



# Chiral anomalies in higher-derivative supersymmetric 6D gauge theories

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## Abstract

We show that the recently constructed higher-derivative 6D SYM theory involves internal chiral anomaly breaking gauge invariance. The anomaly is cancelled when adding to the theory an adjoint matter hypermultiplet.

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## 1. Introduction

We argued recently [1] that the fundamental theory of everything may represent a conventional field theory defined in flat space–time with  $D > 4$ . Our Universe represents then a classical 3-brane solution in this theory. Einstein’s gravity appears as the effective Lagrangian induced in the world-volume of this brane. If this is true, the fundamental higher-dimensional theory should be internally consistent. Two major problems which should be solved here are renormalizability and the existence and stability of quantum vacuum state (the absence of ghosts).

Conventional theories (involving the terms like  $\text{Tr}\{F_{\mu\nu}^2\}$ , etc.) are not renormalizable for  $D > 4$ . However, if we add extra derivatives such that the canonical dimension of the Lagrangian is equal to  $D$ , renormalizability might be achieved. Adding extra derivatives creates the problem of ghosts [2]. However, one may hope that in *supersymmetric* theories, this problem can be handled. Indeed, supersymmetry algebra implies that the energies of all Hamiltonian eigenstates are non-negative. Considering a particular supersymmetric QM model (dimensionally reduced 5D superconformal Yang–Mills theory [4]) involving singularity and, potentially, associated ghost states [3], we showed that though the latter seem to be present in full spectrum of the Hamiltonian, the negative energy states do not possess normalizable superpartners and do not form complete supermultiplets. A reduced Hilbert space involving only complete supermultiplets involves only the states with non-negative energies. In Ref. [3], we presented arguments that 5D superconformal YM theory in the sector with zero v.e.v. of the real scalar field  $\sigma$  might be feasible and internally consistent. However, this theory is very difficult to analyze because its Lagrangian does not involve quadratic terms and conventional perturbative methods do not work. In Ref. [5], we constructed using the methods of harmonic superspace [6] a scale invariant (conformally invariant at the classical level) supersymmetric YM theory in six dimensions. The component Lagrangian has the form

$$\mathcal{L} = -\text{Tr}\left\{(\nabla_\mu F_{\mu\lambda})^2 + i\psi^j\gamma_\mu\nabla_\mu(\nabla)^2\psi_j + \frac{1}{2}(\nabla_\mu D_{jk})^2 + D_{lk}D^{kj}D_j^l - 2iD_{jk}(\psi^j\gamma_\mu\nabla_\mu\psi^k - \nabla_\mu\psi^j\gamma_\mu\psi^k) + (\psi^j\gamma_\mu\psi_j)^2 + \frac{1}{2}\nabla_\mu\psi^j\gamma_\mu\sigma_{\nu\sigma}[F_{\nu\sigma}, \psi_j] - 2\nabla_\mu F_{\mu\nu}\psi^j\gamma_\nu\psi_j\right\} \quad (1)$$

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where  $\sigma_{\mu\nu} = (\tilde{\gamma}_\mu \gamma_\nu - \tilde{\gamma}_\nu \gamma_\mu)/2$ . The Lagrangian (1) involves gauge fields  $A_\mu$ , adjoint Weyl fermions  $\psi^{ka}$ , where  $a = 1, 2, 3, 4$  is the spinor index and the subflavor index  $k$  acquires two values, and adjoint scalar fields  $D_{jk} = D_{kj}$  of canonical dimension 2. (In the conventional SYM theory the fields  $D_{jk}$  are auxiliary and enter without derivatives. But in the higher-derivative theory (1), they propagate.) Here all the fields are Hermitian  $N \times N$  colour matrices. The  $6D$  Weyl  $4 \times 4$  matrices  $(\gamma_\mu)_{ab}$  are antisymmetric. They satisfy the relation

$$\gamma_\mu \tilde{\gamma}_\nu + \gamma_\nu \tilde{\gamma}_\mu = -2\eta_{\mu\nu}, \tag{2}$$

where

$$\tilde{\gamma}_\mu^{ab} = \frac{1}{2} \epsilon^{abcd} (\gamma_\mu)_{cd}.$$

One of the possible explicit representations is

$$\begin{aligned} \gamma_0 = \tilde{\gamma}_0 = i\sigma_2 \otimes \mathbb{1}, & \quad \gamma_1 = -\tilde{\gamma}_1 = i\sigma_2 \otimes \sigma_1, & \quad \gamma_2 = -\tilde{\gamma}_2 = i\mathbb{1} \otimes \sigma_2, \\ \gamma_3 = -\tilde{\gamma}_3 = i\sigma_2 \otimes \sigma_1, & \quad \gamma_4 = \tilde{\gamma}_4 = \sigma_1 \otimes \sigma_2, & \quad \gamma_5 = \tilde{\gamma}_5 = \sigma_3 \otimes \sigma_2. \end{aligned} \tag{3}$$

The property

$$\frac{1}{4} \text{Tr}\{\gamma_\mu \tilde{\gamma}_\nu \gamma_\alpha \tilde{\gamma}_\beta \gamma_\gamma \tilde{\gamma}_\delta\} = -\frac{1}{4} \text{Tr}\{\tilde{\gamma}_\mu \gamma_\nu \tilde{\gamma}_\alpha \gamma_\beta \tilde{\gamma}_\gamma \gamma_\delta\} = \epsilon_{\mu\nu\alpha\beta\gamma\delta} + \text{symmetric part} \tag{4}$$

holds (with the convention  $\epsilon_{012345} = 1$ ).

The fermions  $\psi^{ja}$  belong to the representation  $(0, 1)$  of the group  $SO(5, 1)$ . They satisfy the pseudoreality constraint

$$(\psi^{aj})^* = \epsilon_{jk} (\gamma_0)_{ab} \psi^{bk}. \tag{5}$$

Note that the possibility to impose such a constraint is specific for six dimensions. Indeed, both  $SO(3, 1)$  and  $SO(5, 1)$  involve two different chiral spinor representations. But their behavior under complex conjugation is different for  $D = 4$  and  $D = 6$ . When  $D = 4$ , complex conjugation changes the type of representation of a Minkowski spinor  $(1, 0) \leftrightarrow (0, 1)$ . On the other hand, for  $D = 6$ , the complex conjugated spinor belongs to the same representation as the original one. One can say that the subflavor index  $j = 1, 2$  corresponds to the original spinor and the complex conjugated one.<sup>2</sup>

The Lagrangian (1) involves a dimensionless coupling constant, which suggests renormalizability. In [5], we calculated the beta function in this theory and found that it has the same sign as in the ordinary 4D QED corresponding to the zero charge situation. However (and this is the main observation of this Letter), the theory (1) is *not* renormalizable because it involves chiral anomaly!<sup>3</sup> The reason is the mentioned above chiral nature of Minkowskian fermions in six dimensions, irrespectively whether they belong to a real or to a complex representation of the gauge group. Indeed, the Lagrangian (1) involves only *one* type of spinors  $\psi^a$  while the spinors  $\chi_a$  belonging to the representation  $(1, 0)$  are absent. It is this asymmetry which gives rise to anomaly (cf. the four-dimensional situation, where the SYM Lagrangian involves the structure  $\sim \text{Tr}\{\psi \sigma_\mu \nabla_\mu \bar{\psi}\}$  containing both  $\psi$  and  $\bar{\psi}$ ; there is no asymmetry and no chiral anomaly).

## 2. Anomalies

The anomalies in standard higher-dimensional theories involving fermion kinetic term  $\propto \bar{\psi} \not{D} (1 + \Gamma^{D+1}) \psi$  were thoroughly studied in [8–10]. In six dimensions, one has to calculate the anomalous box graph. Another convenient method uses the Schwinger splitting technique. Consider the theory with the standard fermion kinetic term

$$\mathcal{L} = i \text{Tr}\{\psi^k \not{D} \psi_k\}, \tag{6}$$

where  $\not{D} = \gamma_\mu \nabla_\mu$ ,  $\nabla_\mu = \partial_\mu - i A_\mu$ . The colour current  $J_\mu^A = \text{Tr}\{\psi^k T^A \gamma_\mu \psi_k\}$  ( $T^A$  is the adjoint generator) is covariantly conserved at the classical level. Quantum effects bring about the anomaly:

$$\nabla_\mu J_\mu^A = \frac{1}{3 \cdot 128 \pi^3} \epsilon_{\mu\nu\alpha\beta\gamma\delta} \text{Tr}\{T^A F_{\mu\nu} F_{\alpha\beta} F_{\gamma\delta}\} + \dots, \tag{7}$$

where the dots stand for the terms of higher order in  $A_\mu$  having the form  $\sim \epsilon_{\mu\nu\alpha\beta\gamma\delta} \text{Tr}\{T^a F_{\mu\nu} F_{\alpha\beta} A_\gamma A_\delta\}$ ,  $\sim \epsilon_{\mu\nu\alpha\beta\gamma\delta} \times \text{Tr}\{T^a F_{\mu\nu} A_\alpha A_\beta A_\gamma A_\delta\}$  and  $\sim \epsilon_{\mu\nu\alpha\beta\gamma\delta} \text{Tr}\{T^a A_\mu A_\nu A_\alpha A_\beta A_\gamma A_\delta\}$ . The coefficients of these terms are rigidly related [10] to the coefficient in Eq. (7).

<sup>2</sup> Note, that in Euclidean space the situation is exactly inverse.  $SO(4) \equiv SU(2) \otimes SU(2)$  with two completely independent factors. The spinor represents a doublet under either left or right such  $SU(2)$  factor and complex conjugation does not change this. On the other hand,  $SO(6) \equiv SU(4)$  and complex conjugation transforms “quarks” into “antiquarks”.

<sup>3</sup> The original guess belongs to Soo-Jong Rey [7].

Note that the anomaly is proportional to the symmetrized trace  $\text{Tr}\{T^{(A}T^BT^CT^D)\}$  which does not vanish. If we would try to evaluate the internal chiral anomaly in the 4D SYM theory, it would involve the factor  $\text{Tr}\{T^{(A}T^BT^C)\} = 0$ . In other words, the anomaly vanishes there and this is of course related to the fact that in four dimensions the Minkowski fermion kinetic term involves the fermion fields both in  $(0, 1)$  and  $(1, 0)$  representations, as explained above. Note also that, for the fermions belonging to the representation  $(1, 0)$  of  $SO(5, 1)$ , the result would be the same with the opposite sign, as follows from (4).

In the case under consideration, the Lagrangian is different from (6) and involves higher derivatives. However, as was observed in [9], this does not change the result for the anomaly. The argument of Ref. [9] was the following. Consider the  $U(1)$  theory (non-Abelian structures change nothing) with the Lagrangian, involving both left- and right-handed fermions with different kinetic terms. In Dirac notations,

$$\mathcal{L} = \bar{\psi}A(1 + \Gamma^{D+1})\psi + \bar{\psi}B(1 - \Gamma^{D+1})\psi = \bar{\psi}(A + B)\psi + \bar{\psi}(A - B)\Gamma^{D+1}\psi. \quad (8)$$

For example, one can take  $A = \not{D}$  and  $B = \not{D}^3$ . We are allowed to regularize the theory in the ultraviolet multiplying  $A + B$  in the first term by  $(1 - \mathcal{D}_\mu^2/\Lambda^2)^n$  while keeping the second term intact. This will bring in the factor  $(1 + p^2/\Lambda^2)^n$  in the fermion propagator while the axial current depending only on the second term would not change. Then the Feynman integral for the anomalous box graph would involve four such factors downstairs and not more than three such factors upstairs coming from the vector vertices. As a result, the integral would converge in the ultraviolet meaning the absence of anomaly. And this means that the anomaly coefficients in the theories  $\bar{\psi}A(1 + \Gamma^{D+1})\psi$  and  $\bar{\psi}B(1 + \Gamma^{D+1})\psi$  are the same.

Another argument is the following [7]. Consider a mixed theory

$$\mathcal{L}_{\text{ferm}} = \frac{i}{2}\bar{\psi}(\not{D} - \not{D}^3/M^2)(1 + \Gamma^{D+1})\psi. \quad (9)$$

When ultraviolet regulator is large  $\Lambda \gg M$ , the higher-derivative term dominates in all calculations and the anomaly should be the same as in the pure higher-derivative theory. On the other hand, the anomaly has an infrared face and is related to the number of levels crossing zero in a slowly varying external field with different Chern–Simons numbers at  $t = -\infty$  and  $t = \infty$ . But when the characteristic frequency of this external field is much smaller than  $M$ , one can forget about the higher-derivative term in (9) and the anomaly should be the same as in the theory with the standard Dirac kinetic term.

Still another way to derive the same result is noting that the indices of the Euclidean operators  $\not{D}$ ,  $\not{D}^3$ ,  $\not{D}D_\mu^2$ , etc. coincide.

Finally, one can amuse oneself by the explicit calculation of the anomalous divergence of the gauge current for the Lagrangian (9) using the Schwinger splitting technique and be convinced that the coefficient is, indeed, the same as for the standard Dirac Lagrangian (see Appendix A).

### 3. Discussion

The presence of internal chiral anomaly in the theory means breaking of gauge invariance. Among other things, this makes the theory nonrenormalizable. Though there were attempts to attribute meaning to anomalous theories [11], we prefer to stick to a conservative viewpoint by which such theories are sick. In a “healthy” theory, chiral anomaly must be cancelled. To achieve such cancellation, one has to include in the theory some other fermions in addition to the gluino fields  $\psi^{ka}$ , the superpartners of the gauge potential. In order to keep supersymmetry, these extra fermions should come together with their superpartners. The only 6D supermultiplet besides the vector multiplet which admits off-shell formulation is the hypermultiplet. We considered the problem of coupling 6D hypermultiplet to 6D vector multiplet in Ref. [12]. We found out there that it is difficult to do it in a “symmetric” way such that the kinetic term involves higher derivatives for all fermions present in the theory. The problem is that an off-shell hypermultiplet involves besides physical fields an infinite number of auxiliary components (that is especially clearly seen in the harmonic superspace approach [6]). For a conventional hypermultiplet, these extra components are auxiliary, indeed. They enter in the Lagrangian without derivatives and can be easily integrated over. However, in a HD Lagrangian, the former auxiliary fields acquire derivatives and become propagating.

It is not quite clear for us what does a theory with an infinite number of propagating massless degrees of freedom mean. For example, an infinite number of propagating fields may provide an infinite contribution to the beta function (though it is not known at present whether it is the case or not). It would be interesting to think more in this direction. What we would like to point out now, however, is that an adjoint hypermultiplet gives a *finite* contribution to the anomaly, which exactly cancels the pure SYM contribution (7).

As was discussed in [12], there are many ways to couple the hypermultiplet to the gauge supermultiplet.<sup>4</sup> Consider e.g. the action  $S_2$  given by Eqs. (B.3), (B.6) of Ref. [12]. The fermion kinetic term is

$$\mathcal{L}_{\text{kin}} \sim \int du \text{Tr}\{i\chi_a^A(\tilde{\gamma}^\mu)^{ab}\partial_\mu\chi_{Ab} - 2\chi_a^A\partial^{++}\chi_A^a\}, \quad (10)$$

<sup>4</sup> Incidentally, none of them allows one to preserve the conformal invariance of the classical action (though scale invariance is manifest).

where  $A = 1, 2$  is the subflavor index. The fields  $\chi^A, \lambda_A$ , subject to pseudoreality constraints like in Eq. (5), are expanded over the harmonics  $u_i^\pm, i = 1, 2^5$  as

$$\begin{aligned}\chi(u, x) &= \chi(x) + \chi^{(ij)}(x)u_i^+u_j^- + \chi^{(ijkl)}u_i^+u_j^+u_k^-u_l^- + \dots, \\ \lambda(u, x) &= \lambda^{(ij)}(x)u_i^-u_j^- + \lambda^{(ijkl)}u_i^-u_j^-u_k^-u_l^+ + \dots.\end{aligned}\quad (11)$$

Note that the field  $\chi$  has the spinor indices down and belongs to *another* fermion representation compared to the gluino field  $\psi^{ka}$ . Indeed, the field  $\chi_a^k$  comes from the linear term of the expansion of the hypermultiplet superfield in  $\theta^a$ , while the field  $\psi^{ka}$  comes from the cubic in  $\theta$  term  $\sim \varepsilon_{abcd}\psi^a\theta^b\theta^c\theta^d$  [5,12]. The field  $\lambda^a$  stems from the cubic in  $\theta$  term of the hypermultiplet expansion and has the same chirality as the gluino field.

In each term of the expansion of  $\chi$ , the number of the factors  $u^-$  is equal to the number of the factors  $u^+$  (the eigenvalue of the harmonic charge  $u_i^+\partial/\partial u_i^+ - u_i^-\partial/\partial u_i^-$  for the field  $\chi$  is zero). It exceeds by 2 the number of the factors  $u^+$  for the expansion of  $\lambda$  (the latter has harmonic charge  $-2$ ). The harmonic derivative  $\partial^{++}$  entering (10) is defined as  $u_i^+\partial/\partial u_i^-$ . Harmonic integrals  $\int du$  are nonzero only for the structures of zero harmonic charge,

$$\int du = 1, \quad \int du u_i^+u_j^- = \frac{1}{2}\epsilon_{ij}, \quad \int du u_i^+u_k^+u_j^-u_l^- = \frac{1}{6}(\epsilon_{ij}\epsilon_{kl} + \epsilon_{il}\epsilon_{kj}), \quad \dots \quad (12)$$

Thus, we have an infinite number of physical fields  $\chi(x), \chi^{(ij)}(x), \lambda^{(ij)}(x)$ , etc. with growing isospins. When substituting the expansion (11) into Eq. (10), we obtain

$$\mathcal{L}_{\text{kin}} = i\chi^A\tilde{\gamma}^\mu\partial_\mu\chi_A - \frac{i}{3}\chi_{(ij)}^A\tilde{\gamma}^\mu\partial_\mu\chi_A^{(ij)} + \frac{4}{3}\chi_{(ij)}^A\chi_A^{(ij)}. \quad (13)$$

The first term gives a nonzero contribution to the anomaly, which cancels the contribution in Eq. (7). On the other hand, the contribution of the terms involving  $\chi^{(ij)}$  and  $\lambda^{(ij)}$  vanishes because the fields  $\chi^{(ij)}$  and  $\lambda^{(ij)}$  have opposite chiralities and we argued above that the anomaly does not depend on a particular form of the Lagrangian, but only on the field content. (To illustrate this, one can e.g. lift the box operator in the second and the third terms in Eq. (13). Then variation over  $\lambda^{(ij)}$  gives the equation of motion  $\chi_{(ij)} = 0$  so that the current and its anomalous divergence vanish.) The same is true for the terms involving the components  $\chi^{(ijkl)}$  and  $\lambda^{(ijkl)}$ , etc.

Another ‘‘asymmetric’’ possibility is to couple the HD vector multiplet to a conventional hypermultiplet with the standard kinetic term. In this case, one can get rid of the auxiliary fields, as usual. We are left with only one Weyl fermion  $\chi_a^A$  satisfying the pseudoreality constraint. Its contribution to the anomalous divergence cancels the gluino contribution. The conformal invariance of the classical Lagrangian can be imposed if attributing canonical dimension 2 to scalar components and dimension 5/2 to  $\chi_a^A$ . Unfortunately, classical conformal symmetry is not preserved at the quantum level. Conformal anomaly (alias, beta function) was calculated in [5,12]. We found that the contributions to the beta function coming from interactions with hypermultiplet and from vector multiplet self-interactions have the same sign corresponding to the Landau zero situation, like in the ordinary QED.

Besides logarithmic renormalization of the coupling constant (the coefficient at the structure (1) in the effective Lagrangian), the theory may involve also quadratic ultraviolet divergences at the structure  $\sim \text{Tr}\{F_{\mu\nu}^2\} + \dots$  (the conventional 6D SYM Lagrangian). We found earlier that this coefficient vanishes for the pure HD SYM theory (1), but does not vanish for the theories involving hypermultiplet interactions. In other words, the ‘‘asymmetric’’ anomaly-free theory involves a quadratic divergence, which should be cancelled by a properly chosen counterterm. This theory has thus roughly the same status as the scalar QED or  $\lambda\phi^4$  theory in four dimensions, where renormalization of the scalar mass involves quadratic UV divergences. All these theories are renormalizable, but the eventual cancellation of the divergences implies the presence of the counterterms  $\sim \Lambda_{\text{UV}}^2$  with a fine-tuned coefficient. In all these theories, the effective charge grows at high energies, which means that, in spite of being renormalizable, the theories are not consistently defined nonperturbatively.

Constructing a nontrivial 6D theory that would be internally consistent both perturbatively and nonperturbatively remains a major challenge.

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<sup>5</sup> The harmonics satisfy the relation  $u_i^-u^{+i} = 1$  and represent the complex coordinates on  $\mathbb{CP}^1$ —the coset of the automorphism group of the  $\mathcal{N} = 1$  6D SUSY algebra.

## Appendix A

We will demonstrate here the independence of the anomaly coefficient of the particular form of the Lagrangian by illustrative calculations using Schwinger-splitting UV regularization and background field technique [13]. Let us be interested in the global axial anomaly in the  $U(1)$  vector theory with the Lagrangian

$$\mathcal{L}_{\text{ferm}} = i\bar{\psi}(\not{D} - \not{D}^3/M^2)\psi. \quad (\text{A.1})$$

The corresponding equations of motion are

$$(\not{D} - \not{D}^3/M^2)\psi = 0, \quad \bar{\psi}(\overleftarrow{\not{D}} - \overleftarrow{\not{D}}^3/M^2) = 0. \quad (\text{A.2})$$

The Nöther axial current is

$$\mathcal{A}_\mu = \bar{\psi}\Gamma_\mu\Gamma^{D+1}\psi - \frac{1}{M^2}\bar{\psi}[\Gamma_\mu\not{D}^2 - \overleftarrow{\not{D}}\Gamma_\mu\not{D} + \overleftarrow{\not{D}}^2\Gamma_\mu]\Gamma^{D+1}\psi, \quad (\text{A.3})$$

where  $\Gamma_\mu$  are Dirac gamma matrices in  $D$  dimensions (not to confuse with 6-dimensional Weyl matrices  $\gamma_\mu$ ). It is conserved at the classical level. The Schwinger-split current is

$$\mathcal{A}_\mu^{(\epsilon)} = E\bar{\psi}_+\Gamma_\mu\Gamma^{D+1}\psi - \frac{E}{M^2}\bar{\psi}_+[\Gamma_\mu\not{D}^2 - \overleftarrow{\not{D}}_+\Gamma_\mu\not{D} + \overleftarrow{\not{D}}_+^2\Gamma_\mu]\Gamma^{D+1}\psi, \quad (\text{A.4})$$

where  $\psi_+$  and  $\overleftarrow{\not{D}}_+$  are evaluated at the point  $x + \epsilon$  and

$$E = P \exp \left\{ i \int_x^{x+\epsilon} A_\alpha(y) dy_\alpha \right\} \quad (\text{A.5})$$

is the path-ordered exponent. It is convenient to work in the Fock–Schwinger or fixed point gauge  $x_\mu A_\mu = 0$  [14,15] where the vector potential is expressed via the field density as

$$A_\mu(x) = \frac{1}{2}F_{\nu\mu}x_\nu + \dots, \quad (\text{A.6})$$

the dots standing for the terms involving  $\partial_\alpha F_{\mu\nu}$ ,  $\partial_\alpha\partial_\beta F_{\mu\nu}$ , etc. We can safely neglect them because the sought-for anomaly involves only  $F$ , but not its derivatives. The anomalous divergence is

$$\partial_\mu\mathcal{A}_\mu = \lim_{\epsilon \rightarrow 0} iF_{\mu\nu}\epsilon_\nu \left\{ \Gamma_\mu - \frac{1}{M^2}[\Gamma_\mu\not{D}^2 - \overleftarrow{\not{D}}_+\Gamma_\mu\not{D} + \overleftarrow{\not{D}}_+^2\Gamma_\mu]\Gamma^{D+1}\mathcal{G}(x, x + \epsilon) \right\}, \quad (\text{A.7})$$

where  $\mathcal{G}(x, x + \epsilon)$  is fermion Green's function  $\langle \psi(x)\bar{\psi}(x + \epsilon) \rangle_A$  evaluated in the presence of the background  $A_\mu$ . When deriving (A.7), we took into account the terms where the derivatives act on the fermion fields (and used the equations of motion (A.2)) and also the terms coming from differentiating the factor

$$E \approx 1 + \frac{i}{2}F_{\nu\mu}x_\nu\epsilon_\mu.$$

One can show that these two contributions are equal.

The calculations are trivial in two dimensions where the anomaly is linear in  $F$ . As the factor  $\sim F$  is already present in (A.7), we can trade the covariant derivatives there for the usual ones and substitute tree Green's function

$$G_0(-\epsilon) = \int \frac{d^2p}{(2\pi)^2} G_0(p) e^{ip\epsilon}, \quad G_0(p) = \frac{i\not{p}}{p^2(1 + p^2/M^2)} \quad (\text{A.8})$$

for  $\mathcal{G}(x, x + \epsilon)$ . Using  $\overleftarrow{\not{D}}\Gamma_\mu \equiv 2\partial_\mu - \Gamma_\mu\overleftarrow{\not{D}}$ , antisymmetry of  $F_{\mu\nu}$ , and going into momentum space, one immediately sees that the anomaly (A.7) involves the factor  $\sim 1 + p^2/M^2$  upstairs, which cancels the same factor downstairs in Eq. (A.8). In other words, the result does not depend on  $M$ .

The case  $D = 4$  is slightly less trivial. One has to take into account two contributions: (i) the terms where covariant derivatives in Eq. (A.7) are traded for the usual ones, but Green's function is evaluated in linear order in  $F$ ; (ii) the terms with tree level Green's function (A.8) where the extra power of  $F$  is extracted from

$$\not{D}^2 = \partial^2 - \frac{i}{2}F_{\alpha\beta}\Gamma_\alpha\Gamma_\beta, \quad \overleftarrow{\not{D}}^2 = \partial^2 - \frac{i}{2}F_{\alpha\beta}\Gamma_\alpha\Gamma_\beta. \quad (\text{A.9})$$

The contribution of the second type is convenient to present as

$$\partial_\mu\mathcal{A}_\mu|_{G_0} = -\frac{F_{\mu\nu}\epsilon_\nu}{M^2} \int \frac{d^4p}{(2\pi)^4} e^{ip\epsilon} \text{Tr}\{\Gamma_\mu F_{\alpha\beta}\Gamma_\alpha\Gamma_\beta\Gamma^5 G_0(p)\}. \quad (\text{A.10})$$

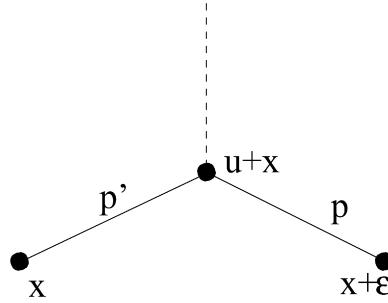


Fig. 1.  $\mathcal{G}_1(x, x + \epsilon)$ .

To fix the contribution of the first type, we have to evaluate Green's function in the first order in  $F$ . It is given by the graph in Fig. 1. We have

$$\mathcal{G}_1(x, x + \epsilon) = i \int d^4u G_0(-u) \Lambda_\alpha A_\alpha(u + x) G_0(u - \epsilon) = -\frac{i}{2} \int d^4u G_0(-u) \Lambda_\alpha F_{\alpha\beta}(u + x)_\beta G_0(u - \epsilon), \quad (\text{A.11})$$

where the vector potential  $A_\alpha(x)$  is chosen in the fixed point gauge form (A.6) and  $\Lambda_\alpha$  is the vertex following from the Lagrangian (A.1). In the momentum representation,

$$\Lambda_\alpha(p', p) = \Gamma_\alpha + \frac{\Gamma_\alpha(p'^2 + p^2) + p' \Gamma_\alpha p}{M^2}. \quad (\text{A.12})$$

Green's function (A.11) involves two terms. The first term coming from the structure  $\sim F_{\alpha\beta} u_\beta$  in the integrand depends only on the difference  $x - (x + \epsilon) = -\epsilon$ . The second term  $\sim F_{\alpha\beta} x_\beta$  depends both on  $x$  and  $\epsilon$ . It is not translationally invariant, which is not surprising as the fixed point gauge condition breaks translational invariance. For the first term, the calculation gives

$$\mathcal{G}_1^{\text{tr. inv.}}(-\epsilon) = \int \frac{d^4p}{(2\pi)^4} G_1(p) e^{ip\epsilon} \quad (\text{A.13})$$

with

$$G_1(p) = \frac{F_{\alpha\beta} \Gamma_\alpha \Gamma_\beta p (1 + 2p^2/M^2)}{2p^4(1 + p^2/M^2)^2} + \dots \quad (\text{A.14})$$

We have explicitly written here only the terms involving an irreducible product of three gamma matrices that contribute to the anomaly. The structure  $\sim F_{\alpha\beta} p_\alpha \Gamma_\beta$  gives zero, when substituted in Eq. (A.7).

The second translationally noninvariant piece is

$$\mathcal{G}_1^{\text{tr. noninv.}}(x, x + \epsilon) = -\frac{i}{2} F_{\alpha\beta} x_\beta \int \frac{d^4p}{(2\pi)^4} e^{ip\epsilon} G_0(p) \Lambda_\alpha(p, p) G_0(p). \quad (\text{A.15})$$

We have

$$G_0(p) \Lambda_\alpha(p, p) G_0(p) = \frac{\Gamma_\alpha}{p^2(1 + p^2/M^2)} + \text{irrelevant for anomaly terms}. \quad (\text{A.16})$$

Substituting the sum of (A.13), (A.15) into Eq. (A.7) and adding to this Eq. (A.10), we derive

$$\begin{aligned} \partial_\mu \mathcal{A}_\mu &= -i \lim_{\epsilon \rightarrow 0} F_{\mu\nu} \epsilon_\nu \int \frac{d^4p e^{ip\epsilon}}{2(2\pi)^4 p^4} \left\{ \frac{1 + 2p^2/M^2}{(1 + p^2/M^2)} + \frac{p^2}{M^2(1 + p^2/M^2)} - \frac{2p^2}{M^2(1 + p^2/M^2)} \right\} \cdot \text{Tr}\{\Gamma_\mu \Gamma_\alpha \Gamma_\beta p \Gamma^5\} F_{\alpha\beta} \\ &= -\frac{1}{16\pi^2} \epsilon_{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \end{aligned} \quad (\text{A.17})$$

the same for all  $M$ . Note that, for the standard Dirac action in the limit  $M \rightarrow \infty$ , the only contribution to the anomaly comes from the translationally invariant piece in  $\mathcal{G}_1$  of Eq. (A.13). But when  $M < \infty$ , all three contributions discussed above are important.

Cancellation of  $M$  dependence in the sum of the three contributions should work also in higher dimensions. Only one has to consider higher terms of the expansion of Green's function in  $F$ . For example, for  $D = 6$ , the relevant contributions come from: (i) substituting in Eq. (A.7), with covariant derivatives traded for the usual ones, the translationally invariant and (ii) translationally noninvariant parts of  $\mathcal{G}_2$ ; (iii) substituting in Eq. (A.7), with the terms  $\sim F$  in Eq. (A.9) taken into account, the translationally invariant part of  $\mathcal{G}_1$ ,

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