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Nonexistence of type II blowup solution for a semilinear heat equation

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ABSTRACT

A solution u of a Cauchy problem for a semilinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbf{R}^N \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases}$$

is said to undergo type II blowup at $t = T < \infty$ if

$$\limsup_{t \to T} (T - t)^{1/(p-1)} \big|u(t)\big|_\infty = \infty.$$

Let p_S and p_{JL} be the exponents of Sobolev and of Joseph and Lundgren, respectively. We prove that when $p_S < p < p_{JL}$, a radial solution u does not exhibit type II blowup if u does not blow up at infinity. Let φ_∞ be the positive singular stationary solution with radial symmetry. It was shown in Matano and Merle (2009) [12] that for $p_S < p < p_{JL}$ if the number of intersections with $\pm\varphi_\infty$ is at most finite, then the radial solution does not undergo type II blowup. We do not impose an assumption on the number of intersections with $\pm\varphi_\infty$. For example, when a radial initial data u_0 is nonnegative and nonincreasing in $r = |x|$, the result in Matano and Merle (2009) [12] does not exclude type II blowup for p in the range, while our result does it.

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1. Introduction

We are concerned with blowup of solutions to a Cauchy problem for a semilinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbf{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbf{R}^N \end{cases} \tag{1.1}$$

with $p > 1$, $T > 0$ and $u_0 \in L^\infty(\mathbf{R}^N)$. Here a solution u of (1.1) is said to blow up at $t = T < \infty$ if $\limsup_{t\to T} |u(t)|_\infty = \infty$ with L^∞-norm $|\cdot|_\infty$. We call T blowup time of u. Let p_S be the Sobolev critical exponent, i.e.,

$$p_S = \begin{cases} \infty & \text{if } N = 1, 2, \\ 1 + \frac{4}{N-2} & \text{if } N \geqslant 3. \end{cases}$$

As for the blowup problem for (1.1), there are phenomena which are quite different in the subcritical case ($p < p_S$) and in the supercritical case ($p > p_S$). We refer to [4,12] and their references for detail. One of such interesting features is the blowup rate.

According to [6], any solution of the Cauchy problem with $p < p_S$ blowing up at $t = T$ fulfills

$$\big|u(t)\big|_\infty \leqslant C(T-t)^{-\frac{1}{p-1}} \quad \text{for } t \in [0,T) \tag{1.2}$$

with some constant $C > 0$. The right-hand side of (1.2) is the blowup rate of a solution to the corresponding ordinary differential equation $u_t = u^p$. The blowup satisfying (1.2) is said to be of type I and of type II otherwise.

We call a point $a \in \mathbf{R}^N$ a blowup point of u if there exist sequences $\{a_n\} \subset \mathbf{R}^N$, $\{t_n\} \subset (0,T)$ with $a_n \to a$ and $t_n \to T$ as $n \to \infty$ such that $u(a_n, t_n) \to \infty$ as $n \to \infty$. In order to investigate asymptotic behavior around a blowup point a of a solution u of (1.1) with blowup time T, a transformation through backward self-similar variables

$$w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad y = (T-t)^{-1/2}(x-a), \ s = -\log(T-t)$$

was used in [5–7]. The function w satisfies

$$\begin{cases} w_s = \Delta w + \dfrac{y}{2}\nabla w + |w|^{p-1}w & \text{in } \mathbf{R}^N \times (s_T, \infty), \\ w(y, s_T) = T^{1/(p-1)} u_0\big(a + T^{1/2}y\big) & \text{in } \mathbf{R}^N, \end{cases} \tag{1.3}$$

where $s_T = -\log T$. An advantage of the transformation is that w is a global solution of (1.3). Asymptotic behavior of u around the blowup point a corresponds to that of w as $s \to \infty$. Type I blowup of u is equivalent to the uniform boundedness of $|w(s)|_\infty$ in $[s_T, \infty)$. Therefore it is important in the study of behavior of w to know whether the blowup of u is of type I or not.

In the supercritical case, there seem to be known results only for radial solutions. If u is radially symmetric, then (1.1) is written as

$$\begin{cases} u_t = u_{rr} + \dfrac{N-1}{r} u_r + |u|^{p-1}u & \text{in } (0,\infty) \times (0,T), \\ u(r,0) = u_0(r) & \text{in } [0,\infty) \end{cases} \tag{1.4}$$

with $r = |x|$. Let φ_∞ be the singular positive stationary solution of (1.4) defined by

$$\varphi_\infty(r) = c_\infty r^{-\frac{2}{p-1}} \quad \text{for } r > 0 \tag{1.5}$$

with

$$c_\infty = \left\{ \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right\}^{\frac{1}{p-1}}.$$

There exists an important exponent p_{JL} due to Joseph and Lundgren given by

$$p_{JL} = \begin{cases} \infty & \text{if } N \leqslant 10, \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{if } N \geqslant 11. \end{cases}$$

The exponent p_{JL} is closely related to the properties of regular stationary solutions of (1.4). For $\alpha > 0$, let φ_α be a solution of

$$\begin{cases} \varphi'' + \dfrac{N-1}{r}\varphi' + |\varphi|^{p-1}\varphi = 0 \quad \text{in } (0,\infty), \\ \varphi(0) = \alpha, \qquad \varphi'(0) = 0. \end{cases} \tag{1.6}$$

In the supercritical case, $\varphi_\alpha(r)$ is positive for all $r \geqslant 0$ by the Pohožaev identity [15]. For a function f on $[0,\infty)$ with $f \not\equiv 0$, let $z(f)$ be the supremum over all j such that there exist $0 < r_1 < r_2 < \cdots < r_{j+1} < \infty$ with $f(r_i) \cdot f(r_{i+1}) < 0$ for $i = 1, 2, \ldots, j$. For each $\alpha > 0$, the following holds [10]:

(i) if $p < p_{JL}$, then $z(\varphi_\alpha - \varphi_\infty) = \infty$;
(ii) if $p > p_{JL}$, then $z(\varphi_\alpha - \varphi_\infty) < \infty$.

When $p > p_{JL}$, there exists a type II blowup solution of (1.4) by [8,9] (also see [14]). On the other hand, it was proved in [11] that when $p_S < p < p_{JL}$, a solution u of (1.4) does not exhibit type II blowup if there exists $t_0 \in [0, T)$ such that

(A1) $z(u_t(t_0)) < \infty$;
(A2) $z(u(t_0) - \varphi_\infty) < \infty$ and $z(u(t_0) + \varphi_\infty) < \infty$.

They improved the result in [12] removing the assumption (A1). However the other assumption (A2) has remained. It was mentioned in [12] that they do not know whether the possibility of type II blowup is eliminated without the assumption (A2).

In this paper, we do not impose the hypothesis on the number of intersections with $\pm\varphi_\infty$.

Theorem 1.1. *Suppose that $p_S < p < p_{JL}$. Let u be a radial solution of* (1.1) *blowing up at $t = T < \infty$. If u does not blow up at infinity, that is, there exist constants $C, K > 0$ such that $|u(x,t)| \leqslant C$ for $|x| \geqslant K$ and $t \in [0, T)$, then the blowup of u is of type I.*

If a radial initial data u_0 is nonnegative and nonincreasing in $r \geqslant 0$, then u does not blow up at infinity [13]. Therefore the following is immediate from Theorem 1.1.

Corollary 1.1. *Let $p_S < p < p_{JL}$. If a radial initial data u_0 is nonnegative and nonincreasing in $r \geqslant 0$, then type II blowup does not occur for* (1.1).

When u_0 is nonnegative and nonincreasing in r, the above result in [12] does not exclude type II blowup for p in the range, while our result does it.

For a solution u of (1.4), each zero of $u_t(t)$ and $u(t) \pm \varphi_\infty$ is isolated for $t > 0$ regardless of the situation of $t = 0$ (e.g. [1,2,11,12,16]). One also find in these papers that $z(u_t(t))$ and $z(u(t) \pm \varphi_\infty)$ are nonincreasing in t. The same way as the proof of Theorem 1.1 shows that at most finitely many intersections between a solution $u(t)$ with blowup time T and $\pm\varphi_\infty$ approach zero as $t \to T$ regardless of the value of p.

Theorem 1.2. *Let u be a radial solution of* (1.1) *with $p > 1$ blowing up at $t = T < \infty$. Suppose that u does not blow up at infinity. For a positive integer i, let $r_i(t)$ be the ith zero of $u(t) - \varphi_\infty$ for $t \in (0, T)$ which does not disappear before $t = T$, where the numbering is done in size of those zeros. Let*

$$m(u) = \sup\Big\{i:\ \liminf_{t\to T} r_i(t) = 0\Big\}.$$

Then we have $m(u) < \infty$. The same conclusion holds for $u(t) + \varphi_\infty$.

In Section 2, we intuitively explain why the value of exponent p is restricted to $p_S < p < p_{JL}$ to eliminate type II blowup. In Section 3, we prove the main theorems.

2. The role of the range $p_S < p < p_{JL}$

In this section, we give heuristic explanation about what role the range $p_S < p < p_{JL}$ plays. One can find the full proof in [11,12], which is very long and complicated.

Let u be a radial solution of (1.1), i.e., a solution of (1.4), blowing up at $t = T < \infty$. Put $M(t) = |u(t)|_\infty$ and

$$U(\eta, \tau) = M(t)^{-1} u\big(M(t)^{-(p-1)/2}\eta, t\big) \quad \text{with } \tau = \int_0^t M(s)^{p-1}\, ds.$$

Then $\tau \to \infty$ as $t \to T$ since $M'(t) \leqslant M(t)^p$ for all $t \in (0, T)$. A straightforward calculation yields

$$U_\tau = U_{\eta\eta} + \frac{N-1}{\eta} U_\eta + |U|^{p-1}U - a(\tau)\left(\frac{\eta}{2}U_\eta + \frac{1}{p-1}U\right) \quad \text{in } (0,\infty) \times (0,\infty),$$

where $a(\tau) = \frac{(p-1)M'(t)}{M(t)^p}$. The blowup of u is of type I if $|a(\tau)| \geqslant C$ for sufficiently large τ with some $C > 0$. Since we are giving intuitive observation, we omit taking a time sequence in convergence in the rest of this section. If u exhibits type II blowup, then $a(\tau) \to 0$ as $\tau \to \infty$. Suppose that $M(t) = u(0, t)$ for $t < T$ sufficiently close to T, which holds if $u_0(r)$ is nonnegative and nonincreasing in $r \geqslant 0$ for example. Then $U(\eta, \tau) \to \varphi_1(\eta)$ locally uniformly in $[0, \infty)$ as $\tau \to \infty$, where φ_1 is the solution of (1.6) with $\alpha = 1$. Therefore

$$u(r,t) \sim M(t)\varphi_1\big(M(t)^{\frac{p-1}{2}} r\big) \quad \text{as } t \to T. \tag{2.1}$$

A similar instinct consideration on type II blowup was given in [9]. The idea of rescaling as above was used to prove type I blowup in subcritical case in [17]. As mentioned in Introduction, we see $z(\varphi_1 - \varphi_\infty) = \infty$ if $p_S < p < p_{JL}$. Therefore it follows from (2.1) that $z(u(t) - \varphi_\infty) = \infty$. Similarly to above, we see $z(u(t) + \varphi_\infty) = \infty$.

For a positive integer i, let $r_i(t)$ be as in Theorem 1.2. Since $r_i(t)$ remains for $t \in (0, T)$, we have

$$u_r\big(r_i(t), t\big) - (\varphi_\infty)_r\big(r_i(t)\big) \neq 0 \quad \text{for } t \in (0, T).$$

If it equals zero at some time, the zero of $u(t) - \varphi_\infty$ disappears at the time [2,11,16]. By the implicit function theorem, we see

$$r_i'(t) = -\frac{u_t(r_i(t), t)}{u_r(r_i(t), t) - (\varphi_\infty)_r(r_i(t))} \quad \text{for } t \in (0, T). \tag{2.2}$$

Let R_i be the ith zero of $\varphi_1 - \varphi_\infty$ for a positive integer i, where the numbering is done in size of those zeros. It follows from (2.1) that

$$r_i(t) \sim M(t)^{-\frac{p-1}{2}} R_i \quad \text{as } t \to T. \tag{2.3}$$

Suppose that $M(t)$ is increasing for $t < T$ sufficiently close T. From (2.3), we see

$$r_i'(t) \sim -\frac{p-1}{2} M(t)^{-\frac{p-1}{2}-1} M'(t) < 0 \quad \text{as } t \to T.$$

It follows from (2.2) that when $t < T$ is sufficiently close to T, $u_t(r_i(t), t) < 0$ if i is even and $u_t(r_i(t), t) > 0$ if i is odd. Since $z(\varphi_1 - \varphi_\infty) = \infty$ for $p_S < p < p_{JL}$, we obtain that $z(u_t(t)) = \infty$.

It seems from above argument that at least one of (A1), (A2) is necessary to exclude type II blowup in the case of $p_S < p < p_{JL}$. However in next section we prove the nonexistence of type II blowup solution in the range of p without (A1) nor (A2).

3. Proof of Theorem 1.1

When a solution u of (1.1) blows up at $t = T < \infty$, $u(x, T) \equiv \lim_{t \to T} u(x, t)$ exists if x is not a blowup point of u by the standard parabolic regularity theory. Since $u(x, T)$ has a singularity, classical backward uniqueness theorem in $\mathbf{R}^N$ cannot be applied to our case. The following result on backward uniqueness in an exterior domain was given in [3], which plays a crucial role to prove the main theorems.

Proposition 3.1. *For positive constants R, T, let $Q_{R,T} = (\mathbf{R}^N \setminus B_R) \times [0, T]$, where $B_R = \{x \in \mathbf{R}^N\colon\ |x| \leqslant R\}$. Assume that u satisfies*

$$|\Delta u + u_t| \leqslant M\big(|u| + |\nabla u|\big) \quad \text{in } Q_{R,T}$$

and

$$\big|u(x, t)\big| \leqslant M \exp\big(M|x|^2\big) \quad \text{in } Q_{R,T}$$

for some constant $M > 0$. If $u(x, 0) = 0$ for all $x \in \mathbf{R}^N \setminus B_R$, then u vanishes identically in $Q_{R,T}$.

The following result was shown in Theorem 1.6 of [11].

Proposition 3.2. *Suppose that $p_S < p < p_{JL}$. Let u be a solution of* (1.4) *blowing up at $t = T < \infty$. If there exists $t_0 \in [0, T)$ such that*

(A1) $z(u_t(t_0)) < \infty$;
(A2) $z(u(t_0) - \varphi_\infty) < \infty$ *and* $z(u(t_0) + \varphi_\infty) < \infty$,

then the blowup of u is of type I.

Under the conditions (A1), (A2) with some $t_0 \in [0, T)$, we get

$$\sup_{t\in[t_0,T)} z\bigl(u_t(t)\bigr) < \infty$$

and

$$\sup_{t\in[t_0,T)} z\bigl(u(t) - \varphi_\infty\bigr) < \infty \quad \text{and} \quad \sup_{t\in[t_0,T)} z\bigl(u(t) + \varphi_\infty\bigr) < \infty.$$

This is essential in the proof of Proposition 3.2. For a function $f \not\equiv 0$ on $[a,b]$ with $0 \leqslant a < b \leqslant \infty$, let $z(f : [a,b])$ be the supremum over all j such that there exist $a < r_1 < r_2 < \cdots < r_{j+1} < b$ with $f(r_i) \cdot f(r_{i+1}) < 0$ for $i = 1, 2, \ldots, j$. We can prove the following lemma by the quite same manner as the proof of Proposition 3.2.

Lemma 3.1. *Suppose that $p_S < p < p_{JL}$. Let u be a solution of* (1.4) *blowing up at $t = T < \infty$. If there exist $R > 0$ and $t_0 \in [0, T)$ such that*

(A1′) $\sup_{t\in[t_0,T)} z(u_t(t); [0, R]) < \infty$;
(A2′) $\sup_{t\in[t_0,T)} z(u(t) - \varphi_\infty; [0, R]) < \infty$ *and* $\sup_{t\in[t_0,T)} z(u(t) + \varphi_\infty; [0, R]) < \infty$,

then the blowup of u is of type I.

Proof of Theorem 1.1. Let u be a solution of (1.4) blowing up at $t = T$ which does not blow up at infinity. Then there exist $C_1, K_1 > 0$ such that

$$\bigl|u(r,t)\bigr| \leqslant C_1 \quad \text{in } [K_1, \infty) \times [0, T). \tag{3.1}$$

Put $v(r,t) = u_t(r,t)$ for $[0,\infty) \times (0,T)$. It is immediate that there exist $C_2 > 0$, $K_2 > K_1$ such that

$$\bigl|v(r,t)\bigr| \leqslant C_2 \quad \text{in } [K_2, \infty) \times (0, T).$$

Then $u(r,T) \equiv \lim_{t\to T} u(r,t)$ and $v(r,T) \equiv \lim_{t\to T} v(r,t)$ exist for $r > K_2$.

It is immediate that

$$v_t = v_{rr} + \frac{N-1}{r} v_r + p|u|^{p-1} v \quad \text{in } (0,\infty) \times (0,T). \tag{3.2}$$

According to Proposition 3.1, we obtain $R > K_2$ such that

$$v(R,T) \neq 0. \tag{3.3}$$

There exists $t_0 \in (0, T)$ such that

(i) if $v(R,T) > 0$, then

$$v(R,t) \geqslant \frac{1}{2} v(R,T) \quad \text{for } t \in [t_0, T);$$

(ii) if $v(R,T) < 0$, then

$$v(R,t) \leqslant \frac{1}{2} v(R,T) \quad \text{for } t \in [t_0, T).$$

Then we see

$$u(R,t)-\varphi_\infty(R)\neq 0 \quad \text{and} \quad u(R,t)+\varphi_\infty(R)\neq 0 \quad \text{for all } t\in(t_1,T) \tag{3.4}$$

with some $t_1 \in [t_0,T)$. Indeed, we treat the case of (i) since we can similarly prove in the second case. The first statement in (3.4) for all $t\in[t_0,T)$ is trivial if $u(R,t)<\varphi_\infty(R)$ for $t\in[t_0,T)$. If $u(R,t_2)\geqslant\varphi_\infty(R)$ for some $t_2\in[t_0,T)$, then $u(R,t)>\varphi_\infty(R)$ for all $t\in(t_2,T)$ since $u_t(R,t)>0$ for $t\in[t_0,T)$. The second one in (3.4) is similarly shown. Then $z(u_t(t);[0,R])$, $z(u(t)-\varphi_\infty;[0,R])$ and $z(u(t)+\varphi_\infty;[0,R])$ are nonincreasing in t [16]. Therefore the conditions (A1′), (A2′) in Lemma 3.1 hold. Consequently the blowup of u is of type I by Lemma 3.1. □

Theorem 1.2 is proved in the same way as Theorem 1.1.

Remark 3.1. It was mentioned in Remark 3.10 of [12] that if u is a solution of (1.4) with $p_S<p<p_{JL}$ which undergoes type II blowup at $t=T<\infty$, then $u(r,T)=\varphi_\infty(r)$ for all $r>0$ or $u(r,T)=-\varphi_\infty(r)$ for all $r>0$. Under an additional assumption that u does not blowup at infinity, then we can also prove Theorem 1.1 applying Proposition 3.1 to $u(r,t)-\varphi_\infty(r)$ or $u(r,t)+\varphi_\infty(r)$. However, they carried out very long and complicated analysis to get the fact in Remark 3.10 of [12]. On the other hand, our method is much simpler since application of Proposition 3.1 to u_t enables us to use Lemma 3.1, which is essentially as same as Theorem 1.6 in [11]. In fact, the proof of Theorem 1.6 in [11] used the hypothesis (A1), while [12] made further effort to remove (A1). We will apply our method in order to show the nonexistence of type II blowup solution for other equation in a forthcoming paper.

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