A generalization of some inequalities for the gamma function

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Abstract

Laforgia (1984) obtained some inequalities of the type

\[
\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} \geq \frac{1}{(k + \alpha)^{1-\lambda}} \quad \text{or} \quad \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} \leq \frac{1}{(k + \alpha)^{1-\lambda}},
\]

according to the values of the positive parameters \(\alpha\) and \(\lambda\), valid for every non-negative real value of \(k\), or at least for \(k\) greater than or equal to a \(k_0\) depending on \(\alpha\) and \(\lambda\). In this paper a complete analysis of the problem is carried out, in order to establish, for fixed \(\alpha\) and \(\lambda\), which of the two former inequalities holds, and for which values of \(k\). © 1997 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many authors studied inequalities for the ratio \(\Gamma(k + \lambda)/\Gamma(k + 1)\), where \(\lambda\) is a positive parameter. In [2] Gautschi, seeking inequalities for the function \(\int_{x}^{+\infty} e^{-t} dt\), proved that

\[
\frac{1}{(k + 1)^{1-\lambda}} \leq \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \frac{1}{k^{1-\lambda}},
\]

where \(0 < \lambda < 1\) and \(k = 1, 2, 3, \ldots\); in the particular case \(\lambda = \frac{1}{2}\), Watson [5] showed that

\[
\frac{\Gamma(k + \frac{1}{2})}{\Gamma(k + 1)} > \frac{1}{(k + 4/\pi - 1)^{1/2}},
\]

which holds for every real \(k > 1\).

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In [4] Lorch found some interesting inequalities for the ultraspherical polynomials; to do this, he first proved that the sequence

$$\beta_k = \frac{(2k + \lambda)^{1-\lambda} \Gamma(k + \lambda)}{\Gamma(k + 1)}$$

where $k$ is a non-negative integer and $0 < \lambda < 1$, converges to the limit $2^{1-\lambda}$ and increases as $k \to \infty$; so he proved that

$$\frac{1}{(k + \lambda)^{1-\lambda}} < \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \frac{1}{(k + \lambda/2)^{1-\lambda}},$$

for $0 < \lambda < 1$ and $k = 0, 1, 2, \ldots$

Indeed, the upper bound in formula (4) directly follows from the inequality

$$\beta_k = \frac{(2k + \lambda)^{1-\lambda} \Gamma(k + \lambda)}{\Gamma(k + 1)} < 2^{1-\lambda};$$

whereas the lower bound is obtained in a similar way, precisely by proving that the sequence

$$\gamma_k = (k + \lambda)^{1-\lambda} \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)}$$

converges to 1 and decreases as $k \to \infty$.

The results obtained by Laforgia in [3], concerning the more general inequalities

$$\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + \alpha)^{\lambda-1} \text{ and } \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > (k + \alpha)^{\lambda-1},$$

where $\alpha$ and $\lambda$ are two positive parameters, were found using a technique similar to Lorch's, but for real values of $k$, not only integer. For example, Laforgia proved that

$$\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( \frac{k + \lambda}{2} \right)^{\lambda-1} \text{ for } 0 < \lambda < 1 \text{ or } \lambda > 2, \ k \geq 0;$$

$$\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \left( \frac{k + \lambda}{2} \right)^{\lambda-1} \text{ for } 1 < \lambda < 2, \ k \geq 0.$$

In some cases the inequality only holds for $k \geq k_0$ ($k_0$ depending on $\alpha$ and $\lambda$); for example

$$\alpha = \frac{2}{3} \lambda \Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \left( \frac{k + \lambda}{2} \right)^{\lambda-1} \text{ for } 0 < \lambda < 1, \ k \geq 1;$$

$$\alpha = \frac{\lambda}{2} + \frac{1}{10} \Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( \frac{k + \frac{\lambda}{2} + \frac{1}{10}}{2} \right)^{\lambda-1} \text{ for } 1 < \lambda < 2, \ k \geq 1.$$
In this paper the problem is completely solved, i.e. if the positive parameters \( \alpha \) and \( \lambda \) are given, we may know which of the two inequalities (5) holds, for \( k \geq 0 \) or for \( k \geq k_0 \). The following technique will be used: let us start from the limit

\[
\lim_{k \to \infty} k^{b-a} \frac{\Gamma(k+a)}{\Gamma(k+b)} = 1,
\]

(6)

see [1, p. 257, formula (6.1.46)], which holds for positive \( a \) and \( b \). Then, if we define the sequence

\[
f_k = \frac{\Gamma(k+\lambda)}{\Gamma(k+1)}(k+\alpha)^{1-\lambda},
\]

where \( \alpha \) and \( \lambda \) are two positive parameters and \( k = 0, 1, 2, 3, \ldots \), it easily follows from (6) that \( \lim_{k \to \infty} f_k = 1 \), independent of \( \alpha \) and \( \lambda \). So, if for our choice of the parameters \( f_k \) is increasing, then \( f_k < 1 \) for any \( k \in \mathbb{N}_0 \), and the inequality

\[
\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} \leq \frac{1}{(k+\alpha)^{1-\lambda}}
\]

(7)

holds for every \( k \in \mathbb{N}_0 \); and if \( f_k \) is ultimately increasing, then \( f_k < 1 \) at least for \( k \) greater than or equal to a suitable \( k_0 \), so that (7) holds for \( k \geq k_0 \). Similarly, if \( f_k \) is decreasing (ultimately decreasing), \( f_k > 1 \) for any \( k \in \mathbb{N}_0 \) (for \( k \geq k_0 \)). Therefore we have

\[
\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} > \frac{1}{(k+\alpha)^{1-\lambda}}
\]

(8)

for any \( k \in \mathbb{N}_0 \) (for \( k \geq k_0 \)). For \( \lambda = 1 \) the two inequalities trivially reduce to \( 1 = 1 \): for this reason we suppose \( \lambda \neq 1 \) in what follows.

It is not easy to study the monotonicity of \( f_k \); However, we may consider the sequence

\[
g_k = f_{k+1}/f_k,
\]

which also tends to 1 as \( k \to \infty \): if \( g_k < 1 \) then \( f_k \) is decreasing, while if \( g_k > 1 \) then \( f_k \) is increasing. Due to the functional equation \( \Gamma(z+1) = z\Gamma(z) \), we see that \( g_k \) does not contain the gamma function explicitly:

\[
g_k = \frac{k+\lambda}{k+1} \left( \frac{k+\alpha+1}{k+\alpha} \right)^{1-\lambda}.
\]

(9)

Let us write \( G(k) \) instead of \( g_k \), considering \( k \) as a continuous variable, \( k \in [0, +\infty) \). It is to be noted that in this case \( G(k) > 1 \) does not imply that the function

\[
F(k) = \frac{\Gamma(k+\lambda)}{\Gamma(k+1)}(k+\alpha)^{1-\lambda}
\]

is increasing (see remark at the end of the paper). Nevertheless, the technique is also suitable for real values of \( k \). Since \( \lim_{k \to +\infty} G(k) = 1 \), we may state that if \( G'(k) > 0 \) in \( (0, +\infty) \), it follows that

\[ G \text{ increasing in } [0, +\infty) \Rightarrow G(k) < 1 \Rightarrow F(k) > 1 \Rightarrow \text{inequality (8) for } k \geq 0, \]
while if \( G'(k) > 0 \) from a value \( k_0 \) on, inequality (8) holds at least for \( k \geq k_0 \). Likewise, if \( G'(k) < 0 \) in \((0, +\infty)\) (or ultimately) we have inequality (7) for \( k \geq 0 \) (or for \( k \geq k_0 \)).

Now,
\[
G'(k) = \frac{(1 - \lambda)((2\alpha - \lambda)k - \lambda + \alpha^2 + \alpha)}{(k + \alpha)^2 - \lambda(k + 1)^2(k + \alpha + 1)^2}.
\]

So we have to study the sign of the function \( A_{\lambda,\alpha}(k) = (1 - \lambda)((2\alpha - \lambda)k - \lambda + \alpha^2 + \alpha) \).

2. Main result

We have seen in the introduction that inequality (7) or (8) holds if the function \( A_{\lambda,\alpha}(k) \) is, respectively, negative or positive. Now we have to consider the three cases

(i) \( 2\alpha - \lambda = 0 \),
(ii) \( 2\alpha - \lambda > 0 \),
(iii) \( 2\alpha - \lambda < 0 \).

Case (i) has been already considered in [3]: if \( 2\alpha - \lambda = 0 \) we have
\[
A_{\lambda,\alpha/2}(k) = \frac{(1 - \lambda)\lambda(\lambda - 2)}{4},
\]
which is positive for \( 1 < \lambda < 2 \) and negative for \( 0 < \lambda < 1 \) or \( \lambda > 2 \). Hence,

- for \( 1 < \lambda < 2 \)
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \left( k + \frac{\lambda}{2} \right)^{\lambda - 1}
  \]
  for every real non-negative \( k \);
- for \( 0 < \lambda < 1 \) and for \( \lambda > 2 \)
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( k + \frac{\lambda}{2} \right)^{\lambda - 1}
  \]
  for every \( k \geq 0 \). \( \lambda = 2 \) has no interest, because in that case \( k + 1 = k + 1 \).

In order to study cases (ii) and (iii) let us define \( P_{\lambda,\alpha}(k) = (2\alpha - \lambda)k - \lambda + \alpha^2 + \alpha \); if \( 2\alpha - \lambda > 0 \) it is \( P_{\lambda,\alpha}(k) \geq 0 \) for \( k \geq (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda) \), while if \( 2\alpha - \lambda < 0 \) \( P_{\lambda,\alpha}(k) \leq 0 \) for \( k \leq (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda) \), i.e., \( P_{\lambda,\alpha}(k) \leq 0 \) for \( k \geq (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda) \). So we have to study the sign of the function \( H(\alpha, \lambda) = (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda) \) in the set \( Q \) of the \( \alpha, \lambda \)-plane in which \( \alpha > 0 \) and \( \lambda > 0 \).

The two curves \( \lambda = 2\alpha \) and \( \lambda = \alpha^2 + \alpha \) divide the set \( Q \) in four regions, which are denoted in the figure by I, II, III and IV. Besides, the straight line \( \lambda = 1 \) further divides in two parts each of the regions I, III and IV. So we have the results given in Fig. 1.
Let \((\alpha, \lambda)\) be a point of region I, which includes the arc of the parabola \(\lambda = \alpha^2 + \alpha\) with \(\alpha > 1\): we have \(2\alpha - \lambda < 0\) and \(\lambda - \alpha^2 - \alpha \geq 0\), so \(H(\alpha, \lambda)\) is non-positive. Since we consider only non-negative values of \(k\), we may state that \(P_{\lambda, a}(k) \leq 0\) for every \(k \geq 0\). Finally

- for each pair \((\alpha, \lambda)\) such that \(0 < \alpha < \frac{1}{2}\) and \(2\alpha < \lambda < 1\), \(A_{\lambda, a}(k) < 0\). Therefore

\[
\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \frac{1}{(k + \alpha)^{1-\lambda}} \quad \text{for every } k \geq 0;
\]

- for each pair \((\alpha, \lambda)\) such that \((0 < \alpha < \frac{1}{2} \text{ and } \lambda > 1)\) or \((\frac{1}{2} < \alpha < 1 \text{ and } \lambda > 2\alpha)\) or \(\alpha > 1 \text{ and } \lambda \geq \alpha^2 + \alpha\) \(A_{\lambda, a}(k) \geq 0\). Therefore

\[
\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > (k + \alpha)^{4-1} \quad \text{for every } k \geq 0.
\]

Let \((\alpha, \lambda)\) be a point of region II. Here we have \(2\alpha < \lambda < \alpha^2 + \alpha\) with \(\alpha > 1\). So \(H(\alpha, \lambda) > 0\). Since \(P_{\lambda, a}(k) < 0\) for every \(k > (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda)\), we have

- for each pair \((\alpha, \lambda)\) such that \(\alpha > 1 \text{ and } 2\alpha < \lambda < \alpha^2 + \alpha\) \(A_{\lambda, a}(k) \geq 0\) for every \(k\) greater than or equal to a \(k_0\) depending on \(\alpha\) and \(\lambda\). Therefore

\[
\frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > (k + \alpha)^{4-1} \quad \text{ultimately, i.e. for every } k \geq k_0 = H(\alpha, \lambda) = \frac{\lambda - \alpha^2 - \alpha}{2\alpha - \lambda};
\]

it is to be noted that we cannot find an uniform bound for \(k\), since \(H(\alpha, \lambda)\) is upperly unbounded in region II (e.g. for \(\alpha = 3, \lambda = 6.00599\) we have \(k_0 > 1000\)).

Let \((\alpha, \lambda)\) be a point of region III, which includes the arc of the parabola \(\lambda = \alpha^2 + \alpha\) with \(0 < \alpha < 1\); for \(0 < \alpha < 1, 0 < \lambda \leq \alpha^2 + \alpha < 2\alpha\); whereas for \(\alpha \geq 1, 0 < \lambda < 2\alpha < \alpha^2 + \alpha\). Anyway, \(H(\alpha, \lambda) \leq 0\), so \(P_{\lambda, a}(k) \geq 0\) for every \(k \geq 0\). Therefore, taking into account that the straight line
\(\lambda = 1\) meets the arc of the parabola \(\lambda = x^2 + x\) in the point \(F\) of abscissa \(\varphi = (\sqrt{5} - 1)/2 \approx 0.618\), we have

- for each pair \((x, \lambda)\) such that \(0 < x < \varphi\) and \(0 < \lambda \leq x^2 + x\), \(A_{\lambda,a}(k) > 0\). Therefore
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \frac{1}{(k + x)^{x-1}} \quad \text{for } k \geq 0;
  \]

- for each pair \((x, \lambda)\) such that \(x \geq \varphi\) and \(0 < \lambda < 1\), \(A_{\lambda,a}(k) > 0\). Therefore
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \frac{1}{(k + x)^{x-1}} \quad \text{for } \bar{k} \geq 0;
  \]

- for each pair \((x, \lambda)\) such that \(\varphi < x \leq 1\) and \(1 < \lambda \leq x^2 + x\), \(A_{\lambda,a}(k) \leq 0\). Therefore
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + x)^{x-1} \quad \text{for } k \geq 0;
  \]

- for each pair \((x, \lambda)\) such that \(x > 1\) and \(1 < \lambda < 2x\), \(A_{\lambda,a}(k) < 0\). Therefore
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + x)^{x-1} \quad \text{for } k \geq 0.
  \]

Finally, let \((x, \lambda)\) be a point of region IV: here we have \(0 < x < 1\) and \(2x < \lambda < x^2 + x\). Since \(H(x, \lambda) > 0\), we have \(P_{\lambda,a}(k) > 0\) for \(k > \frac{\lambda - x^2 - x}{2x - \lambda}\). So

- for each pair \((x, \lambda)\) such that \(0 < x \leq \frac{1}{2}\) and \(x^2 + x < \lambda < 2x \leq 1\), \(A_{\lambda,a}(k) > 0\) for \(k \geq k_0\). So
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \frac{1}{(k + x)^{x-1}} \quad \text{for } k \geq k_0 = H(x, \lambda) = \frac{\lambda - x^2 - x}{2x - \lambda};
  \]

- for each pair \((x, \lambda)\) such that \(\frac{1}{2} \leq x < \varphi\) and \(x^2 + x < \lambda < 1\), \(A_{\lambda,a}(k) > 0\) for \(k \geq k_0\). So
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \frac{1}{(k + x)^{x-1}} \quad \text{for } k \geq k_0;
  \]

- for each pair \((x, \lambda)\) such that \(\frac{1}{2} < x \leq \varphi\) and \(1 < \lambda < 2x\), \(A_{\lambda,a}(k) < 0\) for \(k \geq k_0\). So
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + x)^{x-1} \quad \text{for } k \geq k_0;
  \]

- for each pair \((x, \lambda)\) such that \(\varphi < x < 1\) and \(x^2 + x < \lambda < 2x\), \(A_{\lambda,a}(k) < 0\) for \(k \geq k_0\). So
  \[
  \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + x)^{x-1} \quad \text{for } k \geq k_0.
  \]

Similar to what happens for the points of region II, in these four subcases of region IV we cannot give an uniform bound for \(k\), since \(H(x, \lambda)\) is upperly unbounded.
After all that, the inequality $\frac{F(k + 2)}{F(k + 1)} > 1/(k + \lambda)^{1-\lambda}$ holds in the cases
- $0 < \alpha \leq \frac{1}{2} \land \lambda > 1$ ($k \geq 0$);
- $\frac{1}{2} < \alpha \leq 1 \land \lambda > 2\alpha$ ($k \geq 0$);
- $\lambda > 1 \land \lambda \geq \alpha^2 + \alpha$ ($k \geq 0$);
- $\lambda > 1 \land 2\alpha < \lambda < \alpha^2 + \alpha$ ($k \geq k_0$);
- $0 < \alpha < \phi \land 0 < \lambda \leq \alpha^2 + \alpha$ ($k \geq 0$);
- $\lambda > \phi \land 0 < \lambda < 1$ ($k \geq 0$);
- $0 < \alpha \leq \frac{1}{2} \land \alpha^2 + \alpha < \lambda < 2\alpha$ ($k \geq k_0$);
- $\frac{1}{2} \leq \alpha < \phi \land \alpha^2 + \alpha < \lambda < 1$ ($k \geq k_0$);
- $\lambda = 2\alpha \land 1 < \lambda < 2$ ($k \geq 0$),

while the inequality $\frac{F(k + 2)}{F(k + 1)} < 1/(k + \lambda)^{1-\lambda}$ holds in the cases
- $0 < \alpha < \frac{1}{2} \land 2\alpha < \lambda < 1$ ($k \geq 0$);
- $\phi < \alpha \leq 1 \land 1 < \lambda \leq \alpha^2 + \alpha$ ($k \geq 0$);
- $\lambda > 1 \land 1 < \lambda < 2\alpha$ ($k \geq 0$);
- $\frac{1}{2} < \alpha \leq \phi \land 1 < \lambda < 2\alpha$ ($k \geq k_0$);
- $\phi < \alpha < 1 \land \alpha^2 + \alpha < \lambda < 2\alpha$ ($k \geq k_0$);
- $\lambda = 2\alpha \land 0 < \lambda < 1$ ($k \geq 0$);
- $\lambda = 2\alpha \land \lambda > 2$ ($k \geq 0$).

These results are summarized in Fig. 2 in which "d" means that the inequality with the sign $>$ or $<$ holds for $k \geq k_0$.

3. Some particular cases

Let $\alpha = \frac{3}{2} \lambda$: the line $\lambda = \frac{3}{2} \alpha$ lies in part in the "lunule" $OBCFO$, from $O$ to the point $K = (\frac{1}{2}; \frac{3}{4})$, in part in the region $HGFOI$, from $K$ to $(\frac{3}{4}, 1)$, and a half-line in $ECFGH$, from $(\frac{3}{4}, 1)$ on. Hence we have the following results, which the reader can partly find in [3]:
- for $0 < \lambda < \frac{3}{4}$ $\Rightarrow \frac{F(k + \lambda)}{F(k + 1)} > (k + \frac{3}{2} \lambda)^{1-\lambda}$ for $k \geq k_0 = (\lambda - \alpha^2 - \alpha)/(2\alpha - \lambda) = 1 - \frac{3}{4} \lambda$; since $0 < \lambda < \frac{3}{4}$, we also have $0 < k_0 < 1$, so the inequality holds at least for $k \geq 1$, independent of $\lambda$;
- for $\frac{3}{4} < \lambda < 1$ $\Rightarrow \frac{F(k + \lambda)}{F(k + 1)} > (k + \frac{3}{2} \lambda)^{1-\lambda}$ for $k \geq 0$;
- for $\lambda > 1$ $\Rightarrow \frac{F(k + \lambda)}{F(k + 1)} < (k + \frac{3}{2} \lambda)^{1-\lambda}$ for $k \geq 0$.

Now let us consider the case $\alpha = \frac{3}{2} \lambda + \frac{1}{8}$, that is $\lambda = 2\alpha - \frac{1}{4}$: this straight line can be divided in a segment which lies in the region $HGFOI$, from the point $(\frac{9}{8}, 0)$ to the point $(\frac{9}{8}, 1)$ (note that it is tangent to the parabola $\lambda = \alpha^2 + \alpha$ at the point $K = (\frac{1}{2}, \frac{3}{4})$; this causes no problem, since the arc $OF$ belongs to the region $HGFOI$) and in a half-line which lies in the region $ECFGH$, from $(\frac{9}{8}, 1)$ on. Hence
- for $0 < \lambda < 1$ $\Rightarrow \frac{F(k + \lambda)}{F(k + 1)} > (k + \frac{3}{2} \lambda + \frac{1}{8})^{1-\lambda}$ for $k \geq 0$;
- for $\lambda > 1$ $\Rightarrow \frac{F(k + \lambda)}{F(k + 1)} < (k + \frac{3}{2} \lambda + \frac{1}{8})^{1-\lambda}$ for $k \geq 0$.

The last inequality was also found by Laforgia [3] for $1 < \lambda < 2$.

Let us see the case $\alpha = \frac{1}{2} \lambda + \frac{1}{16}$, i.e., $\lambda = 2\alpha - \frac{1}{4}$. This straight line meets the parabola $\lambda = \alpha^2 + \alpha$ in the two points $(5 - \sqrt{5})/10, (4 - \sqrt{5})/5 \cong (0.2764, 0.3528)$ and $(5 + \sqrt{5})/10, (4 + \sqrt{5})/5 \cong (0.7236, 1.2472)$ and the straight line $\lambda = 2\alpha$ in the point $(\frac{9}{8}, 1)$. Therefore it can be divided in a segment which lies in $HGFOI$, a segment which lies in $OBCFO$, part in $OBF$ and part in $CBF$, and
a half-line which lies in $ECFGH$, so that

- for $0 < \lambda \leq (4 - \sqrt{5})/5 \leq 0.3528 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq 0$;
- for $(4 - \sqrt{5})/5 < \lambda < 1 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq k_0 = (40\lambda - 25\lambda^2 - 11)/500$. Since the maximum of this function for $\lambda \in ((4 - \sqrt{5})/5, 1)$ is $1/100$, the inequality holds at least for $k \geq 1/100$;
- for $1 < \lambda < (4 + \sqrt{5})/5 \leq 1.2472 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq k_0 = (40\lambda - 25\lambda^2 - 11)/500$. Since the maximum of this function for $\lambda \in ((4 - \sqrt{5})/5, 1)$ is $1/100$, the inequality holds at least for $k \geq 1/100$.

Laforgia in [3] proved the inequality $\Gamma(k + 2)/\Gamma(k + 1) < (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $1 < \lambda < 2$ and $k \geq 1$. 

In each of the two cases $\lambda = \lambda_e + \lambda_0$ a straight line parallel to $2 = 2\lambda_e$ has been taken into consideration: let now us see what happens when a straight line having a different slope, e.g. $\lambda = \lambda_e + \lambda_0$ is considered. Here we have a segment, from $O$ to $(2, 6)$, in the region $DAOBCJ$ and a half-line in the region $JCE$, so that

- for $0 < \lambda < 1 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) < (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq 0$;
- for $1 < \lambda < 6 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) < (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq 0$;
- for $\lambda > 6 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq k_0 = (\lambda - 6)/3$. There is no uniform bound, since this function is upperly unbounded.

This case may be generalized by setting $\alpha = \lambda/p$, with fixed $p > 2$. The straight line $\lambda = p\alpha$ lies in part in the region $DAOBCJ$, from $O$ to $(p - 1, p^2 - p)$, and in part in the region $JCE$. So

- for $0 < \lambda < 1 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) < (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq 0$;
- for $1 < \lambda < p^2 - p \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq 0$;
- for $\lambda > p^2 - p \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \frac{1}{2}\lambda + \frac{1}{10})^{k-1}$ for $k \geq k_0 = (p^2 - 1)/p$. Also in this case we cannot find an uniform bound.

The case $\alpha = \frac{1}{2}\lambda + \frac{1}{8}$ may also be generalized, by considering a generic straight line tangent to the parabola $\lambda = \alpha^2 + \alpha$ at a point of the arc $OK$, say $P = (t - \frac{1}{2}, t^2 - \frac{1}{4})$, with $\frac{1}{2} < t \leq 1$. Since $\lambda'(\alpha) = 2\alpha + 1$, the slope of the straight line tangent to the parabola in $P$ is $2t$, so its equation is $\lambda = 2t\alpha - (t - \frac{1}{2})^2$, i.e., $\alpha = \lambda/2t + (2t - 1)^2/8t$. This line lies partly in $HGFOI$, from the $x$-axis to the point of ordinate $1$, partly in $ECFGH$. So

- for $0 < \lambda < 1 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \lambda/2t + (2t - 1)^2/8t)^{k-1}$ for $k \geq 0$;
- for $\lambda > 1 \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) < (k + \lambda/2t + (2t - 1)^2/8t)^{k-1}$ for $k \geq 0$.

Other interesting particular cases may be studied by considering straight lines that lie in two regions in which only one of the two inequalities hold. Let us take for example the bundle of straight lines with centre $B(\frac{1}{2}, 1)$, i.e., $\lambda = m\alpha - \frac{1}{2}m + 1$, which for $m \neq 0$ may be written $\alpha = \lambda/m + \frac{1}{2} - 1/m$. The vertical line of this bundle, having equation $\alpha = \frac{1}{2}$, lies partly in the region $HGFOI$, from the $x$-axis to the point of ordinate $1$, partly in $ECFGH$. So

- for $0 < \lambda < \frac{1}{2} \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > (k + \lambda/2t + (2t - 1)^2/8t)^{k-1}$ for $k \geq 0$;
- for $\lambda > \frac{1}{2} \Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) < (k + \lambda/2t + (2t - 1)^2/8t)^{k-1}$ for $k \geq 0$.

For what concerns the other lines of the bundle, the following cases are to be considered:

$m < 0$:

we have a segment in $DABCJ$, from $(0, 1 - m/2)$ to $B$, a segment in $OBF$, from $B$ to

\[
\left(\frac{m - 1 + \sqrt{m^2 - 4m + 5}}{2}, \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2}\right).
\]
and a segment in HGFOI, from this point to \((\frac{1}{2} - 1/m, 0)\);

0 < \(m < 2\): we have a segment in the triangle \(ABO\), from \((0, 1 - m/2)\) to \(B\), a segment in the region \(BCF\), from \(B\) to
\[
\left( \frac{m - 1 + \sqrt{m^2 - 4m + 5}}{2}, \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2} \right),
\]
and a half-line in \(ECFGH\);

\(m > 2\): the straight line may be divided in a segment in HGFOI, from \((\frac{1}{2} - 1/m, 0)\) to
\[
\left( \frac{m - 1 - \sqrt{m^2 - 4m + 5}}{2}, \frac{m^2 - m\sqrt{m^2 - 4m + 5} - 2m + 2}{2} \right),
\]
a segment in \(OBF\) up to \(B\), a segment in \(DABCJ\) up to
\[
\left( \frac{m - 1 + \sqrt{m^2 - 4m + 5}}{2}, \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2} \right),
\]
and a half-line in \(JCE\). So we have the following results:

\(m < 0\):

for \(1 < \lambda < 1 - \frac{m}{2}\) \(\Rightarrow \Gamma(k + \lambda)/\Gamma(k + 1) > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) for \(k \geq 0\);

for \(\frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2} < \lambda < 1\) \(\Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) ultimately, without uniform bound;

for \(0 < \lambda \leq \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2}\) \(\Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) for \(k \geq 0\);

0 < \(m < 2\):

for \(1 - \frac{m}{2} < \lambda < 1\) \(\Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) for \(k \geq 0\);

for \(1 < \lambda < \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2}\) \(\Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) ultimately, without uniform bound;

for \(\lambda \geq \frac{m^2 + m\sqrt{m^2 - 4m + 5} - 2m + 2}{2}\) \(\Rightarrow \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda-1}\) for \(k \geq 0\);
$m > 2$:

for $0 < \lambda \leq \frac{m^2 - m\sqrt{m^2 - 4m + 5 - 2m + 2}}{2}$

\[ \Gamma(k + \lambda) \Gamma(k + 1) > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda - 1} \]

for $k \geq 0$;

for $\frac{m^2 - m\sqrt{m^2 - 4m + 5 - 2m + 2}}{2} < \lambda < 1$

\[ \Gamma(k + \lambda) \Gamma(k + 1) > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda - 1} \]

ultimately, without uniform bound;

for $1 < \lambda < \frac{m^2 + m\sqrt{m^2 - 4m + 5 - 2m + 2}}{2}$

\[ \Gamma(k + \lambda) \Gamma(k + 1) > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda - 1} \]

ultimately, without uniform bound;

for $\lambda \geq \frac{m^2 + m\sqrt{m^2 - 4m + 5 - 2m + 2}}{2}$

\[ \Gamma(k + \lambda) \Gamma(k + 1) > \left( k + \frac{\lambda}{m} + \frac{1}{2} - \frac{1}{m} \right)^{\lambda - 1} \]

for $k \geq 0$.

4. Numerical results

If we are interested in the practical use of the former inequalities, we may proceed as follows:

given the parameter $\lambda (\lambda > 0$, but $\lambda \neq 1$ and $\lambda \neq 2)$, we want to find the best values $z_1$ and $z_2$ of the positive parameter $\alpha$ for which the double inequality

\[ (k + z_1)^{\lambda - 1} < \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + z_2)^{\lambda - 1} \]

holds for every $k \geq 0$. Let us consider in the $\alpha\lambda$-plane the horizontal straight line corresponding to the fixed value of $\lambda$: it meets the straight line $\lambda = 2\alpha$ at the point $\left(\frac{\lambda}{2}; \lambda\right)$ and the parabola $\lambda = \alpha^2 + \alpha$ at the point $((\sqrt{4\lambda + 1} - 1)/2; \lambda)$; we must distinguish the three following cases:

• $0 < \lambda < 1$: for $0 < \alpha \leq \frac{1}{2} \lambda$ we have

\[ \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < (k + \alpha)^{\lambda - 1} \quad \text{for} \quad \alpha \geq \frac{\sqrt{4\lambda + 1} - 1}{2} \]

\[ \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} > (k + \alpha)^{\lambda - 1}. \]

Hence (10) holds for $k \geq 0$ if we choose $0 < z_2 \leq \lambda/2$ and $z_1 \geq (\sqrt{4\lambda + 1} - 1)/2$. Since the function $T(\alpha) = (k + \alpha)^{\lambda - 1}$ is decreasing, the best inequalities are obtained for $z_2 = \lambda/2$ and $z_1 = (\sqrt{4\lambda + 1} - 1)/2$:

\[ \left( k + \frac{\sqrt{4\lambda + 1} - 1}{2} \right)^{\lambda - 1} < \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} < \left( k + \frac{\lambda}{2} \right)^{\lambda - 1}. \]

(11)
Now (10) holds for \( k \geq 0 \) if \( 0 < x_1 \leq \frac{1}{2} \lambda \) and \( x_2 \geq (\sqrt{4 \lambda + 1} - 1)/2 \). Since the function \( T(\alpha) = (k + \alpha)^{\lambda-1} \) is increasing, we find that the best inequalities are

\[
\left( k + \alpha \right)^{\lambda-1} \leq \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} \leq \left( k + \frac{\sqrt{4 \lambda + 1} - 1}{2} \right)^{\lambda-1}.
\]

\[ (12) \]

\[
\cdot \lambda > 2: \text{proceeding as above, we obtain the same double inequality of the case } 0 < \lambda < 1.
\]

As an example of the case \( 0 < \lambda < 1 \), let us take \( \lambda = 0.11 \): for this value of \( \lambda \), formula (11) becomes

\[
(k + 0.1)^{-0.89} \leq \frac{\Gamma(k + 0.1)}{\Gamma(k + 1)} \leq (k + 0.055)^{-0.89}.
\]

The bounds are not very good for small \( k \): e.g., for \( k = 0.25 \) we have \( 0.35^{-0.89} < \frac{\Gamma(0.36)}{\Gamma(1.25)} < 0.305^{-0.89} \), i.e., with 7 decimal places, \( 2.5455372 < 2.7280760 < 2.8772205 \) (relative errors are 6.70% and 5.47% respectively). But for larger values of \( k \) relative errors decrease: for \( k = 2 \) we have \( 0.5166838 < 0.5258045 < 0.5267414 \) (relative errors 1.73% and 0.18%), and \( k = 10 \Rightarrow 0.1276891 < 0.1281879 < 0.1281976 \) (relative errors 0.39% and 0.0076%).

Similarly, let us take \( \lambda = 1.71 \). For this \( \lambda \) formula (12) becomes

\[
(k + 0.855)^{0.71} \leq \frac{\Gamma(k + 1.71)}{\Gamma(k + 1)} \leq (k + 0.9)^{0.71}.
\]

Similar to the former case, best results are obtained for large values of \( k \): e.g., for \( k = 0.5 \) we have \( 1.2407289 < 1.2500721 < 1.2698456 \) (relative errors 0.75% and 1.58%), while \( k = 10 \Rightarrow 5.4362219 < 5.4368980 < 5.4522130 \) (relative errors 0.012% and 0.28%).

The bound \( (k + (\sqrt{4 \lambda + 1} - 1)/2)^{\lambda-1} \) may be improved if we want to find inequalities which hold true for \( k \geq k_0 \). Let us fix, e.g., \( k_0 = 1 \): the inequality \( (\lambda - x^2 - \lambda)/(2x - \lambda) \geq 1 \) (see Section 2) is equivalent to \( (2\lambda - x^2 - 3x)/(2x - \lambda) \geq 0 \): so we must consider in the \( x \)-plane the straight line \( \lambda = 2x \) and the parabola \( \lambda = \frac{1}{2}x^2/2 + \frac{3}{2}x \) (for \( x \geq 0 \)). The horizontal straight line corresponding to a fixed value of meets the straight line \( \lambda = 2x \) at the point \((\frac{1}{2} \lambda; \lambda)\) and the parabola at the point \( ((\sqrt{8 \lambda + 9} - 3)/2; \lambda)\); so we have, proceeding as above, that for \( 0 < \lambda < 1 \) and for \( \lambda > 2 \) the best inequalities are

\[
\left( k + \frac{\sqrt{8 \lambda + 9} - 3}{2} \right)^{\lambda-1} \leq \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} \leq \left( k + \frac{\sqrt{8 \lambda + 9} - 3}{2} \right)^{\lambda-1},
\]

while for \( 1 < \lambda < 2 \) the best inequalities are

\[
\left( k + \frac{\lambda}{2} \right)^{\lambda-1} \leq \frac{\Gamma(k + \lambda)}{\Gamma(k + 1)} \leq \left( k + \frac{\sqrt{8 \lambda + 9} - 3}{2} \right)^{\lambda-1}.
\]

\[ (\text{13}) \]
5. A final remark

For what concerns the technique used in this paper (the same as [3]), we said in Section 1 that if $f_k$ is a positive sequence such that $\lim_{k \to \infty} f_k = 1$ and $g_k = f_{k+1}/f_k$, then $g_k < 1$ implies that $f_k$ is decreasing, while $g_k > 1 \Rightarrow f_k$ increasing; but if we write $G(k)$ instead of $g_k$, with $k \in [0, + \infty)$, then $G(k) > 1$ does not imply that the function $F(k)$ is increasing. In fact, we may easily show an example of a function $f(x)$ continuous and non-negative on $[0, + \infty)$ such that $\lim_{x \to +\infty} f(x) = 1$ and $f(x + 1) > f(x) \forall x \in [0, + \infty)$, but $f$ is not increasing.

The counterexample is

$$f(x) = \begin{cases} \frac{x}{2^{n+1}} + 1 - \frac{n+2}{2^{n+1}} & \text{for } n \leq x \leq n + \frac{1}{2} \\ -\frac{x}{2^{n+1}} + 1 - \frac{n-1}{2^{n+1}} & \text{for } n + \frac{1}{2} \leq x \leq n + \frac{3}{4} \\ \frac{3x}{2^{n+1}} + 1 - \frac{3n+4}{2^{n+1}} & \text{for } n + \frac{3}{4} \leq x \leq n + 1, \end{cases}$$

where $n = 0, 1, 2, \ldots$

The function $f(x)$ is linear in every interval $[n, n + \frac{1}{2}]$, $[n + \frac{1}{2}, n + \frac{3}{4}]$ and $[n + \frac{3}{4}, n + 1]$, with $n = 0, 1, 2, \ldots$, and it is continuous in $[0, + \infty)$. The inequality $f(x + 1) > f(x)$ comes from the fact that in $[0, 1]$ it is $0 \leq f(x) \leq \frac{1}{2}$, in $[3, 2]$ it is $\frac{1}{2} \leq f(x) \leq \frac{3}{4}$, and so on.

Nevertheless, under these hypothesis we have $f(x) < 1 \forall x \in [0, + \infty)$, and this is the result we need. In fact, if we suppose $f(\alpha) \geq 1$ for an $\alpha \in [0, + \infty)$, and we put $x_1 = \alpha$, $x_2 = \alpha + 1$,
\[ x_3 = x + 2, \ldots, \text{we have} \]
\[ 1 \leq f(x_1) < f(x_2) < \cdots, \]
so we have \( \lim_{n \to \infty} f(x_n) = L > 1 \) or \( +\infty \). This contradicts the hypothesis \( \lim_{x \to +\infty} f(x) = 1 \).

**References**