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Journal of Algebra 311 (2007) 435–462

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

On the minimal free resolution of the universal ring for resolutions of length two

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Received 1 May 2006

Available online 6 March 2007

Communicated by Luchezar L. Avramov

Abstract

Hochster established the existence of a commutative noetherian ring $\tilde{C}$ and a universal resolution U of the form $0 \to \tilde{C}^e \to \tilde{C}^f \to \tilde{C}^g \to 0$ such that for any commutative noetherian ring S and any resolution V equal to $0 \to S^e \to S^f \to S^g \to 0$, there exists a unique ring homomorphism $\tilde{C} \to S$ with $V = U \otimes_{\tilde{C}} S$. In the present paper we assume that $f = e + g$ and we find the minimal resolution of $\tilde{C} \otimes_{\mathbf{Z}} \mathbf{Q}$ by free B-modules, where B is a polynomial ring over the field of rational numbers. The modules of the resolution are described in terms of Schur functors. The graded strands of the differential are described in terms of Pieri maps.

Keywords: Finite free resolution; Pieri map; Schur functor; Universal resolution; Variety of complexes

0. Introduction

Fix positive integers e, f, and g, with $r_1 \geqslant 1$ and $r_0 \geqslant 0$, for r_1 and r_0 defined to be $f - e$ and $g - f + e$, respectively. Hochster [Ho] established the existence of a commutative noetherian ring $\tilde{C}$ and a universal resolution

$$U: \quad 0 \to \tilde{C}^e \to \tilde{C}^f \to \tilde{C}^g$$

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1 Partially supported by NSF grant DMS-0600229.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.02.026

such that for any commutative noetherian ring S and any resolution

$$V:\quad 0 \to S^e \to S^f \to S^g,$$

there exists a unique ring homomorphism $\tilde{C} \to S$ with $V = U \otimes_{\tilde{C}} S$. The ring $\tilde{C}$ is generated as an algebra by the entries of the matrices giving the universal complex with these Betti numbers, together with the universal Buchsbaum–Eisenbud multipliers assuring that the First Structure Theorem from [BE] holds. The generators of $\tilde{C}$ were described in [Hu]. Over a field $\mathbf{K}$ of characteristic zero [PW] gave the presentation and the description of the decomposition of $\tilde{C}$ to the irreducible representations of $\mathrm{GL}_e(\mathbf{K}) \times \mathrm{GL}_f(\mathbf{K}) \times \mathrm{GL}_g(\mathbf{K})$. Finally, the explicit basis of $\tilde{C}$ as a $\mathbf{Z}$-module and its presentation as an algebra over $\mathbf{Z}$ was given in [T].

The ring $\tilde{C}$ is important because of its universality property. It found a remarkable application as Heitmann [He] used it to give a counterexample to the rigidity conjecture.

In the present paper we take the next step, by describing the syzygies of $\tilde{C}$, i.e. the minimal resolution of $\tilde{C}$ as a module over the polynomial algebra of which $\tilde{C}$ is a factor. We do it only in the case $r_0 = 0$ over a field of characteristic zero. These are reasonable assumptions; as, for bigger r_0, the Buchsbaum–Eisenbud multipliers satisfy Plücker relations, so the resolution of $\tilde{C}$ would include the knowledge of (unknown) resolutions of Plücker ideals. Over fields of positive characteristic, the resolution of $\tilde{C}$ would include the knowledge of an (unknown) resolution of a determinantal ideal.

We use the techniques from [W] and the papers mentioned above. We describe the terms of the resolution of $\tilde{C}$. In the case under consideration we know that there is exactly one Buchsbaum–Eisenbud multiplier, which we call a, and we know that a is a nonzerodivisor in $\tilde{C}$; so the resolution of $\tilde{C}$ has the same terms as the analogous resolution of $\tilde{C}/a\tilde{C}$. The last ring is the coordinate ring of a variety of pairs of matrices that form a complex and satisfy certain rank conditions.

The resolution of $\tilde{C}/a\tilde{C}$ has a very nice structure. It is filtered by resolutions of certain maximal Cohen–Macaulay modules supported in a determinantal variety. We describe these complexes in several ways.

Let us set up the notation of the paper. We deal with the universal ring $\tilde{C}$ when $r_0 = 0$. In this case, $f = e + g$ and $\tilde{C} = \tilde{B}/\tilde{J}$, for $\tilde{B}$ equal to the polynomial ring $\mathbf{Z}[\{\phi_{j,i}\}, \{\psi_{k,j}\}, a]$, with $1 \leqslant i \leqslant e$, $1 \leqslant j \leqslant f$, and $1 \leqslant k \leqslant g$, where $\{\phi_{j,i}\} \cup \{\psi_{k,j}\} \cup \{a\}$ is a list of indeterminates over $\mathbf{Z}$. The indeterminate a corresponds to the unique Buchsbaum–Eisenbud multiplier which occurs in the present situation. Let ϕ be the $f \times e$ matrix and ψ be the $g \times f$ matrix with entries $\phi_{j,i}$ and $\psi_{k,j}$, respectively. View the matrices ϕ and ψ as homomorphisms of $\tilde{B}$-modules:

$$\tilde{B}^e \xrightarrow{\phi} \tilde{B}^f \xrightarrow{\psi} \tilde{B}^g.$$

We give $\tilde{J}$ in the language of [T]. For each

$$\text{partition of } \{1, \ldots, f\} \text{ into } I \cup \bar{I} \text{ with } |I| = e \text{ and } |\bar{I}| = g, \tag{0.1}$$

let $\nabla_{\bar{I},I}$ be the sign of the permutation which arranges the elements of $\bar{I}, I$ into increasing order, $\phi(I)$ the submatrix of ϕ consisting of the rows from I, and $\psi(\bar{I})$ the submatrix of ψ consisting of the columns from $\bar{I}$. In this notation, the ideal $\tilde{J}$ which defines the universal ring $\tilde{C}$ is

$$I_1(\psi\phi) + \bigl(\bigl\{\det\psi(\bar{I}) + \nabla_{\bar{I},I}\, a \det\phi(I) \bigm| I \cup \bar{I} \text{ from (0.1)}\bigr\}\bigr). \tag{0.2}$$

One resolution of $\tilde{C}$ by free $\tilde{B}$-modules may be found in [K]. The resolution of [K] is not minimal; but it is straightforward, coordinate free, and independent of characteristic; furthermore, one can use it to calculate $\mathrm{Tor}_\bullet^{\tilde{B}}(\tilde{C}, \mathbf{Z})$. If e and g are both at least 5, then $\mathrm{Tor}_\bullet^{\tilde{B}}(\tilde{C}, \mathbf{Z})$ is not a free abelian group; and therefore (see Roberts [R] or Hashimoto [Ha]), the graded Betti numbers in the minimal resolution of $\tilde{C} \otimes_{\mathbf{Z}} \mathbf{K}$ by free $\tilde{B} \otimes_{\mathbf{Z}} \mathbf{K}$-modules depend on the characteristic of the field $\mathbf{K}$.

Henceforth, we work over a field $\mathbf{K}$ *of characteristic zero.* Consider the vector spaces E, F, G over $\mathbf{K}$ of dimensions e, f, g respectively, with $f = e + g$. Since we will apply the geometric technique of [W], we identify $B = \tilde{B} \otimes_{\mathbf{Z}} \mathbf{K}$ with the coordinate ring of the affine space

$$\mathrm{Hom}_{\mathbf{K}}(E, F) \times \mathrm{Hom}_{\mathbf{K}}(F, G) \times \mathbf{K}.$$

The vector space $\mathrm{Hom}(E, F)$ is naturally equal to $F \otimes E^*$; and therefore, B is the polynomial ring

$$B = \mathrm{Sym}_{\mathbf{K}}(F^* \otimes_{\mathbf{K}} E) \otimes_{\mathbf{K}} \mathrm{Sym}_{\mathbf{K}}(G^* \otimes_{\mathbf{K}} F) \otimes_{\mathbf{K}} \mathbf{K}[a].$$

Let $E \otimes_{\mathbf{K}} B \xrightarrow{\phi} F \otimes_{\mathbf{K}} B \xrightarrow{\psi} G \otimes_{\mathbf{K}} B$ be the natural maps given by

$$\phi(u) = \sum_i v_i \otimes \left(v_i^* \otimes u\right)$$

and

$$\psi(v) = \sum_i w_i \otimes \left(w_i^* \otimes v\right),$$

for each $u \in E$ and $v \in F$. It is not necessary to pick bases; however, if $u_1, \dots, u_e$; $v_1, \dots, v_f$; and $w_1, \dots, w_g$ are bases for the vector spaces E, F, and G; and $u_1^*, \dots, u_e^*$; $v_1^*, \dots, v_f^*$; and $w_1^*, \dots, w_g^*$ are the corresponding dual bases for E^*, F^*, and G^*; then $\sum_i v_i \otimes v_i^*$, which is used in the definition of ϕ, is the element in $F \otimes F^*$ which represents the identity map under the canonical identification of $F \otimes F^*$ with $\mathrm{Hom}(F, F)$. The coordinate functions in B may be identified as $\phi_{i,j} = v_i^* \otimes u_j \in F^* \otimes E$ and $\psi_{i,j} = w_i^* \otimes v_j \in G^* \otimes F$. The matrices which represent the maps ψ and ϕ, with respect to the chosen bases, are the generic matrices $(\psi_{i,j})$ and $(\phi_{i,j})$, respectively. We have $C = \tilde{C} \otimes_{\mathbf{Z}} \mathbf{K}$ and $J = \tilde{J} B$. So, B is the polynomial ring $\mathbf{K}[\{\phi_{i,j}\}, \{\psi_{i,j}\}, a]$, $C = B/J$, and J is given by (0.2). In Corollary 6.2, we produce the modules in the minimal resolution $\mathbf{G}$ of C by free B-modules. The ring B is bigraded with $\phi_{i,j} \in B_{(1,0)}$, $\psi_{i,j} \in B_{(0,1)}$, and $a \in B_{(-e,g)}$. The ideal J and the resolution $\mathbf{G}$ are homogeneous with respect to this bidegree.

Notice that in [W] one uses the notation $L_\lambda E$, $K_\lambda E$ to denote the Schur and Weyl functors. In this paper we work over a field of characteristic zero, so we have our $S_\lambda E$ isomorphic to $L_{\lambda'} E$ or $K_\lambda E$, where λ' is a conjugate partition. The module $S_\lambda E$ is defined for any dominant weight λ (i.e., for any integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_e$) because

$$S_{(\lambda_1, \dots, \lambda_e)} E = S_{(\lambda_1 + t, \dots, \lambda_e + t)} E \otimes \left(\bigwedge^e E^*\right)^{\otimes t}$$

for any integer t.

Let us recall the result from [PW] that gives a natural basis for the universal ring C. We notice that the proper GL-representation in

$$C = \frac{B}{J} = \frac{\mathrm{Sym}_{\mathbf{K}}(F^* \otimes E) \otimes_{\mathbf{K}} \mathrm{Sym}_{\mathbf{K}}(G^* \otimes F) \otimes_{\mathbf{K}} \mathbf{K}[a]}{J}$$

for the multiplier a is $\bigwedge^e E^* \otimes \bigwedge^f F \otimes \bigwedge^g G^*$. Indeed, the representation $S_{1^g} F \otimes S_{1^g} G^*$ is equal to

$$S_\lambda E \otimes S_{(\mu_1, \dots, \mu_{g-1}, 0, -\lambda_e, \dots, -\lambda_1)} F \otimes S_\mu G^* \otimes \left(\bigwedge^e E^* \otimes \bigwedge^f F \otimes \bigwedge^g G^* \right),$$

for $\lambda = 1^e$ and $\mu = 0$. In other words, in C, each maximal minor of ψ is equal to the appropriately signed complementary maximal minor of ϕ times the image of $\bigwedge^e E^* \otimes \bigwedge^f F \otimes \bigwedge^g G^*$.

Proposition 0.3. *The ring C has the following decomposition to representations of* $\mathrm{GL}(E) \times \mathrm{GL}(F) \times \mathrm{GL}(G)$:

$$C = \bigoplus_{\lambda, \mu, t} S_\lambda E \otimes S_{(\mu_1, \dots, \mu_{g-1}, 0, -\lambda_e, \dots, -\lambda_1)} F \otimes S_\mu G^* \otimes \left(\bigwedge^e E^* \otimes \bigwedge^f F \otimes \bigwedge^g G^* \right)^{\otimes t},$$

where we sum over all partitions λ with e parts, partitions μ with $g-1$ parts and $t \geqslant 0$. Note that the representation corresponding to the triple (λ, μ, t) is a factor of $(S_\lambda E \otimes S_\lambda F^*) \otimes (S_\mu F \otimes S_\mu G^*) \otimes a^t$.

Proof. Applying Theorem 1.3 from [PW], or Theorem 5.10 from [T], we get

$$C = \bigoplus_{\lambda, \mu, t} L_\lambda E \otimes L_{(e+g-\lambda_u, \dots, e+g-\lambda_1, \mu_1, \dots, \mu_s)} F \otimes L_\mu G^* \otimes \left(\bigwedge^g G^* \right)^{\otimes t}.$$

Changing Schur functors to Weyl functors (i.e., L's to S's), partitions λ, μ to λ', μ' respectively, and adjusting powers of determinant representations to get a $\mathrm{GL}(E) \times \mathrm{GL}(F) \times \mathrm{GL}(G)$—equivariant statement we get the result. $\square$

Corollary 0.4. *The ring C is a free* $\mathbf{K}[a]$*-module.*

Notation 0.5. The ring C/aC is isomorphic to the factor of $A := \mathbf{K}[\phi_{i,j}, \psi_{i,j}]$ by the ideal I given by the relations $\psi\phi = 0$ and $\bigwedge^g \psi = 0$. The ring $A = B/a$ inherits the bidegree of B with $\phi_{i,j} \in A_{(1,0)}$ and $\psi_{i,j} \in A_{(0,1)}$.

In section one we recapitulate the geometric method for calculating syzygies. Section two contains a brief introduction to the Pieri maps which are used in our description of the differentials in our resolutions. Section two also contains the Comparison Principle which we use to prove the acyclicity of some complexes. The modules in the minimal resolution, $\mathbf{F}_\bullet$, of A/I by free A-modules are given in Theorem 3.4. Theorem 5.13 describes the homogeneous strands of the differential of $\mathbf{F}_\bullet$. The differential of $\mathbf{F}_\bullet$ is viewed as arising from an iterated mapping cone in

Theorem 5.4. In section four, we resolve a family of maximal Cohen–Macaulay modules over the determinantal ring $\bar{A}/I_g(\psi)$, for $\bar{A} = \mathrm{Sym}_\bullet(F \otimes G^*)$. The familiar rank one maximal Cohen–Macaulay modules $\mathrm{Sym}_i(\operatorname{cok}\psi)$, for $0 \leqslant i \leqslant e+1$, which are resolved by the Eagon–Northcott complex, are members of our family. Section six gives the free B-modules in the resolution of the universal ring $C = B/J$.

1. Geometric technique of calculating syzygies

In this section we provide a quick description of the main facts related to the geometric technique of calculating syzygies; see [W] for more details. We work over a field $\mathbf{K}$. The characteristic of $\mathbf{K}$ must be zero for the Bott algorithm; otherwise, in this section, the characteristic of $\mathbf{K}$ is arbitrary.

Let us consider the projective variety V of dimension m. Let $X = A^N_{\mathbf{K}}$ be the affine space. The space $X \times V$ can be viewed as a total space of trivial vector bundle $\mathcal{E}$ of dimension N over V. Let us consider the subvariety Z in $X \times V$ which is the total space of a subbundle $\mathcal{S}$ in $\mathcal{E}$. We denote by q the projection $q : X \times V \to X$ and by q' the restriction of q to Z. Let $Y = q(Z)$. We get the basic diagram

$$\begin{array}{ccc} Z & \subset & X \times V \\ \Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q} \\ Y & \subset & X. \end{array}$$

The projection from $X \times V$ onto V is denoted by p and the quotient bundle $\mathcal{E}/\mathcal{S}$ by $\mathcal{T}$. Thus we have the exact sequence of vector bundles on V

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{E} \longrightarrow \mathcal{T} \longrightarrow 0.$$

The dimensions of $\mathcal{S}$ and $\mathcal{T}$ will be denoted by s, t respectively. The coordinate ring of X will be denoted by A. It is a polynomial ring in N variables over $\mathbf{K}$. We will identify the sheaves on X with A-modules.

The locally free resolution of the sheaf $\mathcal{O}_Z$ as an $\mathcal{O}_{X\times V}$-module is given by the Koszul complex

$$\mathcal{K}_\bullet(\xi) : 0 \to \bigwedge^t (p^*\xi) \to \cdots \to \bigwedge^2 (p^*\xi) \to p^*(\xi) \to \mathcal{O}_{X\times V},$$

where $\xi = \mathcal{T}^*$. The differentials in this complex are homogeneous of degree 1 in the coordinate functions on X. The direct image $p_*(\mathcal{O}_Z)$ can be identified with the sheaf of algebras $\mathrm{Sym}(\eta)$, where $\eta = \mathcal{S}^*$.

The idea of the geometric technique is to use the Koszul complex $\mathcal{K}(\xi)_\bullet$ to construct for each vector bundle $\mathcal{V}$ on V the free complex $\mathbf{F}_\bullet(\mathcal{V})$ of A-modules with the homology supported in Y. In many cases the complex $\mathbf{F}(\mathcal{O}_V)_\bullet$ gives the free resolution of the defining ideal of Y.

For every vector bundle $\mathcal{V}$ on V we introduce the complex

$$\mathcal{K}(\xi, \mathcal{V})_\bullet := \mathcal{K}(\xi)_\bullet \otimes_{\mathcal{O}_{X\times V}} p^*\mathcal{V}.$$

This complex is a locally free resolution of the $\mathcal{O}_{X\times V}$-module $M(\mathcal{V}) := \mathcal{O}_Z \otimes p^*\mathcal{V}$.

Now we are ready to state the basic theorem (Theorem (5.1.2) in [W]).

Theorem 1.1. *For a vector bundle $\mathcal{V}$ on V we define a free graded A-module*

$$\mathbf{F}(\mathcal{V})_i = \bigoplus_{j\geqslant 0} \mathrm{H}^j\left(V, \bigwedge^{i+j} \xi \otimes \mathcal{V}\right) \otimes_{\mathbf{K}} A(-i-j).$$

(a) *There exist minimal differentials*

$$d_i(\mathcal{V}) : \mathbf{F}(\mathcal{V})_i \to \mathbf{F}(\mathcal{V})_{i-1}$$

of degree 0 such that $\mathbf{F}(\mathcal{V})_\bullet$ is a complex of graded free A-modules with

$$\mathrm{H}_{-i}\big(\mathbf{F}(\mathcal{V})_\bullet\big) = \mathcal{R}^i q_* M(\mathcal{V}).$$

In particular, the complex $\mathbf{F}(\mathcal{V})_\bullet$ is exact in positive degrees.

(b) *The sheaf $\mathcal{R}^i q_* M(\mathcal{V})$ is equal to $\mathrm{H}^i(Z, M(\mathcal{V}))$ and it can be also identified with the graded A-module $\mathrm{H}^i(V, \mathrm{Sym}(\eta) \otimes \mathcal{V})$.*

(c) *If $\phi : M(\mathcal{V}) \to M(\mathcal{V}')(n)$ is a morphism of graded sheaves then there exists a morphism of complexes*

$$f_\bullet(\phi) : \mathbf{F}(\mathcal{V})_\bullet \to \mathbf{F}(\mathcal{V}')_\bullet(n).$$

Its induced map $\mathrm{H}_{-i}(f_\bullet(\phi))$ can be identified with the induced map

$$\mathrm{H}^i\big(Z, M(\mathcal{V})\big) \to \mathrm{H}^i\big(Z, M(\mathcal{V}')\big)(n).$$

If $\mathcal{V}$ is a one dimensional trivial bundle on V, then the complex $\mathbf{F}(\mathcal{V})_\bullet$ is denoted simply by $\mathbf{F}_\bullet$.

The next theorem gives the criterion for the complex $\mathbf{F}_\bullet$ to be the free resolution of the coordinate ring of Y.

Theorem 1.2. *Let us assume that the map $q' : Z \to Y$ is a birational isomorphism. Then the following properties hold.*

(a) *The module $q'_* \mathcal{O}_Z$ is the normalization of $\mathbf{K}[Y]$.*
(b) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$, then $\mathbf{F}_\bullet$ is a finite free resolution of the normalization of $\mathbf{K}[Y]$ treated as an A-module.*
(c) *If $\mathcal{R}^i q'_* \mathcal{O}_Z = 0$ for $i > 0$ and $\mathbf{F}_0 = \mathrm{H}^0(V, \bigwedge^0 \xi) \otimes A = A$, then Y is normal and it has rational singularities.*

This is Theorem (5.1.3) in [W].

In all our applications the projective variety V will be a Grassmannian. To fix the notation, let us work with the Grassmannian $\mathrm{Grass}(r, E)$ of subspaces of dimension r in a vector space E of dimension n. Let

$$0 \to \mathcal{R} \to E \times \mathrm{Grass}(r, E) \to \mathcal{Q} \to 0$$

be a tautological sequence of the vector bundles on $\mathrm{Grass}(r, E)$.

Assume that the characteristic of the field $\mathbf{K}$ is zero. Then the vector bundle ξ will be a direct sum of the bundles of the form $S_{\lambda_1,\dots,\lambda_{n-r}}\mathcal{Q} \otimes S_{\mu_1,\dots,\mu_r}\mathcal{R}$. Thus all the exterior powers of ξ will also be the direct sums of such bundles. We will apply repeatedly the following result to calculate cohomology of vector bundles $S_{\lambda_1,\dots,\lambda_{n-r}}\mathcal{Q} \otimes S_{\mu_1,\dots,\mu_r}\mathcal{R}$.

Proposition 1.3 *(Bott's algorithm). Assume that the characteristic of* $\mathbf{K}$ *is zero. The cohomology of the vector bundle* $S_{\lambda_1,\dots,\lambda_{n-r}}\mathcal{Q} \otimes S_{\mu_1,\dots,\mu_r}\mathcal{R}$ *on* $\mathrm{Grass}(r, E)$ *is calculated as follows. We look at the weight*

$$(\lambda, \mu) = (\lambda_1, \dots, \lambda_{n-r}, \mu_1, \dots, \mu_r)$$

and add to it $\rho = (n, n-1, \dots, 1)$. *If the resulting sequence*

$$(\lambda, \mu) + \rho = (\lambda_1 + n, \dots, \lambda_{n-r} + r + 1, \mu_1 + r, \dots, \mu_r + 1)$$

has repetitions, then

$$\mathrm{H}^i\big(\mathrm{Grass}(r, E), S_\lambda\mathcal{Q} \otimes S_\mu\mathcal{R}\big) = 0$$

for all $i \geqslant 0$. *If the resulting sequence has no repetitions, there is a unique permutation* $w \in \Sigma_n$ *that makes this sequence decreasing. Then the sequence* $\nu = w((\lambda, \mu) + \rho) - \rho$ *is again a nonincreasing sequence. Then the sheaf* $S_\lambda\mathcal{Q} \otimes S_\mu\mathcal{R}$ *has only one nonzero cohomology group, the group* H^ℓ, *where* $\ell = \ell(w)$ *is the length of* w. *This cohomology group is isomorphic to the representation* $S_\nu E$ *of* $\mathrm{GL}(E)$ *corresponding to the highest weight* ν.

This is Corollary (4.1.9) in [W].

2. The Pieri maps and the Comparison Principle

Ultimately, the differentials in all of our resolutions are described in terms of Pieri maps. For the purposes of the present paper, it is not important to give an explicit description of the exact action of one these maps on each element in its domain. However, it is possible to record such a description. We will first describe what the Pieri map is and explain why it exists. Then we will point any reader so-inclined in the direction of recording an explicit formula for the Pieri map. We are interested only in a special case that is relevant to our resolutions.

Let E be a finite dimensional vector space over a field $\mathbf{K}$ of characteristic zero. Suppose that $\lambda = (\lambda_1, \dots, \lambda_m)$ is a partition and a and b are integers with $1 \leqslant a \leqslant b \leqslant m$. Define $\mu = (\mu_1, \dots, \mu_m)$ by

$$\mu_i = \begin{cases} \lambda_i & \text{if } i < a \text{ or } b < i, \\ \lambda_i - 1 & \text{if } a \leqslant i \leqslant b. \end{cases}$$

Assume that μ is also a partition. Let $N = b - a + 1$. The $\mathrm{GL}(E)$-module $S_{1^N}E \otimes S_\mu(E)$ is equal to a direct sum of irreducible $\mathrm{GL}(E)$-modules. The Pieri formula, which is a special case of the Littlewood–Richardson rule, see, for example, Corollary (2.3.5) in [W], shows that the irreducible $\mathrm{GL}(E)$-module $S_\lambda(E)$ is a summand of $S_{1^N}E \otimes S_\mu(E)$ with multiplicity one. Hence, there is exactly one nonzero $\mathrm{GL}(E)$-module homomorphism

$$P : S_\lambda(E) \to S_{1^N}E \otimes S_\mu(E),$$

up to multiplication by a unit, and this is the map that we call the Pieri map.

To investigate the action of the Pieri map P, it suffices to take $N = 1$. One obtains the general case by iteration. Inspired by the work of Maliakas and Olver [MO], we notice that the partition λ and the skew partition Λ/ν have exactly the same Ferrers diagram, where $\Lambda = (\lambda_1 + 1, \ldots, \lambda_m + 1, 1)$ and $\nu = 1^{m+1}$; and therefore, S_λ and $S_{\Lambda/\nu}$ are the exact same Schur functor. We also notice that when one box is moved from the right side of row a to the left side row $m+1$ in the Ferrers diagram for Λ/ν the resulting skew partition is $\Lambda - \epsilon_a/\nu - \epsilon_{m+1}$ and

$$S_{\Lambda-\epsilon_a/\nu-\epsilon_{m+1}}E = S_1E \otimes S_\mu E,$$

where ϵ_j represents the $(m+1)$-tuple with 1 in position j and zero everywhere else. Thus, $P : S_\lambda(E) \to S_1E \otimes S_\mu(E)$ is the same as $P : S_{\Lambda/\nu}E \to S_{\Lambda-\epsilon_a/\nu-\epsilon_{m+1}}E$, which moves one box from the right side of the arbitrary row a to the left side of the bottom row. Maliakas and Olver give an explicit formula for the related map that moves a box from the left side of the bottom row of an arbitrary skew-partition to the right side of an arbitrary row. Presumably, one can manipulate the map given in [MO] to make it apply to the present situation. Our approach is to start over and just calculate the explicit formula from scratch in our own notation. The skew-Schur module $S_{\Lambda/\nu}E$ is equal to

$$\frac{\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_t} E}{R(\Lambda/\nu, E)},$$

for $n_i = \lambda'_i - \nu'_i$, as described in Proposition (2.1.9) of [W]. The Pieri map

$$P : S_{\Lambda/\nu}E \to S_{\Lambda-\epsilon_a/\nu-\epsilon_{m+1}}E$$

is induced by a map

$$\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_t} E \to \bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_r-1} E \otimes \cdots \otimes \bigwedge^{n_t+1} E, \tag{2.1}$$

for the appropriate choice of r. The combinatorial description of (2.1) says that one sums over all possible sets of rest stops, $r = s_0 < s_1 < s_2 < \cdots < s_\ell = t$, along the direct route from row r to the bottom row. Once the rest stops are planned, one uses co-multiplication to split off one box at each rest stop, one carries the extra box from row s_0 to row s_1, puts it down and picks up the extra box sitting on row s_1 and moves it to row s_2 etc., and then one uses multiplication to join the new box to the old boxes at the new spot:

$$
\begin{array}{c}
\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_t} E \\
\| \\
\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_{s_0}} E \otimes \cdots \otimes \bigwedge^{n_{s_1}} E \otimes \cdots \otimes \bigwedge^{n_{s_2}} E \otimes \cdots \otimes \bigwedge^{n_{s_\ell}} E \\
\downarrow \\
\bigwedge^{n_1} E \otimes \cdots \otimes \left(\bigwedge^{n_{s_0}-1} E \otimes E\right) \otimes \cdots \otimes \left(\bigwedge^{n_{s_1}-1} E \otimes E\right) \otimes \cdots \otimes \left(\bigwedge^{n_{s_2}-1} E \otimes E\right) \otimes \cdots \otimes \bigwedge^{n_{s_\ell}} E \\
\downarrow \\
\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_{s_0}-1} E \otimes \cdots \otimes \left(E \otimes \bigwedge^{n_{s_1}-1} E\right) \otimes \cdots \otimes \left(E \otimes \bigwedge^{n_{s_2}-1} E\right) \otimes \cdots \otimes \left(E \otimes \bigwedge^{n_{s_\ell}} E\right) \\
\downarrow \\
\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_{s_0}-1} E \otimes \cdots \otimes \bigwedge^{n_{s_1}} E \otimes \cdots \otimes \bigwedge^{n_{s_2}} E \otimes \cdots \otimes \bigwedge^{n_{s_\ell}+1} E \\
\| \\
\bigwedge^{n_1} E \otimes \cdots \otimes \bigwedge^{n_r-1} E \otimes \cdots \otimes \bigwedge^{n_t+1} E.
\end{array}
$$

The coefficient for the term that corresponds to a particular set of rest stops is a quotient of products of hook lengths.

Our approach is a combination of the geometric technique and representation theory. We will use the Comparison Principle to prove the acyclicity of some complexes. In practice, we will know that the complex $\mathbf{H}_\bullet$ is acyclic without explicitly knowing its differential, and we will know an explicit differential on $(\mathbf{H}'_\bullet, d'_i)$. We apply the Comparison Principle to show that $(\mathbf{H}'_\bullet, d'_i)$ is acyclic.

Proposition 2.2 *(The Comparison Principle). Let A be a coordinate ring on some representation W of a linearly reductive group G. Let $\mathbf{H}_\bullet$ and $(\mathbf{H}'_\bullet, d'_i)$ be two finite G-equivariant minimal complexes of A-modules. If conditions* (a)–(e) *all are satisfied, then the complexes $\mathbf{H}_\bullet$ and $\mathbf{H}'_\bullet$ are isomorphic.*

(a) *The terms $\mathbf{H}_i$ and $\mathbf{H}'_i$ are direct sums of modules of type $V_\lambda \otimes A(-j)$, where V_λ is an irreducible representation of G of highest weight λ.*
(b) *For each i, the terms $\mathbf{H}_i$ and $\mathbf{H}'_i$ are isomorphic as graded equivariant $G - A$-bimodules, and, for $i < 0$, we have $\mathbf{H}_i = \mathbf{H}'_i = 0$.*
(c) *The complex $\mathbf{H}_\bullet$ is acyclic.*
(d) *The homology modules* $\mathrm{H}_0(\mathbf{H}_\bullet)$ *and* $\mathrm{H}_0(\mathbf{H}'_\bullet)$ *are isomorphic as $G - A$-bimodules.*
(e) *Denote $\mathbf{H}'_i = \bigoplus_{s=1}^{t} V_{\lambda_s} \otimes A(-j_s)$ with $j_1 \leqslant \cdots \leqslant j_t$. Let v'_{λ_s} be the highest weight vector in V_{λ_s}. Assume that*

($*$) *for all j, the images $d_i'(v'_{\lambda_s}\otimes 1)$ with $j_s=j$ are linearly independent vectors modulo the image $\sum_{j_s<j} d_i'(V_{\lambda_s}\otimes A(-j_s))$ in $\mathbf{H}'_{i-1}$.*

In particular, if the term $V_\lambda\otimes A(-j)$ occurs at most once in the complex $\mathbf{H}'_\bullet$, for each highest weight λ, then it is enough to replace ($*$) with the condition,

($**$) *for all s, the image $d_i'(v'_{\lambda_s}\otimes 1)$ is nonzero modulo the image $\sum_{u<s} d_i'(V_{\lambda_u}\otimes A(-j_u))$ in $\mathbf{H}'_{i-1}$.*

Proof. We construct a G-equivariant isomorphism of complexes $h_\bullet:\mathbf{H}'_\bullet\to\mathbf{H}_\bullet$. We induct on i. For $i=0$ and 1, the maps h_i exist by condition (d). Assume the map h_{i-1} has been constructed. To construct h_i we denote $\mathbf{H}'_i=\bigoplus_{s=1}^t V_{\lambda_s}\otimes A(-j_s)$. Let v'_{λ_s} be the highest weight vector in V_{λ_s}. We notice that the images $d_i'(v'_{\lambda_s}\otimes 1)$ give the cycles which are the highest weight vectors of corresponding weights λ that are linearly independent modulo images generated in lower degrees. Thus, for each s there is exactly one representation $V_{\lambda_s}\otimes A(-j_s)$ with highest vector $v_{\lambda_s}\otimes 1$ of weight λ_s in the appropriate degree in $\mathbf{H}_i$ whose differential equals $h_{i-1}\circ d_i'(v'_{\lambda_s}\otimes 1)$. We define $h_i(v'_{\lambda_s}\otimes 1)$ to be v_{λ_s}. This map extends uniquely to become an equivariant isomorphism $h_i':\mathbf{H}_i'\to\mathbf{H}_i$ and by construction it is obvious that $h_\bullet$ is a map of complexes. $\square$

3. The terms in the minimal resolution of A/I

We apply the geometric technique to calculate the minimal free resolution of A/I as an A-module. The notation is set up in 0.5. Recall that $\mathbf{K}$ is a field of characteristic zero. We use freely the notation of [W]. Denote

$$X=\{(d_2,d_1)\in\operatorname{Hom}_{\mathbf{K}}(E,F)\times\operatorname{Hom}_{\mathbf{K}}(F,G)\}.$$

Therefore we have $A=\mathbf{K}[X]$. Consider the incidence variety

$$Z=\{(d_2,d_1,R)\in X\times\operatorname{Grass}(e+1,F)\mid \operatorname{Im}(d_2)\subseteq R\subseteq\operatorname{Ker}(d_1)\}.$$

Clearly the image $q(Z)$ by the first projection $q:Z\to X$ is equal to the set $Y:=V(I)$. Notice that Z is the desingularization of Y because generically on Y we have $R=\operatorname{Ker}(d_1)$ and the projection q is obviously proper.

We are in the situation described in section one. In this special case we have $\xi=E\otimes\mathcal{Q}^*\oplus\mathcal{R}\otimes G^*$. We also have $\eta=E\otimes\mathcal{R}^*\oplus\mathcal{Q}\otimes G^*$. Let us look at the cohomology groups of the exterior powers of ξ and of symmetric powers of η.

Proposition 3.1. *We have*

(a) $\mathrm{H}^i(\operatorname{Grass}(e+1,F),\operatorname{Sym}_j(\eta))=0$ *for $i>0$,*
(b) $\mathrm{H}^0(\operatorname{Grass}(e+1,F),\operatorname{Sym}_j(\eta))=(A/I)_j$ *for all $j\geqslant 0$.*

Proof. We have

$$\operatorname{Sym}(\eta)=\bigoplus_{\lambda,\mu}S_\lambda E\otimes S_\lambda\mathcal{R}^*\otimes S_\mu\mathcal{Q}\otimes S_\mu G^*,$$

where we sum over partitions λ with e parts and partitions μ with $g-1$ parts. We notice that higher cohomology of the bundles $S_\lambda\mathcal{R}^*\otimes S_\mu\mathcal{Q}$ is zero, with H^0 being just

$$S_{(\mu_1,\ldots,\mu_{g-1},0,-\lambda_e,\ldots,-\lambda_1)}F.$$

Comparing it with Proposition 0.3 we are done. $\square$

Proposition 3.1 implies that the complex $\mathbf{F}_\bullet$ is a minimal free resolution of the coordinate ring of Y.

Let us analyze the cohomology of the exterior powers of ξ. We have

$$\bigwedge^{\bullet}(\xi)=\bigoplus_{\lambda,\mu} S_{\lambda'}E\otimes S_\lambda\mathcal{Q}^*\otimes S_\mu\mathcal{R}\otimes S_{\mu'}G^*.$$

To calculate the cohomology of the summand corresponding to the pair (λ,μ) we need to apply the Bott algorithm, Proposition 1.3, to the sequence

$$(-\lambda_{g-1},\ldots,-\lambda_1,\mu_1,\ldots,\mu_{e+1}).$$

Proposition 3.2.

(a) *The representations of F we get from the above procedure are all of the type $\bigwedge^s F$, for some s with $0\leqslant s\leqslant f$.*
(b) *The ring $\mathbf{K}[Y]$ is normal and Gorenstein and has codimension $eg+1$ as a quotient of $\mathbf{K}[X]$.*

Remark. Assertion (b) is already well known in arbitrary characteristic by work of Kempf [Ke] (characteristic 0) and De Concini and Strickland [DS] on the variety of complexes. Also, $\mathbf{K}[Y]$ clearly is Gorenstein in arbitrary characteristic as a quotient of the Gorenstein ring C by the regular element a; see Corollary 0.4.

Proof. Let us look what will be the highest number in our sequence after applying Bott's algorithm. It clearly is either $-\lambda_{g-1}$ or μ_1-g+1. But $\mu_1\leqslant g$, otherwise the corresponding summand is zero as it involves the factor $S_{\mu'}G^*$. Thus the first number is $\leqslant 1$. Similarly, the last number is either μ_{e+1} or $-\lambda_1+e+1$. Since $\lambda_1\leqslant e$ (otherwise the summand is zero, as it contains factor $S_{\lambda'}E$), we see that the last number is $\geqslant 0$. Thus our weight has to be of the type $(1^s,0^{f-s})$.

Let us look at the top exterior power of ξ. Clearly this is

$$\bigwedge^{\mathrm{top}}\xi=S_{(g-1)^e}E\otimes S_{e^{g-1}}\mathcal{Q}^*\otimes S_{g^{e+1}}\mathcal{R}\otimes S_{(e+1)^g}G^*.$$

To calculate the corresponding term, we need to apply Bott's algorithm to the sequence $(-e^{g-1},g^{e+1})$ which gives the representation $\bigwedge^f F$ in $\mathrm{H}^{(g-1)(e+1)}$. This is the top of the resolution. The representation there is

$$\bigwedge^{\mathrm{top}}\xi=S_{(g-1)^e}E\otimes\bigwedge^{f}F\otimes S_{(e+1)^g}G^*$$

in the homological degree $e(g-1)+g(e+1)-(g-1)(e+1)=eg+1$. The representation is one dimensional, therefore $\mathbf{K}[Y]$ is Gorenstein, of codimension $eg+1$ as claimed. The normality follows because Z is a desingularization of Y. $\square$

The rest of this section is devoted to identifying all of the terms of $\mathbf{F}_\bullet$. The main ideas are contained in the proof of Proposition 3.2: we apply the Bott algorithm many times; however, there are many details to work out. Our answer is recorded as Theorem 3.4 and is expressed in terms of the objects in Definition 3.3. After the proof of Theorem 3.4 is complete, we offer Examples 3.20 and 3.21.

Definition 3.3. Let k be an integer and $\lambda=(\lambda_1,\dots,\lambda_{g-1})$ be a dominant weight. Let $i=\lambda'_k$, which is defined to be the number of indices j with $\lambda_j \geqslant k$. Notice that $\lambda_i \geqslant k > \lambda_{i+1}$. Define $p(\lambda;k)$ to be the dominant weight

$$p(\lambda;k)=(\lambda_1,\dots,\lambda_i,k,\lambda_{i+1}+1,\dots,\lambda_{g-1}+1),$$

$N(\lambda;k)$ to be the integer $g-1-\lambda'_k+k$, and $T_{\lambda;k}$ to be the free A-module

$$T_{\lambda;k}=S_{\lambda'}E\otimes_{\mathbf{K}}\bigwedge^{N(\lambda;k)}F\otimes_{\mathbf{K}}S_{p(\lambda;k)}G^*\otimes_{\mathbf{K}}A.$$

Theorem 3.4. *In the notation of* (0.5)*, the minimal resolution of A/I by free A-modules is*

$$\mathbf{F}_\bullet=\bigoplus_{(\lambda;k)}T_{\lambda;k}\bigl(-|\lambda|,-|\lambda|-N(\lambda;k)\bigr).$$

The sum is taken over all pairs $(\lambda;k)$, where $\lambda=(\lambda_1,\dots,\lambda_{g-1})$ is a partition with $e\geqslant\lambda_1$, and k is integer with $0\leqslant k\leqslant e+1$. The term $T_{\lambda;k}(-|\lambda|,-|\lambda|-N(\lambda;k))$ appears in $\mathbf{F}_{|\lambda|+k}$.

Proof. We know that

$$\mathbf{F}_i=\bigoplus_{\substack{d\geqslant 0\\ |\lambda|+|\mu|=i+d}}\mathrm{H}^d\bigl(\mathrm{Grass}(e+1,F),S_{\lambda'}E\otimes S_\lambda\mathcal{Q}^*\otimes S_\mu\mathcal{R}\otimes S_{\mu'}G\bigr)\otimes A\bigl(-|\lambda|,-|\mu|\bigr).$$

We calculate the cohomology of the vector bundle

$$S_{\lambda'}E\otimes S_\lambda\mathcal{Q}^*\otimes S_\mu\mathcal{R}\otimes S_{\mu'}G^*, \tag{3.5}$$

for partitions $\lambda=(\lambda_1,\dots,\lambda_{g-1})$ and $\mu=(\mu_1,\dots,\mu_{e+1})$ with $\mu_1\leqslant g$ and $\lambda_1\leqslant e$.

We first assume that the contribution of (3.5) is nonzero and we identify k. Start with the weight

$$\alpha(\lambda,\mu)=(-\lambda_{g-1},\dots,-\lambda_1,\mu_1,\dots,\mu_{e+1}),$$

and recalling $\rho=(e+g,\dots,1)$, we have

$$\alpha(\lambda,\mu)+\rho=(-\lambda_{g-1}+e+g,\dots,-\lambda_1+e+2,\mu_1+e+1,\dots,\mu_{e+1}+1).$$

Since $\mu_1 \leqslant g$ and $\lambda_1 \leqslant e$, we see that all coordinates of $\alpha(\lambda,\mu)+\rho$ are integers from the interval $[1, e+g+1]$. Bott's algorithm tells us that the pair (λ,μ) gives a nonzero contribution only if the coordinates of $\alpha(\lambda,\mu)+\rho$ are distinct. Thus, for such a pair (λ,μ), there is a unique $t \in [1, e+g+1]$ such that the coordinates of $\alpha(\lambda,\mu)+\rho$ fill the set $[1, e+g+1] \setminus \{t\}$. The parameter $k =: k(\lambda,\mu)$ is defined to be the cardinality of the set

$$\{s \mid \mu_s + e + 2 - s > t\}.$$

It is clear that $0 \leqslant k \leqslant e+1$. The numbers

$$\mu_1 + e + 1, \ldots, \mu_s + e + 2 - s, \ldots, \mu_{e+1} + 1$$

form a decreasing list; so, as long as one makes the proper interpretation at the boundaries for k, it is convenient to write

$$\mu_k + e + 2 - k > t > \mu_{k+1} + e + 1 - k. \tag{3.6}$$

Now we start with the data $(\lambda; k)$ and we manufacture the corresponding partition μ and integer t. Let

$$A_1 > A_2 > \cdots > A_{e+2} \tag{3.7}$$

be the complement of $\{-\lambda_{g-1} + e + g, \ldots, -\lambda_1 + e + 2\}$ in $\{e+g+1, \ldots, 1\}$. Define t to be A_{k+1} and define μ by

$$(\mu_1 + e + 1, \ldots, \mu_{e+1} + 1) = (A_1, \ldots, \widehat{A_{k+1}}, \ldots, A_{e+2}). \tag{3.8}$$

We see that μ is a partition, $g \geqslant \mu_1$, the coordinates of $\alpha(\lambda,\mu)+\rho$ are distinct, and $k(\lambda,\mu) = k$.

Now that we have manufactured μ and t from the data $(\lambda; k)$, we calculate the contribution of the vector bundle (3.5) to $\mathbf{F}_\bullet$. When $\alpha(\lambda,\mu)+\rho$ has been reordered to become a decreasing sequence, the result is

$$w\big(\alpha(\lambda,\mu)+\rho\big) = (e+g+1, \ldots, \hat{t}, \ldots, 1);$$

therefore, $w(\alpha(\lambda,\mu)+\rho) - \rho = 1^{e+g+1-t}0^{t-1}$. The contribution of the vector bundle (3.5) to $\mathbf{F}_\bullet$ is equal to

$$S_{\lambda'}E \otimes_{\mathbf{K}} \bigwedge^{e+g+1-t} F \otimes_{\mathbf{K}} S_{\mu'}G^* \otimes_{\mathbf{K}} A\big(-|\lambda|, -|\mu|\big).$$

This contribution is a summand of $\mathbf{F}_{|\lambda|+|\mu|-\ell}$, where ℓ is the length of the permutation w. To complete the argument, we show that

$$|\mu| - \ell = k, \tag{3.9}$$

$$e + g + 1 - t = N(\lambda; k), \quad \text{and} \tag{3.10}$$

$$\mu' = p(\lambda; k). \tag{3.11}$$

(It is clear from definition that $|p(\lambda; k)| = |\lambda| + N(\lambda; k)$.)

To accomplish these ends, we introduce the indices $i_1 < \cdots < i_{e+1}$ to which the terms $\mu_s + e + 2 - s$ go when $\alpha(\lambda, \mu) + \rho$ has been reordered to become a decreasing sequence. In other words, if $(B_1, \ldots, B_{e+g})$ is the decreasing sequence

$$(B_1, \ldots, B_{e+g}) = (e + g + 1, \ldots, t + 1, \hat{t}, t - 1, \ldots, 1), \tag{3.12}$$

then the decreasing sequences

$$(B_{i_1}, \ldots, B_{i_{e+1}}) = (\mu_1 + e + 1, \ldots, \mu_{e+1} + 1), \tag{3.13}$$

are equal. To rearrange $\alpha(\lambda, \mu) + \rho$ into decreasing order, one must make

$$\ell = (g - i_1) + (g + 1 - i_2) + \cdots + (g + e - i_{e+1}) = (e + 1)g + \binom{e+1}{2} - \sum_{s=1}^{e+1} i_s$$

exchanges. Equation (3.13) yields

$$B_{i_s} = \mu_s + e + 2 - s \tag{3.14}$$

and Eq. (3.12) gives

$$B_s = \begin{cases} e + g + 2 - s, & \text{if } B_s > t, \\ e + g + 1 - s, & \text{if } t > B_s. \end{cases} \tag{3.15}$$

Recall from (3.6) that $B_{i_s} > t$ if and only if $s \leqslant k$. Combine (3.14) and (3.15) to see that

$$i_s = \begin{cases} g - \mu_s + s, & \text{if } s \leqslant k, \\ g - \mu_s + s - 1, & \text{if } k < s. \end{cases}$$

We now have

$$\sum_{s=1}^{e+1} i_s = (e + 1)g - |\mu| + \binom{e+2}{2} - (e + 1 - k);$$

and therefore, Eq. (3.9) holds.

We establish (3.10) and (3.11) by explicitly recording the values for μ and t in terms of the data $(\lambda; k)$. We think of λ as $e^{n_e}(e-1)^{n_{e-1}} \cdots 1^{n_1} 0^{n_0}$. It is clear that $n_s = \lambda'_s - \lambda'_{s+1}$. We study the entries of the vector

$$[-\lambda_{g-1}, -\lambda_{g-2}, \ldots, -\lambda_1] + [e + g, e + g - 1, \ldots, e + 2]. \tag{3.16}$$

For each integer s, with $0 \leqslant s \leqslant e$, the vector (3.16) contains the following subvector of consecutive integers:

$$[-s,\dots,-s]+\left[e+1+\lambda'_s,\dots,e+2+\lambda'_{s+1}\right];$$

and therefore, the set of entries of (3.16) is

$$\bigcup_{0\leqslant s\leqslant e}\left\{\ell \mid e+2-s+\lambda'_{s+1}\leqslant \ell\leqslant e+1-s+\lambda'_s\right\}. \tag{3.17}$$

The complement of (3.17), in the interval $[1,e+g+1]$, is

$$\left\{e+2-s+\lambda'_s \mid 0\leqslant s\leqslant e+1\right\}. \tag{3.18}$$

The elements of (3.18) were written in decreasing order in (3.7) with

$$A_s=e+3-s+\lambda'_{s-1},$$

for $1\leqslant s\leqslant e+2$. So, $t=A_{k+1}=e+2-k+\lambda'_k$. Equation (3.10) follows immediately. Also, Eq. (3.8) shows that

$$\mu_s=\begin{cases}1+\lambda'_{s-1}, & \text{for } 1\leqslant s\leqslant k,\\ \lambda'_s, & \text{for } k+1\leqslant s\leqslant e+1.\end{cases} \tag{3.19}$$

A quick calculation shows that $p(\lambda;k)'_s$ is also given by the right side of (3.19); thus, (3.11) holds and the proof is complete. □

Example 3.20. Let us take $e=g=2$. We give two versions of our resolution $\mathbf{F}_\bullet$. In the first version, we write $(a,b;c;d,e)$ for $S_{(a,b)}E\otimes\bigwedge^c F\otimes S_{(d,e)}G^*$. Our resolution has the following terms.

$$\begin{gathered}
(1,1;4;3,3)\otimes A(-2,-6)\\
\downarrow\\
(1,0;4;3,2)\otimes A(-1,-5)\oplus(1,1;2;2,2)\otimes A(-2,-4)\\
\downarrow\\
(1,0;3;2,2)\otimes A(-1,-4)\oplus(1,1;1;2,1)\otimes A(-2,-3)\oplus(0,0;4;3,1)\otimes A(0,-4)\\
\downarrow\\
(1,1;0;2,0)\otimes A(-2,-2)\oplus(0,0;3;2,1)\otimes A(0,-3)\oplus(1,0;1;1,1)\otimes A(-1,-2)\\
\downarrow\\
(0,0;2;1,1)\otimes A(0,-2)\oplus(1,0;0;1,0)\otimes A(-1,-1)\\
\downarrow\\
(0,0;0;0,0)\otimes A
\end{gathered}$$

The terms of $\mathbf{F}_\bullet$ are also listed in the following picture, which has the added advantage of giving insight into the maps of $\mathbf{F}_\bullet$. The row which corresponds to the partition λ is $S_{\lambda'}E\otimes_{\mathbf{K}}\mathbf{t}_\lambda$ as described in Theorem 4.7. Each row is acyclic. The Koszul complex map down the column on the right, as described in Proposition 5.12, induces an acyclic sequence on the zeroth homology of the rows; see (5.3). An iterated mapping cone produces the complex $\mathbf{F}_\bullet$; as shown in Theorem 5.4. In other words, there is a map of complexes from the middle row to the bottom row; there is a map of complexes from the top row (shifted up by one against the differential) to the mapping cone formed from the bottom two rows; and $\mathbf{F}_\bullet$ is the mapping cone of this second map of complexes. Notice that it is not correct to think of this picture as a double complex. The "knight move" $T_{2;1}(-2,-3) \to T_{0;2}(0,-3)$ which is induced by $\bigwedge^2 \phi$, see (5.10), is one of the components of the differential of $\mathbf{F}_\bullet$.

$$\begin{tikzcd}
T_{2;3}(-2,-6) \arrow[r] & T_{2;2}(-2,-4) \arrow[r] & T_{2;1}(-2,-3) \arrow[r] & T_{2;0}(-2,-2) \arrow[d] \\
T_{1;3}(-1,-5) \arrow[r] & T_{1;2}(-1,-4) \arrow[r] & T_{1;1}(-1,-2) \arrow[r] & T_{1;0}(-1,-1) \arrow[d] \\
T_{0;3}(0,-4) \arrow[r] & T_{0;2}(0,-3) \arrow[r] & T_{0;1}(0,-2) \arrow[r] & T_{0;0}
\end{tikzcd}$$

Example 3.21. Let us take $e=2$, $g=3$. Our resolution has the following terms where we write $(a,b;c;d,e,f)$ for $S_{(a,b)}E \otimes \bigwedge^c F \otimes S_{(d,e,f)}G^*$.

$$(2,2;5;3,3,3) \otimes A(-4,-9)$$
$$\downarrow$$
$$(2,1;5;3,3,2) \otimes A(-3,-8) \oplus (2,2;2;2,2,2) \otimes A(-4,-6)$$
$$\downarrow$$
$$(2,2;1;2,2,1) \otimes A(-4,-5) \oplus (2,1;3;2,2,2) \otimes A(-3,-6) \oplus (2,0;5;3,2,2) \otimes A(-2,-7)$$
$$\oplus\ (1,1;5;3,3,1) \otimes A(-2,-7)$$
$$\downarrow$$
$$(2,2;0;2,2,0) \otimes A(-4,-4) \oplus (2,0;4;2,2,2) \otimes A(-2,-6) \oplus (1,0;5;3,2,1) \otimes A(-1,-6)$$
$$\oplus\ (1,1;3;2,2,1) \otimes A(-2,-5) \oplus (2,1;1;2,1,1) \otimes A(-3,-4)$$
$$\downarrow$$
$$(1,0;4;2,2,1) \otimes A(-1,-5) \oplus (1,1;2;2,1,1) \otimes A(-2,-4) \oplus (2,1;0;2,1,0) \otimes A(-3,-3)$$
$$\oplus\ (0,0;5;3,1,1) \otimes A(0,-5) \oplus (2,0;1;1,1,1) \otimes A(-2,-3)$$
$$\downarrow$$
$$(0,0;4;2,1,1) \otimes A(0,-4) \oplus (1,0;2;1,1,1) \otimes A(-1,-3) \oplus (1,1;0;2,0,0) \otimes A(-2,-2)$$
$$\oplus\ (2,0;0;1,1,0) \otimes A(-2,-2)$$
$$\downarrow$$
$$(0,0;3;1,1,1) \otimes A(0,-3) \oplus (1,0;0;1,0,0) \otimes A(-1,-1)$$
$$\downarrow$$
$$(0,0;0;0,0,0) \otimes A$$

Also, $\mathbf{F}_\bullet$ is the iterated mapping cone of a picture built using the following modules.

$$
\begin{array}{ccccccc}
T_{2,2;3}(-4,-9) & \longrightarrow & T_{2,2;2}(-4,-6) & \longrightarrow & T_{2,2;1}(-4,-5) & \longrightarrow & T_{2,2;0}(-4,-4)\\
&&&&&& \downarrow\\
T_{2,1;3}(-3,-8) & \longrightarrow & T_{2,1;2}(-3,-6) & \longrightarrow & T_{2,1;1}(-3,-4) & \longrightarrow & T_{2,1;0}(-3,-3)\\
&&&&&& \downarrow\\
T_{1,1;3}(-2,-7) & & T_{1,1;2}(-2,-6) & & T_{1,1;1}(-2,-3) & & T_{1,1;0}(-2,-2)\\
\oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus\\
T_{2,0;3}(-2,-7) & & T_{2,0;2}(-2,-5) & & T_{2,0;1}(-2,-4) & & T_{2,0;0}(-2,-2)\\
&&&&&& \downarrow\\
T_{1,0;3}(-1,-6) & \longrightarrow & T_{1,0;2}(-1,-5) & \longrightarrow & T_{1,0;1}(-1,-3) & \longrightarrow & T_{1,0;0}(-1,-1)\\
&&&&&& \downarrow\\
T_{0,0;3}(0,-5) & \longrightarrow & T_{0,0;2}(0,-4) & \longrightarrow & T_{0,0;1}(0,-3) & \longrightarrow & T_{0,0;0}
\end{array}
$$

4. A family of maximal Cohen–Macaulay modules over a determinantal ring

Our investigation of the differential in the resolution $\mathbf{F}_\bullet$ quickly leads to a family of modules of independent interest.

The parameterization of $\mathbf{F}_\bullet$ given in Theorem 3.4 allows us to write down the terms of $\mathbf{F}_\bullet$ in a different way. One way to do that is to look at the terms with a fixed λ. In order to describe this part of the complex we need another geometric construction related to the Grassmannian of G. Consider $\operatorname{Grass}(g-1, G)$ with the tautological sequence

$$0 \to \bar{\mathcal{R}} \to G \times \operatorname{Grass}(g-1, G) \to \bar{\mathcal{Q}} \to 0. \tag{4.1}$$

We are dealing with the polynomial ring $\bar{A} = \operatorname{Sym}(F \otimes G^*)$ and the modules supported in the determinantal varieties of maps ψ of rank $\leqslant g-1$. We look at twisted complexes $\bar{\mathbf{F}}(S_\lambda\bar{\mathcal{R}}^*)_\bullet = \bar{\mathbf{F}}(\lambda)_\bullet$ which come from taking $\xi = F \otimes \bar{\mathcal{Q}}^*$. Each such complex is the pushdown of the locally free resolution of the sheaf

$$\mathcal{M}(\lambda) := S_\lambda\bar{\mathcal{R}}^* \otimes \operatorname{Sym}(F \otimes \bar{\mathcal{R}}^*).$$

Proposition 4.2. *The sheaf $\mathcal{M}(\lambda)$ has no higher cohomology. Thus the complex $\bar{\mathbf{F}}(\lambda)_\bullet$ is a free resolution of the $\bar{A}$-module*

$$M(\lambda) := \mathrm{H}^0\big(\operatorname{Grass}(g-1, G), \mathcal{M}(\lambda)\big).$$

Assume that $\lambda \subset e^{g-1}$. Then the complex $\bar{\mathbf{F}}(\lambda)_\bullet$ is a complex of length $f-g+1$. Thus the corresponding module $M(\lambda)$ is a maximal Cohen–Macaulay module.

Proof. This is a standard application of the geometric technique, see [W, Chapter 6]. □

Remark. Let us look at the resolution of $M(\lambda)$ more precisely. It is a pushdown of the twisted Koszul complex

$$S_\lambda \bar{\mathcal{R}}^* \otimes \bigwedge^{\bullet}(F \otimes \bar{\mathcal{Q}}^*).$$

Thus we can describe the terms as $\bigwedge^i F$ tensored with the representation $S_{\mu(i)}G$, where $\mu(i)$ is the result of Bott algorithm applied to the weight

$$(-i, -\lambda_{g-1}, \dots, -\lambda_1).$$

The terms we get in H^0 correspond to i satisfying $-i \geqslant -\lambda_{g-1}$. For each such i, the H^0-module is equal to

$$\bigwedge^i F \otimes S_{(-i,-\lambda_{g-1},\dots,-\lambda_1)}G = \bigwedge^i F \otimes S_{(\lambda_1,\dots,\lambda_{g-1},i)}G^*,$$

and it appears in the ith place in the complex $\bar{\mathbf{F}}(\lambda)_\bullet$. The terms we get in H^s for $s \geqslant 1$ correspond to i satisfying the inequalities

$$-\lambda_{g-s} - 1 \geqslant -i + s \geqslant -\lambda_{g-s-1}.$$

For each pair (i, s), the H^s-module is equal to

$$\begin{aligned}
&\bigwedge^i F \otimes S_{(-\lambda_{g-1}-1,\dots,-\lambda_{g-s}-1,-i+s,-\lambda_{g-s-1},\dots,-\lambda_1)}G \\
&\quad = \bigwedge^i F \otimes S_{(\lambda_1,\dots,\lambda_{g-s-1},i-s,\lambda_{g-s}+1,\dots,\lambda_{g-1}+1)}G^* \\
&\quad = \bigwedge^i F \otimes S_{p(\lambda;i-s)}G^*,
\end{aligned}$$

and it appears in the $(i-s)$th place in the complex $\bar{\mathbf{F}}(\lambda)_\bullet$.

Proposition 4.3. *Let λ be a partition contained in the rectangle e^{g-1}. Then the terms of the complex $\mathbf{F}_\bullet$ containing the factor $S_{\lambda'}E$ are identical with the terms of the complex $S_{\lambda'}E \otimes \bar{\mathbf{F}}(\lambda)_\bullet \otimes A(-|\lambda|, -|\lambda|)[|\lambda|]$. Here $[i]$ means homological shift, i.e., the term in position zero of $S_{\lambda'}E \otimes \bar{\mathbf{F}}(\lambda)_\bullet \otimes A(-|\lambda|, -|\lambda|)$ occurs in $\mathbf{F}_{|\lambda|}$.*

Proof. Direct calculation—just look at the pairs $(\lambda; k)$. The lowest term where $S_{\lambda'}E$ occurs corresponds to $k = 0$. Apply Theorem 3.4 to see that $S_{\lambda'}E \otimes \bigwedge^0 F \otimes S_\lambda G^* \otimes A(-|\lambda|, -|\lambda|)$ occurs in the term $\mathbf{F}_{|\lambda|}$. □

This new description of the terms can best be expressed in the language of Definition 3.3. The modules $M(\lambda)$ of Proposition 4.2 are maximal Cohen–Macaulay modules over the determinantal ring $\bar{A}/I_g(\psi)$, for $\bar{A} = \mathrm{Sym}(F \otimes G^*)$, where $\psi : F \otimes_{\mathbf{K}} \bar{A} \to G \otimes_{\mathbf{K}} \bar{A}$ is the natural map. These modules have independent interest. In Theorem 4.7 we record the $\bar{A}$ resolution $\mathbf{t}_\lambda$ of $\mathrm{H}_0(\mathbf{t}_\lambda) = M(\lambda)$ using one parameter k in place of the two parameters i and s that were used to date. Recall that $\mathbf{K}$ is a field of characteristic zero, F and G are vector spaces over $\mathbf{K}$ of dimension f and g, respectively, and $e = f - g$.

Definition 4.4. Let k be an integer and $\lambda = (\lambda_1, \dots, \lambda_{g-1})$ be a dominant weight.

(a) Let $t_{\lambda;k}$ to be the free $\bar{A}$-module

$$t_{\lambda;k} = \bigwedge^{N(\lambda;k)} F \otimes_{\mathbf{K}} S_{p(\lambda;k)} G^* \otimes_{\mathbf{K}} \bar{A}.$$

(b) Define a homomorphism $t_{\lambda;k} \to t_{\lambda;k-1}$. Let $N = 1 + \lambda'_{k-1} - \lambda'_k$. It follows that there exist dominant weights α and β with $\alpha_{\mathrm{last}} \geqslant k > \beta_1$,

$$p(\lambda; k) = \bigl(\alpha, k^N, \beta\bigr) \quad \text{and} \quad p(\lambda; k-1) = \bigl(\alpha, (k-1)^N, \beta\bigr).$$

The homomorphism

$$t_{\lambda;k} \to t_{\lambda;k-1} \tag{4.5}$$

is the composition

$$\begin{aligned} t_{\lambda;k} = \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)} G^* \otimes \bar{A} &\to \bigwedge^{N(\lambda;k)} F \otimes S_{1^N} G^* \otimes S_{p(\lambda;k-1)} G^* \otimes \bar{A} \\ \to \bigwedge^{N(\lambda;k)} F \otimes S_{1^N} F^* \otimes S_{p(\lambda;k-1)} G^* \otimes \bar{A} &\to \bigwedge^{N(\lambda;k)-N} F \otimes S_{p(\lambda;k-1)} G^* \otimes \bar{A} = t_{\lambda;k-1}, \end{aligned}$$

where the first map is the Pieri map, the second is $\bigwedge^N \psi^*$, and the third is the module action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$.

(c) For each dominant weight $\lambda = (\lambda_1, \dots, \lambda_{g-1})$, we define the complex $\mathbf{t}_\lambda$:

$$\cdots \to t_{\lambda;k}\bigl(0, -N(\lambda;k)\bigr) \to t_{\lambda;k-1}\bigl(0, -N(\lambda;k-1)\bigr) \to \cdots,$$

with $t_{\lambda;k}$ in position k.

Remarks. (a) The dominant weight $p(\lambda;k)$ may be interpreted as the result of applying Bott's algorithm to the sequence

$$\lambda_1, \dots, \lambda_{g-1}, N(\lambda;k).$$

(b) If $\lambda_{g-1} \geqslant -1$ and $k < 0$, then $t_{\lambda;k} = 0$.

(c) If $\lambda_{g-1} \geqslant -1$ and $k \geqslant 0$, then $p(\lambda;k)$ is a partition.

(d) The maps and modules of $\mathbf{t}_\lambda$ form a complex because the Littlewood–Richardson rule tells us that the only coordinate free $\mathbf{K}$-vector space map

$$S_{\alpha,k^{1+N},(k-1)^M,\beta}G^* \to S_{1^{2+N+M}}G^* \otimes S_{\alpha,(k-1)^N,(k-2)^{1+M},\beta}G^*$$

is zero, when α and β are dominant weights with $\alpha_{\text{last}} \geqslant k$ and $k-1 > \beta_1$.

Proposition 4.6. *If $\lambda = (\lambda_1, \dots, \lambda_{g-1})$ and $\mu = (e-\lambda_{g-1}, \dots, e-\lambda_1)$ are dominant weights, then the complexes $\mathbf{t}_\lambda$ and $(\mathbf{t}_\mu)^*[-e-1]$ are isomorphic. Furthermore, if λ is contained in $[-1, e+1]^{g-1}$, then μ also sits in $[-1, e+1]^{g-1}$ and $(\mathbf{t}_\lambda)_i = 0$ for $i < 0$ or $e+1 < i$.*

Proof. A way to see duality of the terms is as follows. Let k and ℓ be integers with $k+\ell = e+1$. The modules

$$\bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)}G^* \quad \text{and} \quad \bigwedge^{N(\mu;\ell)} F \otimes S_{p(\mu;\ell)}G^*$$

are dual to one another because $N(\lambda;k) + N(\mu;\ell) = f$ and if

$$p(\lambda;k) = (A_1, \dots, A_g) \quad \text{and} \quad p(\mu;\ell) = (B_1, \dots, B_g),$$

then $A_i + B_{g+1-i} = e+1$. A direct calculation completes the proof. □

Theorem 4.7. *Let k be an integer and $\lambda = (\lambda_1, \dots, \lambda_{g-1})$ be a dominant weight.*

(1) *If $\lambda_{g-1} \geqslant -1$, then*
   (a) *$\mathbf{t}_\lambda$ is a resolution of $\mathrm{H}_0(\mathbf{t}_\lambda)$, and*
   (b) *$\mathrm{H}_0(\mathbf{t}_\lambda)$ is a module over $\bar{A}/I_g(\psi)$.*

(2) *If $\lambda \subset [-1, e+1]^{g-1}$, then*
   (a) *$\mathrm{H}_0(\mathbf{t}_\lambda)$ is a perfect $\bar{A}$-module with*

$$\mathrm{Ext}_{\bar{A}}^{e+1}\big(\mathrm{H}_0(\mathbf{t}_\lambda), \bar{A}\big) = \mathrm{H}_0(\mathbf{t}_\mu)$$

   for $\mu = (e-\lambda_{g-1}, \dots, e-\lambda_1)$, and
   (b) *$\mathrm{H}_0(\mathbf{t}_\lambda)$ is a maximal Cohen–Macaulay $\bar{A}/I_g(\psi)$-module.*

Proof. Apply the Comparison Principle, Proposition 2.2, to the complexes $\mathbf{H}_\bullet = \bar{\mathbf{F}}(S_\lambda\bar{\mathcal{R}}^*)_\bullet$ and $\mathbf{H}'_\bullet = \mathbf{t}_\lambda$. □

Example. In particular, if $\lambda = (i^{g-1})$, then the complex $\mathbf{t}_\lambda$ is isomorphic to the Eagon–Northcott complex $\mathcal{C}^i$, see, for example, [E, Figure A2.6], and

$$\mathrm{H}_0(\mathbf{t}_{i^{g-1}}) = \begin{cases} \bigwedge^{e+1}\operatorname{cok}(\psi^*), & \text{if } i = -1, \\ \bar{A}/I_g(\psi), & \text{if } i = 0, \\ \mathrm{Sym}_i(\operatorname{cok}(\psi)), & \text{if } 1 \leqslant i. \end{cases}$$

5. The homogeneous strands of the differential of $\mathbf{F}_\bullet$

We return to the resolution $\mathbf{F}_\bullet$. The present section has two main results. In Theorem 5.4 we show that $\mathbf{F}_\bullet$ may be obtained as an iterated mapping cone as had been promised in Example 3.20. In Theorem 5.13 we describe the homogeneous strands of the differential of $\mathbf{F}_\bullet$.

The description of the terms of the complex $\mathbf{F}_\bullet$ given in Theorem 3.4 is not accidental. It comes from a pushdown of different Koszul complex. Consider again the Grassmannian $\mathrm{Grass}(g-1, G)$ and the tautological sequence

$$0 \to \bar{\mathcal{R}} \to G \times \mathrm{Grass}(g-1, G) \to \bar{\mathcal{Q}} \to 0$$

of (4.1). Consider the sheaf of algebras

$$\mathcal{B} = \mathrm{Sym}(E \otimes F^*) \otimes \mathrm{Sym}(F \otimes \bar{\mathcal{R}}^*)$$

over $\mathrm{Grass}(g-1, G)$. Obviously, we have linear maps

$$\phi : E \otimes \mathcal{B} \to F \otimes \mathcal{B} \quad \text{and} \quad \psi' : F \otimes \mathcal{B} \to \bar{\mathcal{R}} \otimes \mathcal{B}$$

of sheaves of $\mathcal{B}$-modules. The condition $\psi'\phi = 0$ induces the Koszul complex of sheaves of $\mathcal{B}$-modules given by the entries of the composition:

$$\mathcal{K}_\bullet : 0 \to \mathcal{K}_{e(g-1)} \to \mathcal{K}_{e(g-1)-1} \to \cdots \to \mathcal{K}_1 \to \mathcal{K}_0,$$

with $\mathcal{K}_i = \bigwedge^i (E \otimes \bar{\mathcal{R}}^*) \otimes \mathcal{B}$. Notice that

$$\mathcal{K}_i = \bigoplus_{|\lambda| = i} S_{\lambda'} E \otimes S_\lambda \bar{\mathcal{R}}^* \otimes \mathcal{B}.$$

Lemma 5.1. *The complex $\mathcal{K}_\bullet$ is acyclic.*

Proof. The complex $\mathcal{K}_\bullet$ of $\mathcal{B}$-modules is the relative version of the Koszul complex for the variety of complexes. To be more precise, take three vector spaces E, F, G' of dimensions e, f, $g-1$ respectively. Consider the polynomial ring

$$B = \mathrm{Sym}(E \otimes F^*) \otimes \mathrm{Sym}(F \otimes G'^*).$$

The ring B is the coordinate ring of the affine space X of pairs (ϕ, ψ') of linear maps

$$\phi : E \to F \quad \text{and} \quad \psi' : F \to G'.$$

We want to show that the subvariety Y of pairs of maps (ϕ, ψ') such that $\psi'\phi = 0$ is a complete intersection cut out by the entries of the product matrix $\psi'\phi$. To show this it is enough to show that the codimension of Y in X is $e(g-1)$. Of course $\dim X = ef + f(g-1)$. To calculate dimension of Y we construct its usual desingularization

$$Z = \{(\phi, \psi', S) \in X \times \mathrm{Grass}(e, F) \mid \mathrm{Im}(\phi) \subset S \subset \mathrm{Ker}(\psi')\}.$$

The first projection $(\phi, \psi', S) \mapsto (\phi, \psi')$ is a birational map, as over a general point we have to have $S = \mathrm{Im}(\phi)$, so over an open set where ϕ has a full rank the first projection is an isomorphism. Projecting Z onto the Grassmannian we see that the fibres have dimension $e^2 + (f-e)(g-1)$, so

$$\dim Y = \dim Z = e(f-e) + e^2 + (f-e)(g-1).$$

We conclude that

$$\dim X - \dim Y = e(g-1)$$

which concludes the proof. $\square$

Let us denote $\hat{\mathcal{M}}(\lambda) := S_\lambda \bar{\mathcal{R}}^* \otimes \mathcal{B}$ and $\hat{M}(\lambda) := \mathrm{H}^0(\mathrm{Grass}(g-1, G), \hat{\mathcal{M}}(\lambda))$.

Proposition 5.2. *We have the following properties*:

(a) $\mathrm{H}^j(\mathrm{Grass}(g-1, G), \mathcal{K}_i) = 0$ *for* $j > 0$ *and* $0 \leqslant i \leqslant e(g-1)$,
(b) $\mathrm{H}^j(\mathrm{Grass}(g-1, G), \hat{\mathcal{M}}(\lambda)) = 0$, *for* $j > 0$, *and*
(c) *the resolution of* $\hat{M}(\lambda)$ *as an* A*-module is* $\bar{\mathbf{F}}(\lambda)_\bullet \otimes_{\bar{A}} A$.

Proof. This is clear from the definitions. $\square$

The Koszul complex $\mathcal{K}_\bullet$ induces an acyclic complex of sections

$$K_\bullet\colon \quad 0 \to K_{e(g-1)} \to K_{e(g-1)-1} \to \cdots \to K_1 \to K_0, \tag{5.3}$$

where $K_i := \mathrm{H}^0(\mathrm{Grass}(g-1, G), \mathcal{K}_i) = \bigoplus_{|\lambda|=i} S_{\lambda'} E \otimes \hat{M}(\lambda)$. We can now use the iterated mapping cone construction to construct the resolution $\mathbf{F}'_\bullet$ of the zeroth homology group of $K_\bullet$. The terms of this resolution are the same as the terms of $\mathbf{F}_\bullet$. The whole process can be made $\mathrm{GL}(E) \times \mathrm{GL}(F) \times \mathrm{GL}(G)$-equivariant.

Theorem 5.4. *The resulting complex* $\mathbf{F}'_\bullet$ *is isomorphic to* $\mathbf{F}_\bullet$.

Proof. Both complexes are equivariant resolutions of the same A-module A/I. The resolution $\mathbf{F}_\bullet$ is minimal and the complexes have the same terms. $\square$

Corollary 5.5.

(a) *The* ψ*-component of the complex* $\mathbf{F}_\bullet$ *is the sum of differentials in the complexes* $\bar{\mathbf{F}}(\lambda)_\bullet$.
(b) *The complementary partitions with respect to the rectangle* e^{g-1} *give the parts of* $\bar{\mathbf{F}}(\lambda)_\bullet$ *that are dual to each other. The complex* $\mathbf{F}_\bullet$ *is self-dual and has length equal to* $eg+1$.

Proof. The first part follows from the construction of the mapping cone. The second part was explained in Proposition 4.6. $\square$

We turn now to describing the homogeneous strands of the differential of $\mathbf{F}_\bullet$. For future reference we write the map of (a) from the previous result as

$$d^{\lambda;k}_{\lambda;k-1}(1) : T_{\lambda;k} = S_{\lambda'}E \otimes_{\mathbf{K}} t_{\lambda;k} \xrightarrow{1\otimes(4.5)} S_{\lambda'}E \otimes_{\mathbf{K}} t_{\lambda;k-1} = T_{\lambda;k-1}, \tag{5.6}$$

where "(4.5)" is the map of (4.5). The symbol "(1)" in $d^{\lambda;k}_{\mu;\ell}(1)$ in (5.6) indicates that we consider exactly one map $T_{\lambda;k} \to T_{\mu;\ell}$. In (5.10) we also consider only one map $d^{\lambda;k}_{\mu;\ell}(1) : T_{\lambda;k} \to T_{\mu;\ell}$; however in (5.11) we consider two maps $d^{\lambda;k}_{\mu;\ell}(c)$, with c equal to 1 or 2.

Proposition 5.7. *The differential of the complex $\mathbf{F}_\bullet$ has three components. One involves only the map ϕ, the second only the map ψ, and the third component is of degree $(1,1)$ in ϕ and ψ, and it does not change the F-component of the term. We refer to these components as the ϕ-component, the ψ-component, and the $(\psi\phi)$-component, respectively.*

Proof. Consider two terms $T_{\lambda,k}$ and $T_{\bar\lambda,\bar k}$ satisfying the condition

$$|\lambda| + k = |\bar\lambda| + \bar k + 1. \tag{5.8}$$

The nonzero differential can occur between these two terms only if $\lambda \supset \bar\lambda$ and $p(\lambda;k) \supset p(\bar\lambda;\bar k)$. There are three cases.

In Case 1 we have $\bar k < k$. The conditions $\bar\lambda \subseteq \lambda$ and (5.8) force $\bar k = k-1$ and $\bar\lambda = \lambda$. Let $N = N(\lambda;k) - N(\bar\lambda;\bar k)$. The map $T_{\lambda;k} \to T_{\bar\lambda;\bar k}$ factors through $T_{\bar\lambda;\bar k} \otimes \bigwedge^N F \otimes \bigwedge^N G^*$ and involves only ψ.

In the two remaining cases, $k \leqslant \bar k$. Let $i = \lambda'_k$ (so $\lambda_i \geqslant k > \lambda_{i+1}$) and let $\bar i$ be the analogous number for $(\bar\lambda;\bar k)$; that is, $\bar\lambda_{\bar i} \geqslant \bar k > \lambda_{\bar i+1}$. The inequalities

$$p(\lambda;k)_{i+1} = k \leqslant \bar k = p(\bar\lambda;\bar k)_{\bar i+1} \leqslant p(\lambda;k)_{\bar i+1} \tag{5.9}$$

tell us that $\bar i \leqslant i$.

In Case 2 we have $k \leqslant \bar k$ and $\bar i < i$. The condition (5.8) gives

$$0 = \sum_{s \leqslant \bar i}(\lambda_s - \bar\lambda_s) + (\lambda_{\bar i+1} - \bar k) + \sum_{\bar i+2 \leqslant s \leqslant i}(\lambda_s - \bar\lambda_{s-1}) + \big(k - (\bar\lambda_i + 1)\big) + \sum_{i+1 \leqslant s}(\lambda_s - \bar\lambda_s).$$

The condition $p(\bar\lambda;\bar k) \subseteq p(\lambda;k)$ ensures that $1 \leqslant \lambda_s - \bar\lambda_{s-1}$ for all s with $\bar i + 2 \leqslant s \leqslant i$. The same condition also ensures that all of the other listed differences are nonnegative. Thus, $i \leqslant \bar i + 1$ and all of the remaining listed differences are zero. One may quickly calculate that

$$p(\bar\lambda;\bar k) = p(\lambda;k), \quad \bar k = \lambda_i,\ \bar\lambda_i = k-1,$$

and $\bar\lambda_s = \lambda_s$ for all $s \neq i$. Let $N = N(\bar\lambda;\bar k) - N(\lambda;k)$. The map $T_{\lambda;k} \to T_{\bar\lambda;\bar k}$ factors through $S_{\bar\lambda'}E \otimes \bigwedge^N E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\bar\lambda;\bar k)}G^*$ and involves only ϕ.

In Case 3 we have $k \leqslant \bar k$ and $\bar i = i$. The inequalities of (5.9) tell us that $\bar k = k$. The conditions $\bar\lambda \subseteq \lambda$ and (5.8) force $\bar\lambda$ to differ from λ by exactly one box. It follows that $p(\bar\lambda;\bar k)$ and $p(\lambda;k)$ also differ by exactly one box. The terms $T_{\lambda;k}$ and $T_{\bar\lambda;\bar k}$ have the same $\mathrm{SL}(F)$-coordinate and the map $T_{\lambda;k} \to T_{\bar{\lambda};\bar{k}}$ factors through $T_{\bar{\lambda};\bar{k}} \otimes E \otimes G^*$. The degree of the differential is one in both ψ and ϕ. $\square$

The description of the terms of the complex $\mathbf{F}_\bullet$ given in Theorem 3.4 allows us also to understand the ϕ-component of the differential. Consider two terms of $\mathbf{F}_\bullet$ with the same factor $S_\mu G^*$, but occurring in neighboring terms of the complex. In other words, we are given the data $(\lambda; k)$ and $(\bar{\lambda}, \bar{k})$, from Case 2 of the proof of Proposition 5.7, with $k \geqslant 1$ and $\lambda'_k \geqslant 1$. One may check that $N(\bar{\lambda}; \bar{k}) = N(\lambda; k) + \bar{k} - k + 1$. The map

$$d^{\lambda;k}_{\bar{\lambda};\bar{k}}(1) : T_{\lambda;k} \to T_{\bar{\lambda};\bar{k}} \tag{5.10}$$

is the composition

$$\begin{aligned} T_{\lambda;k} = S_{\lambda'} E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)} G^* &\to S_{\bar{\lambda}'} E \otimes S_{(\bar{k}-k+1)'} E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)} G^* \\ &\to S_{\bar{\lambda}'} E \otimes \bigwedge^{\bar{k}-k+1} F \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)} G^* \to S_{\bar{\lambda}'} E \otimes \bigwedge^{N(\bar{\lambda};\bar{k})} F \otimes S_{p(\lambda;k)} G^* = T_{\bar{\lambda};\bar{k}}, \end{aligned}$$

where the first map is the Pieri map, the second is $\bigwedge^{\bar{k}-k+1} \phi$, and the third is exterior multiplication.

Finally, we can also describe the terms between which we have a $(\psi\phi)$-component map. Consider the term

$$T_{\lambda;k} = S_{\lambda'} E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)} G^*.$$

Consider a corner box of the partition λ' such that we can also subtract the corresponding box from $p(\lambda; k)$ in such way that we get another nonzero term, with the same cohomology group, in the complex $\mathbf{F}_\bullet$. The exterior power $\bigwedge^{N(\lambda;k)} F$ will be unaffected. The new term will occur in degree by one smaller in $\mathbf{F}_\bullet$ than the original term (we decreased λ' by one box, but the homogeneous degree from ψ and the number of cohomology group stayed the same). Between these two terms we have a $(1, 1)$ degree map from $(\psi\phi)$-component. In other words, let ϵ_j represent the $(g - 1)$-tuple with 1 in position j and zero everywhere else. The maps

$$d^{\lambda;k}_{\lambda-\epsilon_j;k}(c) : T_{\lambda;k} \to T_{\lambda-\epsilon_j;k}, \tag{5.11}$$

with $c = 1$ or 2, are defined provided $\lambda - \epsilon_j$ is a partition and $\lambda_j \neq k$. The hypothesis ensures that

$$p(\lambda - \epsilon_j; k) = p(\lambda; k) - \epsilon_J,$$

where

$$J = \begin{cases} j, & \text{if } \lambda_j > k, \\ j+1, & \text{if } k > \lambda_j. \end{cases}$$

The map is the composition:

$$T_{\lambda;k} = S_{\lambda'}E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda;k)}G^* \to S_{(\lambda-\epsilon_j)'}E \otimes S_1E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_1G^* \otimes S_{p(\lambda-\epsilon_j;k)}G^*$$
$$\to S_{(\lambda-\epsilon_j)'}E \otimes \bigwedge^{N(\lambda;k)} F \otimes S_{p(\lambda-\epsilon_j;k)}G^* = T_{\lambda-\epsilon_j;k}.$$

The first arrow is two Pieri maps to split one box from each of λ' and $p(\lambda;k)$. The second arrow has two components. The first component ($c=1$) uses the map $E \otimes G^* \to A$ given by the composition $\psi\phi$. The second component ($c=2$) uses the maps ψ and ϕ separately. To be more explicit, notice that the representation $E \otimes \bigwedge^i F \otimes G^*$ occurs with multiplicity 2 in

$$\bigwedge^i F \otimes (E \otimes F^*) \otimes (F \otimes G^*).$$

The two components of the second arrow involve the two possible embeddings of $\bigwedge^i F$ into $\bigwedge^i F \otimes F^* \otimes F$. Let us describe these two embeddings explicitly. We define two linear maps $\mathrm{tr} : \mathbf{K} \to F^* \otimes F$ sending 1 to $\sum_{i=1}^f v_i^* \otimes v_i$ for some basis $\{v_1, \ldots, v_f\}$ of F. The other is the evaluation $\mathrm{ev} : F \otimes F^* \to \mathbf{K}$. Two embeddings of $\bigwedge^i F$ into $\bigwedge^i F \otimes F^* \otimes F$ are then defined as follows. One is just

$$i_1 := 1 \otimes \mathrm{tr} : \bigwedge^i F \to \bigwedge^i F \otimes F^* \otimes F,$$

the other is the composition

$$i_2 : \bigwedge^i F \xrightarrow{\Delta\otimes\mathrm{tr}} \bigwedge^{i-1} F \otimes F \otimes F^* \otimes F \xrightarrow{\sigma(2,4)} \bigwedge^{i-1} F \otimes F \otimes F^* \otimes F \xrightarrow{m\otimes 1\otimes 1} \bigwedge^i F \otimes F^* \otimes F,$$

where $\sigma(2,4)$ switches the second and fourth factor, and m denotes the exterior multiplication.

Thus the ϕ and ψ components of our differential are easy to identify (up to scalar). The only problem is the $(\psi\phi)$-component where we do not know which linear combination of maps i_c, with c equals 1 or 2, to choose. This problem can be solved, however, by looking at the construction of the complex $\mathbf{F}_\bullet$ given in Theorem 5.4.

Let us choose two partitions λ and ν such that $\nu \subset \lambda$, $|\lambda/\nu| = 1$. We have the induced map of sheaves

$$S_{\lambda'}E \otimes \hat{\mathcal{M}}(\lambda) \to S_{\nu'}E \otimes \hat{\mathcal{M}}(\nu)$$

which is a component of the differential of $\mathcal{K}_\bullet$. The induced map of sections is the equivariant homomorphism of A-modules

$$f(\lambda, \nu) : S_{\lambda'}E \otimes \hat{M}(\lambda) \to S_{\nu'}E \otimes \hat{M}(\nu).$$

The category of $\mathrm{GL}(E) \times \mathrm{GL}(F) \times \mathrm{GL}(G)$-modules is semi-simple, so we know that there is an equivariant map

$$\hat{f}(\lambda, \nu) : S_{\lambda'}E \otimes \bar{\mathbf{F}}(\lambda)_\bullet \otimes_{\bar{A}} A \to S_{\nu'}E \otimes \bar{\mathbf{F}}(\nu)_\bullet \otimes_{\bar{A}} A$$

of the minimal resolutions covering the map $f(\lambda, \nu)$.

Proposition 5.12.

(a) *The* $(\psi\phi)$*-components of the differential of the complex* $\mathbf{F}_\bullet$ *are the corresponding components of the maps* $\hat{f}(\lambda,\nu)$.
(b) *The strand of the complex* $\mathbf{F}_\bullet$ *with the* $\mathrm{SL}(F)$*-component* $\bigwedge^0 F$ *is*

$$\bigoplus_{\lambda} S_{\lambda'}E\otimes S_\lambda G^*\otimes A\big(-|\lambda|,-|\lambda|\big),$$

where the sum is taken over all partitions λ *contained in* e^{g-1}. *This is a subcomplex of* $\mathbf{F}_\bullet$ *and is isomorphic to the corresponding subcomplex of the Koszul complex*

$$\bigwedge^{\bullet}(E\otimes G^*)=\bigoplus_{\lambda\subset e^g} S_{\lambda'}E\otimes S_\lambda G^*\otimes A\big(-|\lambda|,-|\lambda|\big),$$

on the composition $\psi\phi$.

Proof. Assertion (a) follows from the definition of iterated cone construction; and (b) is a consequence of (5.11) because $d^{\lambda;k}_{\lambda-\epsilon_j;k}(2)=0$ when the $\mathrm{SL}(F)$-component of $T_{\lambda;k}$ is $\bigwedge^0 F$. □

Theorem 5.13. *There exists a family of constants* $\{s^{\lambda;k}_{\mu;\ell}(c)\}$ *such that the differential of* $\mathbf{F}_\bullet$ *is equal to*

$$\sum s^{\lambda;k}_{\mu;\ell}(c)\, d^{\lambda;k}_{\mu;\ell}(c),$$

where the maps $d^{\lambda;k}_{\mu;\ell}(c):T_{\lambda;k}\to T_{\mu;\ell}$ *have been previously defined at* (5.6), (5.10), *and* (5.11). *Furthermore, if* $\{s^{\lambda;k}_{\mu;\ell}(c)'\}$ *is a family of constants for which*

$$\Big(\mathbf{F}_\bullet,\sum s^{\lambda;k}_{\mu;\ell}(c)' d^{\lambda;k}_{\mu;\ell}(c)\Big) \tag{5.14}$$

is a complex, and such that for every pair $(\lambda;k)$ *there exist a pair* $(\mu;\ell)$ *such that* $\{s^{\lambda;k}_{\mu;\ell}(c)'\}$ *is nonzero*; *then* (5.14) *is acyclic and there is equivariant homotopy between* (5.14) *and* $\mathbf{F}_\bullet$.

Proof. We saw in Proposition 5.7 that the differential of $\mathbf{F}$ has three components. Furthermore, if we ignore two of the components of the differential, then we have shown that the third component is given by (5.6), (5.10), or (5.11), up to constant. The final assertion is an application of the Comparison Principle. □

6. The resolution of B/J

Now that we have some data involving the resolution of A/I, we apply it to find the terms of the resolution of B/J.

Theorem 6.1. *We have the isomorphisms* $\mathrm{Tor}^A_i(A/I,\mathbf{K})=\mathrm{Tor}^B_i(B/J,\mathbf{K})$ *preserving the* $\mathrm{SL}(E)\times\mathrm{SL}(F)\times\mathrm{SL}(G)$ *representation structure and homogeneous bidegree.*

Proof. Consider the minimal graded free resolution of B/J as an B-module:

$$\mathbf{G}_\bullet : 0 \to \mathbf{G}_{eg+1} \to \cdots \to \mathbf{G}_1 \to \mathbf{G}_0.$$

The complex $\mathbf{G}_\bullet \otimes_B B/aB$ has the ith homology module equal to $\mathrm{Tor}_i^B(B/J, B/aB)$. On the other hand, the long exact sequence of homology which is obtained by applying $B/J \otimes_B -$ to the short exact sequence

$$0 \to B \xrightarrow{a} B \to B/aB \to 0$$

yields $\mathrm{Tor}_i^B(B/J, B/aB) = 0$ for $i \geqslant 2$ and yields the exact sequence

$$0 \to \mathrm{Tor}_1^B(B/J, B/aB) \to B/J \xrightarrow{a} B/J \to B/(a, J) \to 0.$$

We know from Corollary 0.4 that a is a nonzerodivisor on B/J; so, $\mathrm{Tor}_1^B(B/J, B/aB)$ is also zero and $\mathbf{G}_\bullet \otimes_B B/aB$ is an A-free resolution of A/I. This resolution is minimal because the matrices of the maps in this complex are obtained from those of maps of $\mathbf{G}_\bullet$ by specializing a to zero. The terms of both minimal resolutions $\mathbf{G}_\bullet$ and $\mathbf{G}_\bullet \otimes_B B/aB$ are the same, and they (after tensoring with $\mathbf{K}$) give us the Tor groups mentioned in the theorem. □

Corollary 6.2. *The terms in the minimal graded free resolution, $\mathbf{G}_\bullet$, of the universal ring $C = B/J$ as a B-module are exactly the same as the terms of the resolution $\mathbf{F}_\bullet$ of Theorem* 3.4*, once "A" is replaced by "B."*

Theorem 6.1 continues to hold over $\mathbf{Z}$; however, the resolutions $\mathbf{F}_\bullet$ and $\mathbf{G}_\bullet$ of Corollary 6.2 requires that $\mathbf{K}$ be a field of characteristic zero.

Acknowledgment

Thank you to Alexandre Tchernev for getting us started on this project. Thank you also to the referee for making many interesting remarks.

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