour and such that this colour appears nowhere else on $N(v)$. We let W_v be the set of colours used to colour these pairs. We let $Z_v = |U_v| = |W_v|$. Clearly this is at most the number of repeated colours on $N(v)$.

By Lemma 17, there are at least $\Delta^2/32-\Delta/4$ pairs of non-adjacent vertices in $N(v)$. For any particular such pair, $\{u, w\}$, both vertices receive the same colour with probability $1/(\Delta-1)$. If u, w both receive the same colour, then they will form one of the pairs counted by Z_v provided they both retain that colour in the partial colouring and no other vertex in $N(v)$ receives the colour, i.e., if and only if no other vertex in $N(u) \cup N(w) \cup N(v)$ receives that colour. The probability that no such vertex receives that colour is $(1-1/(\Delta-1))^{|N(u) \cup N(w) \cup N(v)|-3} \geqslant (1-1/(\Delta-1))^{3\Delta-3} \geqslant 3^{-3}$. It follows that $\mathbf{Exp}(Z_v) \geqslant (\Delta^2/32-\Delta/4) \times 1/(\Delta-1) \times 3^{-3} \geqslant \Delta/1000$.

Thus, for each vertex, the expected number of colours which appear twice in its neighbourhood in the partial colouring is high, and so it seems hopeful that with positive probability, every vertex will have two pairs of neighbours which recieve the same colour in the partial colouring. To prove this requires two more steps: First, we would like to show that for each v, Z_v is highly concentrated, i.e., that the probability that Z_v differs from its expected value by a significant amount is small. This will establish that for any one particular v, with high probability $N(v)$ will contain two repeated colours. The second step is to strengthen this statement, showing that with positive probability *every* vertex will have many such pairs in its neighbourhood.

The first step requires the use of the following:

19 (Azuma's Inequality [2]). *Let $Y = Y_1, Y_2, ..., Y_n$ be a sequence of random events. Let $X = X(Y_1, Y_2, ..., Y_n)$ be a random variable defined by these Y_i. If for each i,*

$$\max |\mathbf{Exp}(X \mid Y_1, Y_2, ..., Y_{i+1}) - \mathbf{Exp}(X \mid Y_1, Y_2, ..., Y_i)| \leqslant c_i$$

then the probability that $|X - \mathbf{Exp}(X)| > a$ is at most $2e^{-a^2/(2\sum c_i^2)}$.

Using which, we can obtain:

20. $\mathbf{Pr}(Z_v < 2) < \Delta^{-7}$.

Proof. To apply Azuma's Inequality, we must be careful about the order in which we assign the random colours to $V(G)$. We order the vertices of G as $w_1, ..., w_n$ so that the vertices adjacent to v are a suffix of this order. We let w_s be the last vertex of $V-N(v)$ under this order.

For each of these choices we now bound the potential effect on $\mathbf{Exp}(Z_v)$. Note that changing the colour of w_i from j to k will not affect $W_v - j - k$. Thus, choosing a different colour for w_i can affect the value of $_v$ by at most 2. Furthermore, for each w_i not in $N(v)$, the probability that changing w_i's colour will affect Z_v is at most the probability that one of the two colours assigned to w_i is also assigned to one of its neighbours in $N(v)$. If we denote by d_i the number of neighbours w_i has in N_v, then this is at most $2d_i/(\Delta-1)$. Thus, the maximum effect that colouring such a w_i can have on the expected value of Z_v is $c_i = 4(d_i/(\Delta-1))$. Clearly $\sum_{i=1}^{s} d_i \leqslant \Delta^2$, and so $\sum_{i=1}^{s} c_i \leqslant 4\Delta+5$. Furthermore, each c_i is at most 5, and so $\sum_{i=1}^{s} c_i^2 < 21\Delta$. Thus, the sum of all the c_i^2 is at most 25Δ. Our claim now follows immediately from Azuma's Inequality, by setting $a = \Delta/1000 - 2$. ▮

The second step in our proof that (i) holds requires:

Lemma 21 (The Lovász Local Lemma [6]). *Consider a set $\mathscr{E}$ of (typically bad) events such that for each $A \in \mathscr{E}$*

(1) $\mathbf{Pr}(A) \leqslant p$, *and*

(2) *A is mutually independent of a set of all but at most d of the other events.*

If $\mathrm{e}p(d+1) < 1$ *then with positive probability, none of the events in $\mathscr{E}$ occur.* (*Here* $\mathrm{e} = 2.71\ldots$.)

Now, for each vertex v in L, we let A_v be the event that there are fewer than two repeated colours in v's neighbourhood. By (20), each A_v holds with probability less than Δ^{-7}. So, the Local Lemma will imply that no A_v holds, i.e., that there is a colouring satisfying (i), provided we can show that each A_v is mutually independent of a set of all but at most Δ^6 other A_v. But, it is easy to see that each A_v is mutually independent of the set

$$\{A_u \mid \textit{there is no } uv \textit{ path with at most four edges}\}.$$

The complement of this set has size less than $\Delta^4 + 1$. So, we are done.

The rest of the proof uses two similar applications of our two probabilistic tools.

4. SOME MORE DETAILS

We now complete the proof of Theorem 4. To do so, we need to define two new kinds of events. For each near-clique S_i, we let E_i be the event that (ii) fails to hold on S_i. For each S_i, we let F_i be the event that (iii) fails to hold on C_i. We note that if none of the events in the set $\mathscr{E} = (\bigcup A_v) \cup (\bigcup E_i) \cup (\bigcup F_i)$ hold then the random colouring satisfies (i), (ii), and (iii) of Lemma 18. To finish the proof we use the Lovász Local Lemma to show that this occurs with positive probability.

We note that each E_i depends only on the colour of the vertices in S_i and within distance one of it. Similarly, each F_i depends only on the colour of the vertices in S_i and within distance two of it. It follows that each event in $\mathscr{E}$ is independent of a set of all but at most $3\Delta^5$ other events. So, we need only show that each event in $\mathscr{E}$ holds with probability at most Δ^{-6}.

We have already established this bound for the A_v. We note that for each near clique S_i, there is a set R_i of $3\Delta/4$ neighbours of v_i in C_i. If E_i is to hold, then we must have assigned at least $3\Delta/4-2$ different colours to R_i. However, the expected number of colours we assign to R_i is less than $(\Delta-1)(1-1/e)$, which exceeds the number of colours we expect to assign to a set of size $\Delta-1$. Furthermore, the number of colours N_i assigned to R_i depends only on the colour choices for the elements of R_i and each choice can affect the total number of colours used by at most 1. So, applying Azuma's inequality, we see that $\mathbf{Pr}(|N_i - Ex(N_i)| > t) < e^{-t^2/\Delta}$. Setting $t = \frac{3}{4}\Delta - (1-1/e)(\Delta-1) - 2$ yields that each E_i occurs with probability less than Δ^{-6}.

To compute the probability bound on F_i, we consider the set $\mathscr{T}_i$ of $\Delta/15$ disjoint triples guaranteed to exist by Lemma 16. We let M_i be the number of these triples for which the vertex in C_i is uncoloured, both the other vertices are coloured with a colour which is also used to colour a vertex of C_i, and no vertex of the triple is assigned a colour assigned to any vertex in any of the other triples. This last condition is present to ensure that changing the colour of a vertex can only affect the value of M_i by two.

To begin, we compute the expected value of M_i. To this end, we let T_i be the union of the vertex set of the triples in $\mathscr{T}_i$. We note that M_i counts the number of triples (a, b, c) in $\mathscr{T}_i$ with $c \in C_i$ such that there are colours j, k, l and vertices x, y, z with $x \in C_i - T_i - N(a)$, $y \in C_i - T_i - N(b)$, $z \in N(c) - T_i$, such that

(1) j is assigned to a and x but to none of the rest of $T_i \cup N(a) \cup N(x)$,

(2) k is assigned to b and y but to none of the rest of $T_i \cup N(b) \cup N(y)$,

(3) l is assigned to z and c but on none of the rest of T_i.

To begin, we fix a triple $\{a, b, c\}$ in $\mathscr{T}_i$. We let $A_{j,k,l,x,y,z}$ be the event that (1), (2), and (3) hold. This is clearly at least $(\Delta-1)^{-6}\, 1/e^5$. Furthermore, two such events with different sets of indices are disjoint. Now, there are $13\Delta/30$ choices for both x and y. There are at least $4\Delta/5$ choices for z and $(\Delta-1)(\Delta-2)(\Delta-3)$ choices for distinct j, k, l. So, a straightforward calculation shows that the probability that (1), (2), and (3) hold for some choice of $\{j, k, l, x, y, z\}$ is at least $(\Delta-1)^{-6}(\Delta-1)(\Delta-2)(\Delta-3)(13\Delta^2/30)(4\Delta/5)(1/e^5) \geqslant 1/7e^5$. Since, there are $\Delta/15$ triples in $\mathscr{T}_i$, M_i is at least $(\Delta/15)(1/7e^5) \geqslant \Delta/105e^5$.

We now prove that M_i is concentrated around this mean. We will colour the vertices of $V-T_i$ first and then the vertices of T_i. We note that changing the colouring of a vertex w_j of $V-T_i$ with d_j will only affect M_i if some neighbour of w_j in T_i receives the same colour as one of the two colours chosen for w_j. This occurs with probability at most $2d_j/\varDelta$ where d_j is the number of neighbours of w_j in T_i. Hence the expected change c_j in M_i which occurs when we colour w_j is at most $4d_j/\varDelta$. Thus, since the d_j sum to at most $\varDelta^2/5$ and each c_j is at most 4/5, we see that the sum of c_j^2 over w_j not in T_i is at most $4\varDelta/25$. Thus, the sum of all the c_j^2 is at most $\varDelta$. Applying, Azuma's inequality with $t=\varDelta/105e^5-2$ yields the desired result.

5. SOME CONCLUDING REMARKS

We note that fairly straightforward modifications yield the value $\varDelta_0=10^8$. We are certain that the proof can be modified to yield $\varDelta_0=10^6$. We expect that a more careful analysis could bring the bound on $\varDelta_0$ down to 1000. However, we doubt if the proof technique can be used to prove any bound better than $\varDelta_0=100$.

On the other hand, Lemma 7 may prove useful in resolving the conjecture. For example, suppose the following were true:

22. *Every graph G of maximum degree nine all of whose cliques of size eight are disjoint, can be partitioned into two graphs of maximum degree four, neither containing a clique of size five* (*and hence by Brook's Theorem, such a G is eight colourable*).

Then, no minimal counterexample to the Borodin–Kostochka conjecture could have maximum degree nine, which by a result of Kostochka, Borodin, and Toft, implies that the conjecture holds. This would be a very attractive way of proving the conjecture. Even if (22) is not true, it may be possible to prove the Borodin–Kostochka conjecture using a similar statement where the hypotheses are strengthened (i.e., by considering the intersections of the cliques of size seven).

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REFERENCES

1. N. Alon and J. Spencer, "The Probabilistic Model," Wiley, New York, 1992.
2. K. Azuma, Weighted sums of certain dependent random variables, *Tôku Math. J.* **19** (1967), 357–367.
3. A. Beutelspacherf and P. Hering, Minimal graphs for which the chromatic number equals the maximal degree, *Ars Combin.* **18** (1984), 201–216.
4. O. Borodin and A. Kostochka, On an upper bound on a graph's chromatic number, depending on the graph's degree and density, *J. Combin. Theory Ser. B* **23** (1977), 247–250.
5. R. L. Brooks, On colouring the nodes of a network, *Math. Proc. Cambridge Philos. Soc.* **37** (1941), 194–197.
6. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *in* "Infinite and Finite Sets" (A. Hajnal *et al.*, Eds.), Colloq. Math. Soc. J. Bolyai, Vol. 11, pp. 609–627, North-Holland, Amsterdam, 1975.
7. T. Jensen and B. Toft, "Graph Coloring Problems," Wiley, New York, 1995.
8. A. Kostochka, Degree, clique number, and chromatic number, *Metody Diskret. Anal.* **35** (1980), 45–70. [In Russian]
9. M. Molloy and B. Reed, A bound on the strong chromatic index of a graph, *J. Combin. Theory Ser. B* **69** (1997), 103–109.
10. M. Molloy and B. Reed, A bound on the total chromatic number, *Combinatorica* **18** (1998), 241–280.
11. M. Molloy and B. Reed, Assymptotically better list colourings, manuscript.
12. M. Molloy and B. Reed, An algorithmic version of the Lovasz local lemma, *in* "Proceedings of STOC, 1988."
13. M. Molloy and B. Reed, Graph colouring via the probabilistic method, to appear.
14. M. Molloy and B. Reed, Colouring graphs whose chromatic number is near their maximum degree, "Proceedings of LATIN, 1998," Lecture Notes in Comput. Sci., Vol. 1380, pp. 216–225, Springer-Verlag, New York/Berlin, 1998.
15. B. Reed, χ, Δ, and w, *J. Graph Theory* **27** (1998), 177–213.