

## Strongly Regular Graphs with Strongly Regular Decomposition

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

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### 1. INTRODUCTION AND PRELIMINARY RESULTS

The title refers to strongly regular graphs  $\Gamma_0$  which admit a partition  $\{X_1, X_2\}$  of the vertex set such that each of the induced subgraphs  $\Gamma_1$  and  $\Gamma_2$  on  $X_1$  and  $X_2$  respectively is strongly regular, a clique, or a coclique. A central role is played by the design  $D$  having point set  $X_1$ , block set  $X_2$ , and incidence given by adjacency in  $\Gamma_0$ . If  $\Gamma_1$  is a clique or a coclique and  $\Gamma_0$  is primitive,  $D$  must be a quasisymmetric design. If  $\Gamma_1$  and  $\Gamma_2$  are both strongly regular,  $D$  is a strongly regular design in the sense of D. G. Higman [14], except possibly when  $\Gamma_0$  is the graph of a regular conference matrix. Conversely, a quasisymmetric or strongly regular design with suitable parameters gives rise to a strongly regular graph with strongly regular decomposition. Moreover, if  $\Gamma_0$  and  $\Gamma_1$  are strongly regular with suitable parameters, then  $\Gamma_2$  must be strongly regular too. We give several examples and some nonexistence results. We include a table of all feasible parameter sets up to 300 vertices. For most of the cases in the table existence or nonexistence is settled. Some of the results in this paper are old, due to M. S. Shrikhande [17], W. G. Bridges and M. S. Shrikhande [3], and W. H. Haemers [13].

We mainly use eigenvalue techniques. We need results on interlacing eigenvalues (see [13]). Two sequences  $\rho_1 \geq \dots \geq \rho_n$  and  $\sigma_1 \geq \dots \geq \sigma_m$

$(n > m)$  are said to *interlace* whenever

$$\rho_i \geq \sigma_i \geq \rho_{n-m+i} \quad \text{for } i = 1, \dots, m.$$

Interlacing is *tight* if there exists an integer  $k$  such that

$$\begin{aligned} \rho_i &= \sigma_i & \text{for } i = 1, \dots, k, \\ \rho_{n-m+i} &= \sigma_i & \text{for } i = k + 1, \dots, m. \end{aligned}$$

RESULT 1.1. *Let  $A_0$  be a symmetric matrix partitioned as follows:*

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\top & A_2 \end{pmatrix}.$$

Let  $B$  be the  $2 \times 2$  matrix whose entries are the average row sums of the blocks of  $A_0$ .

- (i) *Cauchy interlacing.* The eigenvalues of  $A_1$  interlace the eigenvalues of  $A_0$ . If the interlacing is tight, then  $C = 0$ .
- (ii) *The eigenvalues of  $B$  interlace the eigenvalues of  $A_0$ .* If the interlacing is tight, then  $A_1$ ,  $A_2$ , and  $C$  have constant row and column sums. Conversely, if  $A_1$ ,  $A_2$ , and  $C$  have constant row and column sums, both eigenvalues of  $B$  are also eigenvalues of  $A_0$ .

Our main tool is the following lemma. It is a kind of mixture of Theorem 5.1 in [3] and Theorem 1.3.3 in [13] ( $J$  denotes the all-one matrix).

LEMMA 1.2. *For  $i = 0, 1, 2$  let  $A_i$  be a symmetric  $v_i \times v_i$  matrix such that*

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\top & A_2 \end{pmatrix} \quad \text{and} \quad A_1 C + C A_2 = \alpha C + \beta J \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

Let  $A_1$ ,  $A_2$ ,  $C$ , and  $C^\top$  have constant row sums  $k_1$ ,  $k_2$ ,  $r$ , and  $k$  respectively.

For  $i = 0, 1, 2$  denote the eigenvalues of  $A_i$  by  $\rho_{i,1}, \dots, \rho_{i,v_i}$ . Denote the singular values of  $C$  by  $\sqrt{\gamma_1}, \dots, \sqrt{\gamma_m}$ , where  $m = \text{rank } C$ . Then we can order the  $\rho_{ij}$ 's and  $\gamma_j$ 's so that:

(i)  $\rho_{1,1} = k_1, \rho_{2,1} = k_2$  with all-one eigenvector,  $\gamma_1 = rk, k_1 + k_2 = \alpha + \beta v_1/k$ , and  $\rho_{0,1}, \rho_{0,2}$  are the roots of  $(x - k_1)(x - k_2) = rk$ .

(ii)  $\rho_{1,j} + \rho_{2,j} = \alpha$  with eigenvectors in the range of  $C$  and  $C^\top$ , respectively, and  $\rho_{0,2j}, \rho_{0,2j-1}$  are the roots of  $(x - \rho_{1,j})(x - \rho_{2,j}) = \gamma_j$  for  $j = 2, \dots, m$ .

(iii)  $\rho_{1,j}$  has an eigenvector in the kernel of  $C^\top$ ,  $\rho_{1,j} = \rho_{0,m+j}$ , for  $j = m + 1, \dots, v_1$ ;  $\rho_{2,j}$  has an eigenvector in the kernel of  $C$ ,  $\rho_{2,j} = \rho_{0,v_1+j}$ , for  $j = m + 1, \dots, v_2$ .

*Proof.* We have

$$A_1CC^\top = \alpha CC^\top + \beta rJ - CA_2C^\top.$$

The right-hand side is a symmetric matrix; hence  $A_1CC^\top = CC^\top A_1$ . So  $A_1$  and  $CC^\top$  commute, and therefore they have a common orthonormal bases of eigenvectors  $u_1, \dots, u_{v_1}$  (say), ordered so that  $A_1u_j = \rho_{1,j}u_j$  for  $j = 1, \dots, v_1$ ,  $CC^\top u_j = \gamma_j u_j$  for  $j = 1, \dots, m$ ,  $CC^\top u_j = 0$  for  $j = m + 1, \dots, v_1$ , and  $u_1$  is the all-one vector. Now the first two equations of (i) are obvious. Furthermore

$$A_2C^\top u_j = \alpha C^\top u_j + \beta Ju_j - C^\top A_1 u_j = (\alpha - \rho_{1,j})C^\top u_j \quad \text{for } j = 2, \dots, m,$$

proving the first equation of (ii). Define

$$w_j = \begin{pmatrix} \gamma_j u_j \\ (x - \rho_{1,j})C^\top u_j \end{pmatrix} \quad \text{for } j = 1, \dots, m.$$

Then it is easily verified that  $A_0 w_j = x w_j$  whenever  $(x - \rho_{1,j})(x - \rho_{2,j}) = \gamma_j$ . Thus (i) and (ii) are proved. Next define

$$w_j = \begin{pmatrix} u_j \\ 0 \end{pmatrix} \quad \text{for } j = m + 1, \dots, v_1.$$

Then  $A_0 w_j = \rho_{1,j} w_j$ , proving the first part of (iii). The second part of (iii) follows by interchanging  $A_1$  and  $A_2$ . ■

We assume the reader to be familiar with the theory of designs and strongly regular graphs. Some references are Beth, Jungnickel, and Lenz [1], Cameron and Van Lint [9], and Seidel [16]. We recall some result about strongly regular designs (see Higman [14]).

**DEFINITION 1.3.** A design  $D$  with  $v_1$  points and  $v_2$  blocks and incidence matrix  $C$  is strongly regular whenever there exist graphs  $\Gamma_1$  and  $\Gamma_2$  (not complete or void) with adjacency matrices  $A_1$  and  $A_2$  respectively, such that the following hold:

- (i)  $CC^\top = w_1I + y_1J + z_1A_1$  for integers  $w_1$ ,  $y_1$ , and  $z_1$  ( $z_1 \neq 0$ ),
- (ii)  $C^\top C = w_2I + y_2J + z_2A_2$  for integers  $w_2$ ,  $y_2$ , and  $z_2$  ( $z_2 \neq 0$ ),
- (iii)  $CC^\top C = \gamma C + \delta J$  for integers  $\gamma$  and  $\delta$ .

It is easily seen that  $C$  has constant row sum  $r = w_1 + y_1$  and column sum  $k = w_2 + y_2$ , and that  $\delta = k(kr - \gamma)/v_1$ . The graph  $\Gamma_1$  is the *point graph* of  $D$ , and  $\Gamma_2$  is the *block graph* of  $D$ . It is straightforward that  $\Gamma_i$  ( $i = 1, 2$ ) is strongly regular with eigenvalues

$$k_i = \frac{kr - y_i v_i - w_i}{z_i}, \quad \rho_i = \frac{\gamma - w_i}{z_i}, \quad \sigma_i = \frac{-w_i}{z_i}$$

of multiplicity 1,  $m - 1$ , and  $v_i - m$ , respectively, where  $m = \text{rank } C$ . The eigenspaces of the eigenvalues  $\sigma_1$  and  $\sigma_2$  are the kernels of  $C$  and  $C^\top$ , respectively. (The point and block graph are determined up to taking complements. To avoid this ambiguity one often requires that  $z_i > 0$ . However, for our purposes it is not convenient to do so.) Bose, Bridges, and Shrikhande [2] proved that (iii) may be replaced by:

(iii') *The singular values  $\sqrt{\gamma_1}, \dots, \sqrt{\gamma_m}$  of  $C$  satisfy*

$$\gamma_1 = rk, \quad \gamma_2 = \dots = \gamma_m = \gamma.$$

In case  $z_1 = 0$ ,  $D$  is a quasisymmetric block design. A strongly regular design is the same as a quasisymmetric special partially balanced incomplete block design (see Shrikhande [18]).

We finish this section with some notation. For a graph  $\Gamma_i$ ,  $v_i$  denotes the number of vertices, and the adjacency matrix is denoted by  $A_i$ . If  $A_i$  has eigenvalues  $\rho_1, \dots, \rho_n$  with respective multiplicities  $\varphi_1, \dots, \varphi_n$ , we write

$$\text{spec } \Gamma_i = \{ \rho_1^{\varphi_1}, \dots, \rho_n^{\varphi_n} \}.$$

If  $\Gamma_i$  is regular, the degree is denoted by  $k_i$ , and if  $\Gamma_i$  is strongly regular, we write

$$\text{spec } \Gamma_i = \{ k_i, r_i^{f_i}, s_i^{g_i} \} \quad \text{with } r_i \geq 0 > s_i.$$

Throughout the paper  $\Gamma_0$  denotes a graph decomposed into subgraphs  $\Gamma_1$  and  $\Gamma_2$ , that is, the respective adjacency matrices  $A_0$ ,  $A_1$ , and  $A_2$  satisfy

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\tau & A_2 \end{pmatrix},$$

where  $C$  is the incidence matrix of some structure  $D$  (say). For regular  $\Gamma_0$  the decomposition is called *regular* if also  $\Gamma_1$  and  $\Gamma_2$  are regular. For strongly regular  $\Gamma_0$  the decomposition is *strongly regular* if  $\Gamma_1$  and  $\Gamma_2$  are strongly regular, a clique, or a coclique.

## 2. THEORY

If  $\Gamma_0$  or the complement is the disjoint union of two or more cliques of equal size, then  $\Gamma_0$  is a so-called *imprimitive* strongly regular graph. In this case the strongly regular decompositions are obvious. Therefore we restrict ourselves to a primitive  $\Gamma_0$ .

LEMMA 2.1. *If  $\Gamma_0$  is strongly regular with a regular decomposition, then*

$$CJ = (k_0 - k_1)J, \quad C^\tau J = (k_0 - k_2)J,$$

$$A_1^2 + CC^\tau = (\tau_0 + s_0)A_1 - \tau_0 s_0 I + (k_0 + \tau_0 s_0)J,$$

$$A_2^2 + C^\tau C = (\tau_0 + s_0)A_2 - \tau_0 s_0 I + (k_0 + \tau_0 s_0)J,$$

$$A_1 C + C A_2 = (\tau_0 + s_0)C + (k_0 + \tau_0 s_0)J.$$

*Proof.* The first line reflects the fact that the decomposition is regular. If  $\Gamma_0$  is strongly regular, then  $A_0^2 - (\tau_0 + s_0)A_0 + \tau_0 s_0 I = (k_0 + \tau_0 s_0)J$ . Thus the block structure of  $A_0$  gives the remaining formulas. ■

**THEOREM 2.2.** *Suppose  $\Gamma_0$  is strongly regular and  $\Gamma_1$  is regular. Then*

$$s_0 \leq \frac{k_1 v_0 - k_0 v_1}{v_0 - v_1} \leq r_0.$$

*The decomposition is regular if and only if equality holds on the left- or right-hand side. If the left-hand [right-hand] inequality is met, then*

$$k_2 = k_0 - k_1 + s_0 \quad [k_2 = k_0 - k_1 + r_0].$$

*Proof.* We apply Result 1.1(ii). The matrix of the average row sums,

$$B = \begin{pmatrix} k_1 & k_0 - k_1 \\ (k_0 - k_1)v_1/v_2 & k_0 - (k_0 - k_1)v_1/v_2 \end{pmatrix},$$

has eigenvalues  $k_0$  (row sum) and  $\rho$  (say). From  $k_0 + \rho = \text{trace } B$  it follows that  $\rho = (k_1 v_0 - k_0 v_1)/(v_0 - v_1)$ , which gives the desired inequalities. Equality on either side means that the interlacing is tight, and hence the decomposition must be regular. If the decomposition is regular, the eigenvalues of  $B$  are  $k_0$  and  $\rho = k_1 + k_2 - k_0$ . These are also eigenvalues of  $A_0$ ; hence  $\rho = s_0$  or  $\rho = r_0$ . ■

It is easily verified that if equality holds on one side, then the corresponding decomposition of the complement of  $\Gamma_0$  satisfies equality on the other side. If  $\Gamma_1$  is a coclique (i.e.  $k_1 = 0$ ) the above result gives

$$v_1 \leq \frac{-v_0 s_0}{k_0 - s_0}.$$

This is Hoffman's coclique bound. Another bound is the following one.

**THEOREM 2.3.** *If  $\Gamma_1$  is a coclique and  $\Gamma_0$  is primitively strongly regular, then*

$$v_1 \leq \min\{f_0, g_0\}.$$

*Proof.* Define  $A = A_0 - v_0^{-1}(k_0 - s_0)J - s_0 I$ . Then  $\text{rank } A = f_0$ . Since  $A_1 = 0$ ,  $A$  has a submatrix  $-v_0^{-1}(k_0 - s_0)J - s_0 I$  of size  $v_1 \times v_1$ , which is nonsingular ( $s_0 \neq 0$ , since  $\Gamma_0$  is primitive). Hence  $v_1 \leq f_0$ . Similarly we get  $v_1 \leq g_0$ . ■

Theorems 2.2 and 2.3 are special cases of theorems of Haemers [13] and Cvetcović [10], respectively.

**THEOREM 2.4.** *Suppose  $\Gamma_0$  and  $\Gamma_1$  are strongly regular, let  $\Gamma_0$  be primitive, and suppose the decomposition is regular. Put  $\varepsilon$  equal to 0 or 1, according to whether the left- or the right-hand side is tight in Theorem 2.2 (e.g.  $k_2 = k_0 - k_1 + \varepsilon r_0 + (1 - \varepsilon)s_0$ ). Then one of the following holds:*

(i)  $s_1 > s_0, r_1 < r_0, v_1 \leq \min\{f_0 + 1 - \varepsilon, g_0 + \varepsilon\},$

$$\text{spec } \Gamma_2 = \{k_2, (r_0 + s_0 - r_1)^{f_1}, (r_0 + s_0 - s_1)^{g_1}, r_0^{f_0 - v_1 + 1 - \varepsilon}, s_0^{g_0 - v_1 + \varepsilon}\}.$$

(ii)  $s_1 = s_0, r_1 < r_0, v_1 \leq g_0 + \varepsilon,$

$$\text{spec } \Gamma_2 = \{k_2, (r_0 + s_0 - r_1)^{f_1}, r_0^{f_0 - f_1 - \varepsilon}, s_0^{g_0 - v_1 + \varepsilon}\}.$$

(iii)  $s_1 > s_0, r_1 = r_0, v_1 \leq f_0 + 1 - \varepsilon,$

$$\text{spec } \Gamma_2 = \{k_2, (r_0 + s_0 - s_1)^{g_1}, r_0^{f_0 - v_1 + 1 - \varepsilon}, s_0^{g_0 - g_1 - 1 + \varepsilon}\}.$$

*Proof.* By Lemmas 1.2 and 2.1 it follows that  $k_2, r_0 + s_0 - r_1, r_0 + s_0 - s_1, r_0,$  and  $s_0$  are the only possible eigenvalues of  $\Gamma_2$ , and that  $r_0 + s_0 - r_1$  [ $r_0 + s_0 - s_1$ ] has multiplicity  $f_1$  [ $g_1$ ] whenever  $r_1 \neq r_0$  [ $s_1 \neq s_0$ ]. From  $\text{trace } A_2 = 0$  one finds that the multiplicity of  $s_0$  [ $r_0$ ] equals  $g_0 - v_1 + \varepsilon$  [ $f_0 - v_1 + 1 - \varepsilon$ ], which must be a nonnegative number. The inequalities  $s_1 \geq s_0$  and  $r_1 \leq r_0$  follow from Cauchy interlacing [Result 1.1(i)]. What remains to be proved is that  $s_1 = s_0$  and  $r_1 = r_0$  do not both occur. Suppose they do. Define  $\alpha = (k_0 - \varepsilon r_0 - (1 - \varepsilon)s_0)/v_0$ ; then the matrix  $A_0 - \alpha J$ , which has eigenvalues  $r_0$  and  $s_0$  only, has principal submatrix  $A_1 - \alpha J$ , having only eigenvalues  $r_0$  and  $s_0$  too. So, by Result 1.1(i),  $C - \alpha J = 0$  and hence  $\Gamma_0$  is imprimitive: a contradiction. ■

The regular graph  $\Gamma_2$  is strongly regular, a clique, or a coclique whenever it has at most two distinct eigenvalues, except for the degree  $k_2$ . This leads to the following result.

**COROLLARY 2.5.** *With the hypotheses of Theorem 2.4, the decomposition is strongly regular if and only if one of the following holds:*

- (i)  $v_1 = f_0 + 1 - \varepsilon = g_0 + \varepsilon,$
- (ii)  $s_0 = s_1$  and  $f_0 = f_1 + \varepsilon,$
- (iii)  $s_0 = s_1$  and  $v_1 = g_0 + \varepsilon,$
- (iv)  $r_0 = r_1$  and  $g_0 = g_1 + 1 - \varepsilon,$
- (v)  $r_0 = r_1$  and  $v_1 = f_0 + 1 - \varepsilon.$

A strongly regular decomposition is called *improper* if  $\Gamma_1$  or  $\Gamma_2$  is a clique or a coclique. Without loss of generality we may assume then that  $\Gamma_1$  is a coclique. If  $\Gamma_0$  is strongly regular and  $\Gamma_1$  is a coclique, then also Theorem 2.4(i) holds with  $r_1 = 0$  and  $g_1 = 0$ . Thus we find the following result of Haemers [13]:

**THEOREM 2.6.** *Let  $\Gamma_0$  be primitively strongly regular, and let  $\Gamma_1$  be a coclique. Then  $v_1 = g_0 = -v_0 s_0 / (k_0 - s_0)$  (i.e., both Hoffman's bound and Cvetcović's bound are tight) if and only if  $\Gamma_2$  is strongly regular.*

*Proof.* Hoffman's bound is tight if and only if the decomposition is regular. Theorem 2.4(i) gives

$$\text{spec } \Gamma_2 = \{ k_2, (r_0 + s_0)^{v_1 - 1}, r_0^{f_0 - v_1 + 1}, s_0^{g_0 - v_1} \},$$

since  $\varepsilon = 0$  if  $\Gamma_1$  is a coclique. By Theorem 2.3 we have  $f_0 - v_1 + 1 > 0$ ; hence  $\Gamma_2$  is strongly regular if and only if  $g_0 = v_1$ . ■

We call a proper strongly regular decomposition *exceptional* if  $s_1 \neq s_0$  and  $r_1 \neq r_0$ , which is by Theorem 2.4(i) equivalent to  $s_2 \neq s_0$  and  $r_2 \neq r_0$ .

**THEOREM 2.7.** *If  $\Gamma_0$  is primitively strongly regular and admits an exceptional strongly regular decomposition, then  $\Gamma_0$  is the graph of a regular symmetric conference matrix, that is,  $\Gamma_0$  or its complement satisfies*

$$v_0 = 4r_0^2 + 4r_0 + 2, \quad k_0 = 2r_0^2 + r_0, \quad s_0 = -r_0 - 1 \quad \text{for integer } r_0.$$

Moreover, one of the following holds:

(i)  $\Gamma_1$  and  $\Gamma_2$  are so-called conference graphs, that is,

$$\begin{aligned} v_1 = v_2 &= 2r_0^2 + 2r_0 + 1, & k_1 = k_2 &= r_0^2 + r_0, \\ r_1 = r_2 &= \frac{-1 + \sqrt{v_1}}{2}, & s_1 = s_2 &= \frac{-1 - \sqrt{v_1}}{2}, \end{aligned}$$

and  $D$  is a symmetric  $2-(v_1, r_0^2, r_0(r_0 - 1)/2)$  design, or the complement.



(ii) *We have*

$$v_1 = v_2 = 2r_0^2 + 2r_0 + 1, \quad k_1 = k_2 = r_0^2 + r_0,$$

$$r_2 = \frac{k_1 - r_1}{2r_1 + 1}, \quad s_1 = -r_2 - 1, \quad s_2 = -r_1 - 1,$$

$$r_1 \neq r_2, \quad r_1 < r_0, \quad r_2 < r_0,$$

and  $r_1, r_2$ , and  $(2k_1^2 + k_1)/(k_1 + 2r_1^2 + 2r_1 + 1)$  are integers.

*Proof.* Take without loss of generality  $\varepsilon = 0$ . Then Corollary 2.5(i) gives  $f_0 + 1 = g_0$ , and the remaining parameters of  $\Gamma_0$  follow straightforwardly (see [9]). Also by 2.5(i) we have  $2v_1 = v_0$ , so  $v_1 = v_2$  and  $k_1 = k_2$ . By Theorem 2.2 it follows that  $k_1 = k_2 = (k_0 + s_0)/2 = r_0^2 + r_0$ . If  $\Gamma_1$  is a conference graph, then so is  $\Gamma_2$  (by Theorem 2.4), and by Lemma 2.1 we find

$$CC^T = \frac{r_0(r_0 + 1)}{2}I + \frac{r_0(r_0 - 1)}{2}J,$$

proving (i). If  $\Gamma_1$  and  $\Gamma_2$  are not conference graphs, then  $r_1 \neq -1 - s_1$  and the eigenvalues are integers. The remaining formulas of (ii) follow easily from Theorem 2.4 and the well-known identities for strongly regular graphs. ■

The Petersen graph partitioned into two pentagons is an example for (i). We give another example in the next section. For case (ii) it seems hopeless to find an example: The smallest feasible solution has  $r_1 = 554, r_0 = 731, v_0 = 2,140,370$ .

The next theorem relates strongly regular decompositions to strongly regular designs. The result is due to W. G. Bridges and M. S. Shrikhande [3].

**THEOREM 2.8.**  $\Gamma_0$  is primitively strongly regular with a strongly regular decomposition which is proper and not exceptional, if and only if  $D$  is a strongly regular design with point graph  $\Gamma_1$  and block graph  $\Gamma_2$ , whose parameters satisfy

- (i)  $k_1 + r = k_2 + k$ ,
- (ii)  $k_1 - k \in \{\sigma_1, \rho_1 + \rho_2 - \sigma_1\}$ ,
- (iii)  $\rho_2 = \sigma_1 + z_1, \rho_1 = \sigma_2 + z_2$ .

*Proof.* Let  $D$  be a strongly regular design. From Definition 1.3 it follows that  $A_1C + CA_2 \in \langle C, J \rangle$ ; hence Lemma 1.2 applies. Clearly  $\Gamma_0$  is regular (of degree  $k_0 = k_1 + r$ ) whenever  $k_1 + r = k_2 + k$ . For the remaining eigenvalues of  $\Gamma_0$  we get  $k_1 + k_2 - k_0 = k_2 - r$  [by Lemma 1.2(i)],  $\sigma_1$  and  $\sigma_2$  [by Lemma 1.2(iii)], and the roots of

$$(x - \rho_1)(x - \rho_2) = \gamma \tag{*}$$

(by 1.2(ii)). These five eigenvalues take only two values if and only if  $\sigma_1, \sigma_2$ , and  $k_2 - r$  are roots of (\*). By use of  $\rho_i - \sigma_i = \gamma_i/z_i$  we find that  $\sigma_i$  is a root of (\*) for  $i = 1, 2$  if and only if (iii) holds. Suppose  $\sigma_1$  is a root of (\*); then  $\rho_1 + \rho_2 - \sigma_1$  is the other root; hence  $k_2 - r$  is a root of (\*) if and only if (ii) holds. The decomposition is clearly proper, and it is not exceptional, since  $\sigma_1$  and  $\sigma_2$  are eigenvalues of  $\Gamma_0$ .

Next assume  $\Gamma_0$  has the required properties. Then  $r_0 = r_1$  or  $s_0 = s_1$ , since the decomposition is not exceptional. Take without loss of generality  $r_0 = r_1$ . Then Lemma 1.2(ii) gives

$$\gamma_j = (r_0 - s_0)(s_1 - s_0) \quad \text{for } j = 2, \dots, m.$$

So  $D$  satisfies (iii') of Definition 1.3. By Lemma 2.1 and the strong regularity of  $\Gamma_1$  and  $\Gamma_2$  we have

$$CC^T \in \langle A_1, I, J \rangle, \quad C^T C \in \langle A_2, I, J \rangle.$$

Moreover, the coefficient of  $A_i$  equals  $r_0 + s_0 - r_i - s_i \neq 0$  for  $i = 1, 2$ . Hence also (i) and (ii) of Definition 1.3 are satisfied, so  $D$  is a strongly regular design. ■

From the above proof we have that a strongly regular  $\Gamma_0$  has eigenvalues  $\sigma_1$  and  $\rho_1 + \rho_2 - \sigma_1$ ; one of the two must be equal to  $\sigma_2$ . The following result can be regarded as a special case of the above theorem (therefore a proof is superfluous).

**THEOREM 2.9.**  $\Gamma_0$  is primitively strongly regular with an improper strongly regular decomposition (where  $\Gamma_1$  is a coclique) if and only if  $D$  is a quasisymmetric 2-design, having  $\Gamma_2$  as block graph, whose parameters satisfy

$$r = k + k_2, \quad \sigma_2 + z_2 = 0, \quad k = -\gamma/z_2.$$

From  $k = -\gamma/z_2$  it follows that  $z_2 < 0$ . This means that if (as usual) adjacency in the block graph corresponds to the larger intersection number, then  $\Gamma_2$  is the complement of the block graph of  $D$ . M. S. Shrikhande [17] (see also [3]) proved that the conditions for  $D$  in Theorem 2.9 are equivalent to the following:  $D$  is a quasisymmetric  $2-(1 + z_2(k - 1)/(k - z_2^2), k, k(k - z_2^2)/(z_2 + 1))$  design with intersection numbers  $k - z_2^2$  and  $k - z_2^2 - z_2$ .

### 3. CONSTRUCTIONS

In this section we give constructions and some nonexistence results for strongly regular graphs with strongly regular decompositions. With the help of the results of the previous section we have made a table of feasible parameters up to 300 vertices (Table 1). For all cases in the table we indicate existence or nonexistence if known (to us).

EXAMPLE 3.1. The vertices of the *triangular graph*  $T(m)$  are all pairs of a given set  $M$  of cardinality  $m$  ( $m > 3$ ); two vertices are adjacent whenever the pairs are not disjoint.  $T(m)$  is strongly regular with

$$\text{spec } T(m) = \{2(m - 2), (m - 4)^{m - 1}, (-2)^{m(m - 3)/2}\}.$$

For a fixed  $x \in M$ , partition the vertices into the pairs containing  $x$  and pairs not containing  $x$ . It is easily seen that this gives an improper strongly regular decomposition of  $T(m)$  into a clique of size  $m - 1$  and  $T(m - 1)$ .

The next result has often been observed before.

THEOREM 3.2. *The block graph of a quasisymmetric 3-design  $E$  admits a strongly regular decomposition. The decomposition is improper if and only if  $E$  is the extension of a symmetric 2-design.*

*Proof.* Fix a point  $x$  of  $E$ . Partition the blocks of  $E$  to the blocks containing  $x$  and the blocks not containing  $x$ . This gives a partition of the block graph of  $E$  into the block graphs of the derived and the residual design of  $E$  (with respect to  $x$ ) respectively. The derived or residual design of  $E$  is symmetric whenever  $E$  is the extension of a symmetric design; otherwise both designs are quasisymmetric. This proves the result. ■

The design whose blocks are just all pairs of points can be seen as a degenerate quasisymmetric 3-design. This leads to Example 3.1. We know of

TABLE I  
ALL FEASIBLE PARAMETER SETS FOR PRIMITIVE STRONGLY REGULAR GRAPHS WITH STRONGLY REGULAR DECOMPOSITION UP TO  
300 VERTICES<sup>a</sup>

Case	$v_0$	$k_0$	$r_0$	$s_0$	$f_0$	$g_0$	$\lambda_0$	$\mu_0$	$v_1$	$k_1$	$r_1$	$s_1$	$f_1$	$g_1$	$\lambda_1$	$\mu_1$	
									$v_2$	$k_2$	$r_2$	$s_2$	$f_2$	$g_2$	$\lambda_2$	$\mu_2$	
+	1	15	6	1	-3	9	5	1	3	6	3	0	-3	4	1	0	3
									9	4	1	-2	4	4	1	2	
+	2	16	5	1	-3	10	5	0	2	8	1	1	-1	3	4	0	0
									8	1	1	-1	3	4	0	0	
+	3	63	30	3	-5	35	27	13	15	28	15	1	-5	20	7	6	10
									35	18	3	-3	14	20	9	9	
+	4	63	30	3	-5	35	27	13	15	27	10	1	-5	20	6	1	5
									36	15	3	-3	15	20	6	6	
-	5	69	20	5	-3	23	45	7	5	24	5	5	-1	3	20	4	0
									45	12	3	-3	20	24	3	3	
+	6	70	27	6	-3	20	49	12	9	21	6	6	-1	2	18	5	0
									49	18	4	-3	18	30	7	6	
-	7	76	21	2	-7	56	19	2	7	19	0						
									57	14	2	-5	38	18	1	4	
+	8	77	16	2	-6	55	21	0	4	21	0						
									56	10	2	-4	35	20	0	2	
-	9	85	14	4	-3	34	50	3	2	35	4	4	-1	6	28	3	0
									50	7	2	-3	28	21	0	1	

TABLE 1 (Continued)

Case	$v_0$	$k_0$	$r_0$	$s_0$	$f_0$	$g_0$	$\lambda_0$	$\mu_0$	$v_1$	$k_1$	$r_1$	$s_1$	$f_1$	$g_1$	$\lambda_1$	$\mu_1$
									$v_2$	$k_2$	$r_2$	$s_2$	$f_2$	$g_2$	$\lambda_2$	$\mu_2$
-	10	95	40	2	-10	75	19	12	20	10	0	-10	18	1	0	10
-	11	95	40	2	-10	75	19	12	20	19	0	-8	56	18	10	16
+	12	100	22	2	-8	77	22	0	6	30	2	-8	57	18	8	14
+	13	112	30	2	-10	90	21	2	10	7	2	-3	28	21	0	1
-	14	126	60	12	-3	21	104	33	24	7	2	-3	28	21	0	1
+	15	162	56	2	-16	140	21	10	24	56	10	-4	35	20	0	2
+	16	176	70	2	-18	154	21	18	34	56	10	-4	35	20	0	2
-	17	208	75	10	-5	64	143	30	25	120	42	-12	99	20	8	18
-	18	217	66	4	-11	154	62	15	22	144	55	-5	55	88	22	20
?	19	220	84	14	-4	44	175	38	28	63	22	-11	55	7	1	11
										154	48	-8	98	55	12	16
										45	14	-1	2	42	13	0
										175	66	-4	42	132	29	22

TABLE 1 (Continued)

Case	$v_0$	$k_0$	$t_0$	$s_0$	$f_0$	$g_0$	$\lambda_0$	$\mu_0$	$v_1$	$v_2$	$k_1$	$k_2$	$r_1$	$r_2$	$s_1$	$s_2$	$f_1$	$f_2$	$g_1$	$g_2$	$\lambda_1$	$\lambda_2$	$\mu_1$	$\mu_2$	
+	20	221	64	12	-4	51	169	24	16	52	12	12	12	12	-1	3	48	11	0						
										169	48	9	-4	48	120	17	12								
?	21	236	55	11	-4	59	176	18	11	60	11	11	-1	4	55	10	0								
										176	40	8	-4	55	120	12	8								
?	22	245	52	3	-13	195	49	3	13	49	0														
										196	39	3	-10	147	48	2	9								
-	23	246	85	3	-17	204	41	20	34	41	0														
										205	68	3	-14	164	40	15	26								
+	24	253	112	2	-26	230	22	36	60	77	16	2	-6	55	21	4									
										176	70	2	-18	154	21	18	34								
+	25	255	126	7	-9	135	119	61	63	120	63	3	-9	84	35	30	36								
										135	70	7	-5	50	84	37	35								
+	26	255	126	7	-9	135	119	61	63	119	54	3	-9	84	34	21	27								
										136	63	7	-5	51	84	30	28								
-	27	261	84	21	-3	29	231	39	21	29	28														
										232	77	19	-3	28	203	36	20								
?	28	265	96	6	-10	159	105	32	36	105	32	2	-10	84	20	4	12								
										160	54	6	-6	75	84	18	18								
?	29	266	45	3	-12	209	56	0	9	56	0														
										210	33	3	-9	154	55	0	6								
?	30	287	126	3	-21	245	41	45	63	42	21	0	-21	40	1	0	21								
										245	108	3	-18	204	40	39	54								
?	31	287	126	3	-21	245	41	45	63	41	0														
										246	105	3	-18	205	40	36	51								

<sup>a</sup>The triangular graphs (Example 3.1) and the exceptional decompositions (Theorem 2.7) are omitted. For each case it is indicated whether the decomposition exists (+), does not exist (-), or is not settled (?).

just three other quasisymmetric 3-designs (up to taking complements and except for the Hadamard 3-designs, which have imprimitive block graphs): the famous 4-(23, 7, 1) design (see [9]), its derived design, and its residual design. These three 3-designs are cases 8, 16, and 24 in the table. The first one provides an improper decomposition ( $D$  is the extension of the projective plane of order 4). In fact, this decomposition and the ones of Example 3.1 are the only improper decompositions we know.

**THEOREM 3.3.** *Let  $\Gamma_1$  and  $D$  be as in Theorem 2.7(i). Suppose their matrices  $A_1$  and  $C$  commute, and let  $\Gamma_2$  be the complement of  $\Gamma_1$ . Then  $\Gamma_0$  is strongly regular with an exceptional strongly regular decomposition.*

*Proof.* We have

$$A_i^2 = -A_i + \frac{1}{2}r_0(r_0 + 1)J + \frac{1}{2}r_0(r_0 + 1)I \quad \text{for } i = 1, 2,$$

$$CC^T = C^TC = \frac{1}{2}r_0(r_0 - 1)J + \frac{1}{2}r_0(r_0 + 1)I,$$

$$A_1C + CA_2 = A_1C - CA_1 + CJ - C = r_0^2J - C.$$

This yields

$$A_0^2 = -A_0 + r_0^2J + r_0(r_0 + 1)I,$$

which proves the result. ■

If  $r_0 = 1$ , then  $\Gamma_1$  is the pentagon,  $D$  is the degenerate 2-(5, 1, 0) design, and  $\Gamma_0$  is the Petersen graph. For  $r_0 = 2$ , the desired graph and design are known:

$$A_1 = \text{cycle } (1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0),$$

$$C = \text{cycle } (1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0).$$

Since  $A_1$  and  $C$  are both cyclic, they commute. Thus by the above theorem we find a strongly regular  $\Gamma_0$  with  $(v_0, k_0, r_0, s_0) = (26, 10, 2, -3)$ , decomposed into the strongly regular  $\Gamma_1$  and  $\Gamma_2$  with  $(v_1, k_1, r_1, s_1) = (v_2, k_2, r_2, s_2) = (13, 6, (-1 + \sqrt{13})/2, (-1 - \sqrt{13})/2)$ . These are all the exceptional strongly regular decompositions we know. More graphs and designs with suitable parameters are known, but it is not known whether there exists a pair with commuting matrices.

**THEOREM 3.4.** *If  $q$  is the order of a projective plane, and if there exist  $q - 1$  mutually orthogonal Latin squares of order  $q^2 + q + 1$ , then there exists a strongly regular decomposition with*

$$(v_0, k_0, r_0, s_0) = ((q^2 + q + 1)(q^2 + 2q + 2), (q + 1)^3, q^2 + q, -q - 1),$$

$$(v_1, k_1, r_1, s_1) = ((q^2 + q + 1)(q + 1), q(q + 1), q(q + 1), -1),$$

$$(v_2, k_2, r_2, s_2) = ((q^2 + q + 1)^2, q(q + 1)^2, q^2, -q - 1).$$

*Proof.* A set of  $q - 1$  mutually orthogonal Latin squares is the same as a transversal design with  $q + 1$  groups of size  $q^2 + q + 1$ . Let

$$B_2 = \begin{pmatrix} N_1^\top & N_2^\top & \cdots & N_{q+1}^\top \end{pmatrix}^\top$$

be the incidence matrix of the transversal design, where the  $N_i$ 's correspond to the groups. Let  $M$  be the incidence matrix of a projective plane of order  $q$ , and define  $B_1 = I \otimes M$  ( $\otimes$  denotes the tensor product), and  $B = (B_1 \ B_2)$ . Then  $B$  is the incidence matrix of a  $2 - ((q^2 + q + 1)(q + 1), q + 1, 1)$  design (which is obviously quasisymmetric) with block graph  $\Gamma_0$ . Clearly the block graph  $\Gamma_1$  of  $B_1$  is imprimitively strongly regular. Also the block graph of a transversal design is strongly regular. So the decomposition is strongly regular and the eigenvalues readily follow. ■

For many values of  $q$  the conditions of Theorem 3.4 are fulfilled—for instance, if  $q$  and  $q^2 + q + 1$  are both prime powers (e.g.  $q = 1, 2, 3, 5, 8$ ), but also (see Brouwer [4]) if  $q$  and  $q + 1$  are both prime powers (e.g.  $q = 1, 2, 3, 4, 7, 8$ ). Cases 1, 6, and 20 in the table can be constructed in this manner. We do not know if the theorem provides an infinite family. The following example, however, does give infinitely many proper strongly regular decompositions.

**EXAMPLE 3.5.** For every integer  $m > 1$ , the symplectic graph  $\Gamma_0$  with

$$(v_0, k_0, r_0, s_0) = (2^{2m} - 1, 2^{2m-1} - 2, 2^{m-1} - 1, -2^{m-1} - 1)$$



admits two strongly regular decompositions: one with

$$(v_1, k_1, r_1, s_1) = (2^{2m-1} + 2^{m-1} - 1, 2^{2m-2} + 2^{m-1} - 2, 2^{m-1} - 1, -2^{m-2} - 1),$$

$$(v_2, k_2, r_2, s_2) = (2^{2m-1} - 2^{m-1}, 2^{2m-2} - 1, 2^{m-2} - 1, -2^{m-1} - 1),$$

and one with

$$(v_1, k_1, r_1, s_1) = (2^{2m-1} - 2^{m-1} - 1, 2^{2m-2} - 2^{m-1} - 2, 2^{m-2} - 1, -2^{m-1} - 1),$$

$$(v_2, k_2, r_2, s_2) = (2^{2m-1} + 2^{m-1}, 2^{2m-2} - 1, 2^{m-1} - 1, -2^{m-2} - 1).$$

In both cases  $\Gamma_1$  is the orthogonal graph, defined on the points of an orthogonal quadric in  $PG(2m-1, 2)$ . The symplectic and orthogonal graphs are described in Seidel [15]. For  $m = 2$  the decompositions coincide with Theorem 3.4 ( $q = 1$ ) and Example 3.1 ( $m = 6$ ), respectively. For larger  $m$ , the decompositions are proper and  $\Gamma_1$  and  $\Gamma_2$  are both primitive. Cases 1, 3, 4, 25, and 26 in the table are of this type.

Next we shall give some sporadic examples (making use of the table).

**EXAMPLE 3.6.** Case 2 in the table exists, that is, the Clebsch graph has a strongly regular decomposition with

$$A_1 = A_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad C = \begin{pmatrix} I & J-I \\ J-I & I \end{pmatrix}.$$

**EXAMPLE 3.7.** Case 12 in the table exists, that is, the Higman-Sims graph admits a strongly regular decomposition into two Hoffman-Singleton graphs (see Sims [19]).

**EXAMPLE 3.8.** A hemisystem (see Cameron, Delsarte, and Goethals [8]) is a strongly regular decomposition of the point graph of a generalized quadrangle of order  $(q^2, q)$ . The only known hemisystem has  $q = 3$ , where the point graph is decomposed into two Gewirtz graphs. This produces case 13 of the table.

**EXAMPLE 3.9.** Goethals and Seidel [11] give a construction of  $\Gamma_0$  of case 15 from which the strongly regular decomposition is obvious.

Finally some nonexistence results are considered. Case 7 in the table is impossible, since Wilbrink and Brouwer [22] showed that  $\Gamma_2$  does not exist. For cases 17 and 18,  $\Gamma_1$  does not exist because of the absolute bound. By Theorem 2.9 the existence of an improper strongly regular decomposition is equivalent to the existence of a quasisymmetric 2-design with suitable parameters. For quasisymmetric designs many nonexistence results are known. These results lead to nonexistence of cases 11, 14, 23 (due to Calderbank [6, 7]), and 27 (due to Haemers [12]; see also Tonchev [20]) in the table. The remaining cases are more complicated.

**THEOREM 3.10.** *No strongly regular graphs with strongly regular decomposition exist for the parameter sets numbered 5 and 9 in Table 1.*

*Proof.* In both cases  $\Gamma_1$  is imprimitive. Therefore  $D$  is a group divisible design. Take  $C$  in canonical form, that is,

$$C = \left( C_1^\tau \quad \dots \quad C_n^\tau \right)^\tau,$$

where the  $C_i$ 's correspond to the groups. For case 5 we define

$$B_1 = \text{cycle } (J - I \quad I \quad I \quad I),$$

wherein the blocks are  $6 \times 6$  matrices. Then by straightforward verification it follows that  $(C \ B_1)$  is the incidence matrix of a quasi-symmetric 2-(24, 8, 7) design with intersection numbers 4 and 2. Brouwer and Calderbank [5] showed that such a design does not exist. Similarly, for number 9 we define

$$B_1 = \text{cycle } (J - I \quad I \quad I \quad 0 \quad I \quad 0 \quad 0),$$

wherein the blocks have size  $5 \times 5$ . Then  $(C \ B_1)$  is the incidence matrix of a quasisymmetric 2-(35, 7, 3) design with intersection numbers 3 and 1. Calderbank [6] has proved the nonexistence of such a design. ■

**THEOREM 3.11.** *Strongly regular decomposition 10 in Table 1 does not exist.*

*Proof.* We use Seidel switching (see [9] or [16]). Since  $k_0 = -2r_0s_0$ , we obtain a strong graph by extending  $\Gamma_0$  with an isolated vertex. Next we

isolate, by switching, a vertex of  $\Gamma_1$ . The graph on the remaining vertices is strongly regular with the same parameters as the original  $\Gamma_0$ . However, the remaining vertices of  $\Gamma_1$  now induce a coclique of size 19. Therefore, by Theorem 2.6 we have constructed strongly regular decomposition 11, which is impossible. ■

The smallest unsettled case in the table is 19. Tonchev constructed a strongly regular graph with the parameters of  $\Gamma_0$  (see [21]), but it has no cliques of size 15, so it does not admit a strongly regular decomposition. (In fact, Tonchev's graph has maximal clique and coclique size equal to 10.)

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