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On binormality in non-separable Banach spaces

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ABSTRACT

We study binormality, a separation property of the norm and weak topologies of a Banach space. We show that every Banach space which belongs to a $\mathcal{P}$-class is binormal. We also show that the asplundness of a Banach space is equivalent to a related separation property of its dual space.

1. Introduction and main results

Let σ and τ be two topologies on a set X. We say that (X,σ,τ) is *binormal* if, for every disjoint σ-closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. We say that a Banach space X is binormal if X is binormal with respect to its norm and weak topologies.

It is possible to meet the notion of binormality of (X,σ,τ) in the real analysis where it is more likely called Lusin–Menchoff property of τ in the case that the "second topology" τ is finer than σ. For example, it is known that both the density topology and the fine topology have the Lusin–Menchoff property with respect to the Euclidean topology (see, e.g., [10]). The situation in Banach spaces is somewhat opposite to that of real analysis because the finer topology is the metrizable one.

The question whether the weak topology has the corresponding "Lusin–Menchoff property" with respect to the norm topology was posed by L. Zajíček. This question was studied later by P. Holický who proved in [7] that every separable Banach space is binormal and that the space ℓ^∞ is not binormal. But it was not possible to decide what was the answer for many other non-separable Banach spaces, e.g. for non-separable Hilbert spaces.

In this paper, we show that many non-separable Banach spaces are binormal. We prove the following result (see Theorem 5.2 and Theorem 4.2).

Theorem 1.1. *Every Plichko space is binormal. Every dual to an Asplund space is binormal. Generally, any Banach space which belongs to a $\mathcal{P}$-class is binormal.*

We give the necessary definitions below. Note that the class of Plichko spaces is quite wide and it contains all reflexive spaces or, more generally, all weakly compactly generated spaces. On the other hand, we show that there is a Banach space which admits a LUR norm but it is not binormal (Example 5.3).

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Some results in this paper are formulated for a general locally convex topology instead of the weak topology. If X is a Banach space and τ is a locally convex topology which is weaker than the norm topology, we say that X is τ*-binormal* if X is binormal with respect to its norm topology and τ. We prove characterizations of τ-binormality by another separation property and by an in-between condition (Proposition 2.6).

We are interested in the case of the w^*-topology. We prove the following theorem (which is covered by Theorem 6.3). Note that the separability of the set A cannot be dropped (Example 6.6).

Theorem 1.2. *A Banach space E is Asplund if and only if, for every disjoint separable and closed $A \subset E^*$ and w^*-closed $B \subset E^*$, there are disjoint open $D \subset E^*$ and w^*-open $C \subset E^*$ with $A \subset C$ and $B \subset D$.*

Furthermore, our methods lead to the characterization of scattered compact spaces by a separation property (Theorem 6.8).

2. A characterization of binormality

We start with a well-known variant of the Urysohn lemma. The lemma follows from [10, Theorem 3.11] in the case that the topologies are comparable (which will be our case) but it holds in the general situation as well (see [10, exercise 3.B.5(e)]).

Lemma 2.1. *Let (X,σ,τ) be binormal. If σ-closed $A \subset X$ and τ-closed $B \subset X$ are disjoint, then there is a lower σ-semicontinuous and upper τ-semicontinuous function h on X such that*

$$0 \leqslant h \leqslant 1, \quad h = 0 \text{ on } A, \quad h = 1 \text{ on } B.$$

We now prove an abstract version of our characterization.

Lemma 2.2. *Let Y be a set with two topologies σ_Y and τ_Y with τ_Y weaker than σ_Y. Let*

$$X = Y \times \mathbb{R}$$

and let the products of σ_Y and τ_Y with the standard topology on $\mathbb{R}$ be denoted by σ and τ.

If the condition

$$\forall U \in \tau,\ \exists \{U_n\}_{n\in\mathbb{N}},\ U_n \in \tau\colon\quad U = \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \overline{U_n}^{\sigma} \tag{$*$}$$

is satisfied, then the following assertions are equivalent:

- (i) *(X,σ,τ) is binormal.*
- (iia) *If $F_1 \supset F_2 \supset \cdots$ are σ_Y-closed subsets of Y with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ_Y-open subsets of Y, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma_Y} = \emptyset$.*
- (iib) *If $F_1 \supset F_2 \supset \cdots$ are σ-closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ-open subsets of X, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$.*
- (iii) *If $f : X \to (0,\infty)$ is lower σ-semicontinuous, then there exists $g : X \to (0,\infty)$, lower σ-semicontinuous and upper τ-semicontinuous, such that $g < f$.*

Remark 2.3. Binormality of (Y,σ_Y,τ_Y) is not sufficient for binormality of (X,σ,τ). If we take $Y = [0,1]$, σ_Y the discrete topology on Y and τ_Y the standard topology, then (Y,σ_Y,τ_Y) is clearly binormal. Let us show that it does not satisfy (iia). Take pairwise distinct numbers $a_1, a_2, \ldots \in [0,1]$ which form a countable dense subset of $[0,1]$ and put

$$F_n = \{a_n, a_{n+1}, \ldots\}, \quad n \in \mathbb{N}.$$

Note that F_n is dense in $[0,1]$ for every $n \in \mathbb{N}$. We have $\bigcap_{n=1}^{\infty} F_n = \emptyset$ but the Baire theorem guarantees that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ whenever $G_1, G_2, \ldots \subset [0,1]$ are open sets with $F_n \subset G_n$, $n \in \mathbb{N}$.

We will use this simple idea in a general situation later (proof of Lemma 6.2).

Before proving the lemma, we prove

Claim 2.4. *(Cf. proof of [7, Theorem 1].) Let σ and τ be two topologies on a set X and let the condition $(*)$ from Lemma 2.2 be satisfied. Let $A \subset X$ be σ-closed and $B \subset X$ be τ-closed. If there are σ-open $D_n \subset X$, $n \in \mathbb{N}$, such that $B \subset \bigcup_{n=1}^{\infty} D_n$ and $\overline{D_n}^{\tau} \cap A = \emptyset$ for all $n \in \mathbb{N}$, then there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*

Proof. By ($*$), there are τ-open sets $C_m \subset X, m \in \mathbb{N}$, such that $X \setminus B = \bigcup_{m=1}^{\infty} C_m$ and $\overline{C_m}^{\sigma} \cap B = \emptyset$ for all $m \in \mathbb{N}$. In particular, $A \subset \bigcup_{m=1}^{\infty} C_m$. Define

$$D = \bigcup_{n=1}^{\infty} \left(D_n \setminus \bigcup_{m=1}^{n} \overline{C_m}^{\sigma} \right),$$

$$C = \bigcup_{m=1}^{\infty} \left(C_m \setminus \bigcup_{n=1}^{m} \overline{D_n}^{\tau} \right).$$

It can be easily checked that C is τ-open, D is σ-open, $A \subset C$, $B \subset D$ and $C \cap D = \emptyset$. □

Proof of Lemma 2.2. (i) ⇒ (iia) Put

$$A = \bigcup_{n=1}^{\infty} F_n \times [1/n, \infty), \qquad B = Y \times \{0\}. \tag{1}$$

Clearly, A is σ-closed, B is τ-closed and $A \cap B = \emptyset$. By the assumption, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. We have $A \cap \overline{D}^{\tau} \subset A \setminus C = \emptyset$. We define H_n as the set of points $y \in Y$ such that there is a σ_Y-open neighbourhood $U \ni y$ with $U \times [0, 1/n] \subset D$. Let G_n be defined as $Y \setminus \overline{H_n}^{\tau_Y}$. We have $\bigcup_{n=1}^{\infty} H_n = Y$, and so $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma_Y} \subset \bigcap_{n=1}^{\infty} \overline{Y \setminus H_n}^{\sigma_Y} = \bigcap_{n=1}^{\infty} (Y \setminus H_n) = \emptyset$. Clearly, $G_1 \supset G_2 \supset \cdots$. For $n \in \mathbb{N}$, we have $\overline{H_n}^{\tau_Y} \times [0, 1/n] \subset \overline{D}^{\tau} \subset X \setminus A$, and so $F_n \times \{1/n\} = A \cap (Y \times \{1/n\}) \subset (Y \times \{1/n\}) \setminus (\overline{H_n}^{\tau_Y} \times [0, 1/n]) = G_n \times \{1/n\}$.

(iia) ⇒ (iib) For $n \in \mathbb{N}$ and $i \in \mathbb{Z}$, we define

$$F_n^i = \bigl\{ y \in Y\colon\ (y, r) \in F_n \text{ for some } r \in [i - 1/2, i + 1/2] \bigr\}. \tag{2}$$

Due to the compactness of $[i - 1/2, i + 1/2]$, the sets F_n^i are σ_Y-closed and $\bigcap_{n=1}^{\infty} F_n^i = \emptyset$ for all $i \in \mathbb{Z}$. By the assumption, there are, for all $i \in \mathbb{Z}$, τ_Y-open $G_1^i \supset G_2^i \supset \cdots$ such that $F_n^i \subset G_n^i$ and $\bigcap_{n=1}^{\infty} \overline{G_n^i}^{\sigma_Y} = \emptyset$. Then the choice

$$G_n = \bigcup_{i \in \mathbb{Z}} \bigl(G_n^i \times (i - 1, i + 1) \bigr), \quad n \in \mathbb{N},$$

works. (We have $F_n \subset \bigcup_{i \in \mathbb{Z}} F_n^i \times [i - 1/2, i + 1/2] \subset G_n$ for $n \in \mathbb{N}$. Suppose that $(y, r) \in \bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma}$. Put $U = Y \times (r - 1, r + 1)$. We have $U \cap (G_n^i \times (i - 1, i + 1)) = \emptyset$ whenever $|i - r| \geqslant 2$. There is $n \in \mathbb{N}$ such that $y \notin \overline{G_n^i}^{\sigma_Y}$ for all i with $|i - r| < 2$. If we take $V = (Y \setminus \bigcup_{|i-r|<2} \overline{G_n^i}^{\sigma_Y}) \times \mathbb{R}$, then $U \cap V$ is a σ-open neighbourhood of (y, r) which does not intersect G_n. This contradicts $(y, r) \in \overline{G_n}^{\sigma}$.)

(iib) ⇒ (i) Let σ-closed $A \subset X$ and τ-closed $B \subset X$ satisfy $A \cap B = \emptyset$. We need to find disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$. By ($*$), there are τ-open sets $H_n \subset X$, $n \in \mathbb{N}$, such that $X \setminus B = \bigcup_{n=1}^{\infty} H_n$ and $\overline{H_n}^{\sigma} \cap B = \emptyset$ for all $n \in \mathbb{N}$. We may assume that $H_1 \subset H_2 \subset \cdots$. The sets H_n are σ-open in particular. We put

$$F_n = A \setminus H_n \tag{3}$$

for $n \in \mathbb{N}$. The sets F_n, $n \in \mathbb{N}$, are σ-closed, $F_1 \supset F_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} F_n = A \setminus \bigcup_{n=1}^{\infty} H_n = A \setminus (X \setminus B) = \emptyset$. By the assumption, there are τ-open $G_1 \supset G_2 \supset \cdots$ such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$. For $n \in \mathbb{N}$, we put

$$C_n = G_n \cup H_n, \qquad D_n = X \setminus \overline{C_n}^{\sigma}.$$

We obtain $A = F_n \cup (A \cap H_n) \subset G_n \cup (A \cap H_n) \subset C_n$, and so $\overline{D_n}^{\tau} \cap A \subset (X \setminus C_n) \cap C_n = \emptyset$, for $n \in \mathbb{N}$. Considering Claim 2.4, it remains to prove that $B \subset \bigcup_{n=1}^{\infty} D_n$. For $n \in \mathbb{N}$, we have

$$B \setminus D_n = B \cap \overline{C_n}^{\sigma} = \bigl(B \cap \overline{G_n}^{\sigma} \bigr) \cup \bigl(B \cap \overline{H_n}^{\sigma} \bigr) = B \cap \overline{G_n}^{\sigma},$$

and so $B \setminus \bigcup_{n=1}^{\infty} D_n = \bigcap_{n=1}^{\infty} (B \setminus D_n) = \bigcap_{n=1}^{\infty} (B \cap \overline{G_n}^{\sigma}) = \emptyset$.

(iib) ⇒ (iii) We have already proved (iib) ⇒ (i). Therefore, assuming (iib), we can assume (i) as well.

We put $F_n = \{x \in X\colon\ f(x) \leqslant 1/n\}$. By (iib), we take τ-open $G_1 \supset G_2 \supset \cdots$ such that $F_n \subset G_n$ and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$. By (i) and Lemma 2.1, there is, for every $n \in \mathbb{N}$, lower σ-semicontinuous and upper τ-semicontinuous function $g_n : X \to [0, 1]$ such that $g_n = 0$ on F_n and $g_n = 1$ on $X \setminus G_n$. We have $g_n/n < f$ on X. Putting

$$g = \sum_{n=1}^{\infty} \frac{g_n}{2^n n},$$

we have $0 < g < f$ on X.

(iii) ⇒ (iib) We may assume $F_1 = X$. We define $f(x) = 1/n$ for every $x \in F_n \setminus F_{n+1}$ (this defines a lower σ-semicontinuous function on whole space X). By (iii), there exists $g : X \to (0,\infty)$, lower σ-semicontinuous and upper τ-semicontinuous, such that $g < f$. For $n \in \mathbb{N}$, we take τ-open $G_n = \{x \in X: g(x) < 1/n\}$. We have $F_n = \{x \in X: f(x) \leqslant 1/n\} \subset \{x \in X: g(x) < 1/n\} = G_n$. At the same time, $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} \subset \bigcap_{n=1}^{\infty} \{x \in X: g(x) \leqslant 1/n\} = \{x \in X: g(x) \leqslant 0\} = \emptyset$. □

By an inspection of the proof of Lemma 2.2, we get the following modification.

Lemma 2.5. *Let $Y, \sigma_Y, \tau_Y, X, \sigma, \tau$ be as in Lemma 2.2 and let $(*)$ be satisfied. Moreover, let σ be metrizable. Then the following assertions are equivalent*:

(i) *For every disjoint σ-separable and σ-closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint σ-open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*

(iia) *If $F_1 \supset F_2 \supset \cdots$ are σ_Y-separable and σ_Y-closed subsets of Y with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ_Y-open subsets of Y, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma_Y} = \emptyset$.*

(iib) *If $F_1 \supset F_2 \supset \cdots$ are σ-separable and σ-closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ-open subsets of X, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n}^{\sigma} = \emptyset$.*

Proof. The lemma can be proved in the same way as Lemma 2.2. The following should be mentioned.

- In the proof of (i) ⇒ (iia), we realize that the set A defined by (1) is σ-separable because $F_1, F_2, \ldots$ are assumed to be σ_Y-separable.
- In the proof of (iia) ⇒ (iib), we realize that the sets F_n^i defined by (2) are σ_Y-separable because $F_1, F_2, \ldots$ are assumed to be σ-separable (we use the metrizability of σ).
- In the proof of (iib) ⇒ (i), we realize that the sets F_n defined by (3) are σ-separable because A is assumed to be σ-separable (we use the metrizability of σ again). □

The desired characterization and its variant follow.

Proposition 2.6. *Let X be a Banach space and τ be a Hausdorff locally convex topology on X, weaker than the norm topology. Then the following assertions are equivalent*:

(i) *X is τ-binormal.*

(ii) *If $F_1 \supset F_2 \supset \cdots$ are closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ-open subsets of X, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$.*

(iii) *If $f : X \to (0,\infty)$ is lower semicontinuous, then there exists $g : X \to (0,\infty)$, continuous and upper τ-semicontinuous, such that $g < f$.*

Proof. We may suppose that $X \neq \{0\}$. Then, by the Hahn–Banach theorem, there is a τ-continuous linear functional $f \neq 0$ on X. We define Y as the kernel of f, σ as the norm topology of X, σ_Y as the norm topology of Y and τ_Y as the restriction of τ on Y. We want to show that we are in the situation of Lemma 2.2. Fix an $x_0 \in X$ with $f(x_0) = 1$. We will identify a couple $(y,r) \in Y \times \mathbb{R}$ with the point $y + rx_0 \in X$ (then $x \in X$ is identified with $(x - f(x)x_0, f(x)) \in Y \times \mathbb{R}$). It is easy to check that the mapping $(y,r) \in Y \times \mathbb{R} \mapsto y + rx_0$ is $(\tau_Y \times |\cdot|)$-τ-continuous and $(\sigma_Y \times |\cdot|)$-$\sigma$-continuous and that the mapping $x \in X \mapsto (x - f(x)x_0, f(x))$ is τ-$(\tau_Y \times |\cdot|)$-continuous and σ-$(\sigma_Y \times |\cdot|)$-continuous. So the products of σ_Y and τ_Y with the standard topology on $\mathbb{R}$ are σ and τ indeed.

It remains to show that $(*)$ is satisfied. Let $U \subset X$ be τ-open. We prove first that every $x \in U$ has a τ-open neighbourhood V such that $\operatorname{dist}(V, X \setminus U) > 0$. There are τ-continuous seminorms $p_1, p_2, \ldots, p_n$ and $\varepsilon > 0$ such that $y \in U$ whenever $p_i(y - x) < \varepsilon$ for all $i \in \{1, 2, \ldots, n\}$. The seminorms are continuous in particular, so we can take $C > 0$ such that $p_i(z) \leqslant C\|z\|$ for all $z \in X$ and $i \in \{1, 2, \ldots, n\}$. We define τ-open

$$V = \bigl\{y \in X: \ p_i(y - x) < \varepsilon/2 \text{ for } i = 1, 2, \ldots, n\bigr\}.$$

We are going to show that $\operatorname{dist}(V, X \setminus U) \geqslant \varepsilon/(2C)$. Let $a \in V$ and $b \in X \setminus U$. By the choice of $p_1, p_2, \ldots, p_n$ and ε, there is $i \in \{1, 2, \ldots, n\}$ such that $p_i(b - x) \geqslant \varepsilon$. We are computing $\|b - a\| \geqslant (1/C)p_i(b - a) \geqslant (1/C)(p_i(b - x) - p_i(a - x)) > (1/C)(\varepsilon - \varepsilon/2) = \varepsilon/(2C)$. So $\operatorname{dist}(V, X \setminus U) \geqslant \varepsilon/(2C)$.

Now, we define U_n as the set of all $x \in U$ for which there is a τ-open neighbourhood $V \ni x$ such that $\operatorname{dist}(V, X \setminus U) \geqslant 1/n$. This is clearly a τ-open set. We know that every $x \in U$ belongs to U_n for a sufficiently large n. At the same time, $\overline{U_n} \subset U$ since $\operatorname{dist}(U_n, X \setminus U) \geqslant 1/n$. This completes the verification of $(*)$. □

Proposition 2.7. *Let X be a Banach space and τ be a Hausdorff locally convex topology on X, weaker than the norm topology. Then the following assertions are equivalent*:

(i) *For every disjoint separable and closed $A \subset X$ and τ-closed $B \subset X$, there are disjoint open $D \subset X$ and τ-open $C \subset X$ with $A \subset C$ and $B \subset D$.*
(ii) *If $F_1 \supset F_2 \supset \cdots$ are separable and closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, τ-open subsets of X, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$.*

Proof. This has the same proof as Proposition 2.6 with the only difference that we use Lemma 2.5 instead of Lemma 2.2. □

3. A stronger property

We are going to introduce a property which is stronger than binormality. The notion of strong binormality plays a key role for us because our only method how to prove that a space is binormal is to prove that it is strongly binormal. Although we proved a characterization of binormality in the previous section, we still do not know too much about binormality itself. For example, we do not know whether $X \times Y$ is necessarily binormal when X and Y are binormal. However, there is no such a problem with strong binormality (Proposition 4.1).

Let X be a Banach space and τ be a locally convex topology on X, weaker than the norm topology. We say that X is *strongly τ-binormal* if there exists a system of τ-open neighbourhoods $U_x^n \ni x$, $x \in X$, $n \in \mathbb{N}$, such that

$$\bigcap_{n=1}^{\infty} \left(U_{x_n}^n + \varepsilon_n B_X\right) \neq \emptyset \quad \Rightarrow \quad \{x_n \colon n \in \mathbb{N}\} \text{ is relatively compact}$$

whenever $\varepsilon_n \searrow 0$. We say that a Banach space X is strongly binormal if it is strongly w-binormal (where w denotes the weak topology of X).

We prove three easy lemmata about strong binormality.

Lemma 3.1. *If X is strongly τ-binormal, then it is τ-binormal.*

We do not know anything about the converse implication. The problem of the existence of a binormal space which is not strongly binormal does not seem to be easy.

Proof. We will use Proposition 2.6. Let $F_1 \supset F_2 \supset \cdots$ be closed in X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$. We need to find τ-open $G_n \supset F_n$ with $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$ (the inclusions $G_1 \supset G_2 \supset \cdots$ can be arranged by taking $\bigcap_{m \leqslant n} G_m$ instead of G_n). Let $U_x^n \ni x$, $x \in X$, $n \in \mathbb{N}$, be a system witnessing the strong τ-binormality of X. Put

$$G_n = \bigcup_{x \in F_n} U_x^n, \quad n \in \mathbb{N}.$$

If now $a \in \bigcap_{n=1}^{\infty} \overline{G_n}$, then we find $a_n \in G_n$ with $\|a - a_n\| \leqslant 1/n$ for every $n \in \mathbb{N}$. For some $x_n \in F_n$, we have $a_n \in U_{x_n}^n$. By the triangle inequality,

$$a \in \bigcap_{n=1}^{\infty} \left(U_{x_n}^n + (1/n)B_X\right).$$

It follows that $\{x_n \colon n \in \mathbb{N}\}$ is relatively compact. So we have a convergent subsequence $x_{n(k)}$. Its limit is an element of $\bigcap_{n=1}^{\infty} F_n$, which is a contradiction. □

Lemma 3.2. *Assume that there exist a dense subset Z of X and a system of τ-open neighbourhoods $U_z^n \ni z$, $z \in Z$, $n \in \mathbb{N}$, such that, for any sequence z_n, $n \in \mathbb{N}$, in Z,*

$$\bigcap_{n=1}^{\infty} \left(U_{z_n}^n + \varepsilon_n B_X\right) \neq \emptyset \quad \Rightarrow \quad \{z_n \colon n \in \mathbb{N}\} \textit{ is relatively compact}$$

whenever $\varepsilon_n \searrow 0$. Then X is strongly τ-binormal.

In other words, in the definition of strong τ-binormality, it is possible to require the neighbourhoods U_x^n for the elements of a dense set only.

Proof. Let $x \in X$ and $n \in \mathbb{N}$. There is some $z(x,n) \in Z$ for which $\|x - z(x,n)\| \leqslant 1/n$. Put

$$V_x^n = U_{z(x,n)}^n + (1/n)B_X.$$

This is a τ-open neighbourhood of x. Now, suppose that $\varepsilon_n \searrow 0$ and that $a \in X$ and a sequence $x_n \in X$, $n \in \mathbb{N}$, satisfy

$$a \in \bigcap_{n=1}^{\infty} \bigl(V^n_{x_n} + \varepsilon_n B_X\bigr).$$

We obtain

$$a \in \bigcap_{n=1}^{\infty} \bigl(U^n_{z(x_n,n)} + (\varepsilon_n + 1/n) B_X\bigr).$$

By the property of the system U^n_z, $z \in Z$, $n \in \mathbb{N}$, the set $\{z(x_n,n)\colon n \in \mathbb{N}\}$ is relatively compact. Since $\|x_n - z(x_n,n)\| \leqslant 1/n$, the set $\{x_n\colon n \in \mathbb{N}\}$ is relatively compact, too. $\square$

Lemma 3.3. *If X is separable and B_X is τ-closed, then X is strongly τ-binormal.*

Proof. Let $B_1, B_2, \ldots$ be closed balls such that their interiors form a basis of the norm topology. Put

$$U^n_x = X \setminus \bigcup_{m \leqslant n,\, x \notin B_m} B_m, \quad x \in X,\ n \in \mathbb{N}.$$

These sets are τ-open, as $B_1, B_2, \ldots$ are τ-closed. Assume

$$a \in \bigcap_{n=1}^{\infty} \bigl(U^n_{x_n} + \varepsilon_n B_X\bigr).$$

We have to show that $\{x_n\colon n \in \mathbb{N}\}$ is relatively compact. We show that even $x_n \to a$. Let $m \in \mathbb{N}$ be such that a lies in the interior of B_m. Then there is n_0 such that $x_n \in B_m$ for $n \geqslant n_0$. Indeed, take n_0 with $n_0 \geqslant m$ and $\varepsilon_{n_0} < \operatorname{dist}(a, X \setminus B_m)$. Let $n \geqslant n_0$. There is $b \in U^n_{x_n}$ such that $\|b - a\| \leqslant \varepsilon_n$. Since $\|b - a\| \leqslant \varepsilon_n \leqslant \varepsilon_{n_0} < \operatorname{dist}(a, X \setminus B_m)$, we have $b \in B_m$. Also, $x_n \in B_m$ (in the other case, $b \in U^n_{x_n} \subset X \setminus B_m$ because $n \geqslant n_0 \geqslant m$). So the choice of U^n_x works. $\square$

4. Binormality via decomposition

Let X be a non-separable Banach space, and let μ be the first ordinal with cardinality $\operatorname{dens}(X)$. We call a transfinite collection $\{P_\alpha\}_{\omega \leqslant \alpha \leqslant \mu}$ of projections in X a *projectional resolution of identity* (*PRI*) if

- $\|P_\alpha\| \leqslant 1$ for $\alpha \in [\omega, \mu]$,
- $\operatorname{dens}(P_\alpha X) \leqslant \operatorname{card}(\alpha)$ for $\alpha \in [\omega, \mu]$,
- $P_\alpha P_\beta = P_\beta P_\alpha = P_{\min\{\alpha,\beta\}}$ for $\alpha, \beta \in [\omega, \mu]$,
- $P_\omega = 0$ and P_μ is the identity on X,
- $\alpha \mapsto P_\alpha x$ is continuous on $[\omega, \mu]$ for every $x \in X$.

If the first condition is weakened to $\sup\{\|P_\alpha\|\colon \omega \leqslant \alpha \leqslant \mu\} < \infty$, we obtain the notion of a *bounded projectional resolution of identity*.

Our main tool for proving that a non-separable Banach space is binormal follows.

Proposition 4.1. *Let X be a Banach space and let $\{P_\alpha\}_{\omega \leqslant \alpha \leqslant \mu}$ be a bounded PRI in X. If $(P_{\alpha+1} - P_\alpha)X$ is strongly binormal for every $\alpha \in [\omega, \mu)$, then X is strongly binormal.*

Proof. We will denote

$$X_\alpha = (P_{\alpha+1} - P_\alpha)X, \quad \alpha \in [\omega, \mu),$$
$$Z = \bigoplus_{\omega \leqslant \alpha < \mu} X_\alpha,$$
$$x(\alpha) = (P_{\alpha+1} - P_\alpha)x, \quad x \in X,\ \alpha \in [\omega, \mu),$$

where the direct sum $\oplus$ is meant in the algebraic sense (so Z is the linear span of $\bigcup_{\omega \leqslant \alpha < \mu} X_\alpha$). We take some $M > 0$ such that $\|P_\alpha\| \leqslant M$ for any $\alpha \in [\omega, \mu]$. By the assumption, there is, for every $\alpha \in [\omega, \mu)$, a system of weak neighbourhoods $U^n_{x,\alpha} \ni x$, $x \in X_\alpha$, $n \in \mathbb{N}$, in X_α, such that

$$\bigcap_{n=1}^{\infty} \bigl(U^n_{x_n,\alpha} + \varepsilon_n B_{X_\alpha}\bigr) \neq \emptyset \quad \Rightarrow \quad \{x_n\colon n \in \mathbb{N}\} \text{ is relatively compact}$$

whenever $\varepsilon_n \searrow 0$.

Since Z is dense in X, considering Lemma 3.2, it is enough to find appropriate neighbourhoods on Z. Put

$$U_x^n = \bigcap_{\alpha\in S(x)} (P_{\alpha+1}-P_\alpha)^{-1}\big(U^n_{x(\alpha),\alpha}\big) \cap \bigcap_{\gamma\leqslant\beta;\ \beta,\gamma\in S(x)} (P_{\beta+1}-P_\gamma)^{-1}\big(X\setminus\big(\big\|(P_{\beta+1}-P_\gamma)x\big\|/2\big)B_X\big)$$
$$\text{for } x=\sum_{\alpha\in S(x)} x(\alpha)\in Z,\quad n\in\mathbb{N},$$

where $S(x)=\{\alpha\colon\ x(\alpha)\neq 0\}$ is finite.

Let us prove that the choice works. Let $\varepsilon_n\searrow 0$, let x_n, $n\in\mathbb{N}$, be a sequence in Z and let $a\in X$ satisfy

$$a\in\bigcap_{n=1}^{\infty}\big(U^n_{x_n}+\varepsilon_n B_X\big).$$

To show that $\{x_n\colon\ n\in\mathbb{N}\}$ is relatively compact, we prove by induction on $\lambda\in[\omega,\mu]$ that $\{P_\lambda x_n\colon\ n\in\mathbb{N}\}$ is relatively compact. This is clear for $\lambda=\omega$ because then $P_\lambda x_n=0$ for $n\in\mathbb{N}$.

Let $\lambda=\alpha+1$ for some $\alpha\in[\omega,\mu)$ and let $\{P_\alpha x_n\colon\ n\in\mathbb{N}\}$ be relatively compact. We have to show that $\{P_\lambda x_n\colon\ n\in\mathbb{N}\}$ is relatively compact. It is sufficient to show that $\{x_n(\alpha)\colon\ n\in\mathbb{N}\}$ is relatively compact because $P_\lambda x_n=P_\alpha x_n+x_n(\alpha)$ for $n\in\mathbb{N}$. Let us verify that, for every $n\in\mathbb{N}$,

$$x_n(\alpha)\neq 0\quad\Rightarrow\quad a(\alpha)\in\big(U^n_{x_n(\alpha),\alpha}+(2M\varepsilon_n)B_{X_\alpha}\big).$$

Assume $x_n(\alpha)\neq 0$, i.e., $\alpha\in S(x_n)$. Choose $b\in U^n_{x_n}$ satisfying $\|b-a\|\leqslant\varepsilon_n$. We have $b\in(P_{\alpha+1}-P_\alpha)^{-1}(U^n_{x_n(\alpha),\alpha})$, and so $b(\alpha)\in U^n_{x_n(\alpha),\alpha}$. Since $\|b(\alpha)-a(\alpha)\|=\|(P_{\alpha+1}-P_\alpha)(b-a)\|\leqslant 2M\|b-a\|\leqslant 2M\varepsilon_n$, we get $a(\alpha)\in U^n_{x_n(\alpha),\alpha}+(2M\varepsilon_n)B_{X_\alpha}$, and the verification is completed. Now, for $n\in\mathbb{N}$, we put

$$y_n=\begin{cases} x_n(\alpha), & x_n(\alpha)\neq 0,\\ a(\alpha), & x_n(\alpha)=0.\end{cases}$$

We obtain

$$a(\alpha)\in\bigcap_{n=1}^{\infty}\big(U^n_{y_n,\alpha}+(2M\varepsilon_n)B_{X_\alpha}\big).$$

Therefore, $\{y_n\colon\ n\in\mathbb{N}\}$ is relatively compact. As $\{x_n(\alpha)\colon\ n\in\mathbb{N}\}\subset\{0\}\cup\{y_n\colon\ n\in\mathbb{N}\}$, the set $\{x_n(\alpha)\colon\ n\in\mathbb{N}\}$ is relatively compact, too. The inductive step $\alpha\to\alpha+1$ is finished.

Let $\lambda\in(\omega,\mu]$ be a limit ordinal number and let $\{P_\alpha x_n\colon\ n\in\mathbb{N}\}$ be relatively compact for every $\alpha\in[\omega,\lambda)$. We have to show that $\{P_\lambda x_n\colon\ n\in\mathbb{N}\}$ is relatively compact. It is sufficient, given an $\varepsilon>0$, to find n_0 and a sequence x'_n such that $\|P_\lambda x_n-x'_n\|<\varepsilon$ for $n\geqslant n_0$ and $\{x'_n\colon\ n\in\mathbb{N}\}$ is relatively compact. We show that the choice $x'_n=P_\alpha x_n$, $n\in\mathbb{N}$, for an $\alpha<\lambda$ so that

$$\|P_\lambda a-P_\beta a\|<\varepsilon/8,\quad \alpha\leqslant\beta\leqslant\lambda,$$

works. Fix such an α. We know that $\{P_\alpha x_n\colon\ n\in\mathbb{N}\}$ is relatively compact. It remains to find n_0 such that $\|P_\lambda x_n-P_\alpha x_n\|<\varepsilon$ for $n\geqslant n_0$. We choose n_0 so that $\varepsilon_{n_0}\leqslant\varepsilon/(8M)$. Let $n\geqslant n_0$ be given. If $S(x_n)\subset[\omega,\alpha)\cup[\lambda,\mu]$, then $P_\alpha x_n=P_\lambda x_n$, and so $\|P_\lambda x_n-P_\alpha x_n\|=0<\varepsilon$. Assume that $S(x_n)\cap[\alpha,\lambda)\neq\emptyset$ and denote by β and by γ the greatest and the least element of $S(x_n)\cap[\alpha,\lambda)$. We have

$$\begin{aligned} P_\lambda x_n-P_\alpha x_n &= \sum_{\nu\in S(x_n),\ \alpha\leqslant\nu<\lambda} x_n(\nu)\\ &= \sum_{\nu\in S(x_n),\ \gamma\leqslant\nu<\beta+1} x_n(\nu)=P_{\beta+1}x_n-P_\gamma x_n.\end{aligned}$$

Since $a\in U^n_{x_n}+\varepsilon_n B_X$, we can choose $b\in U^n_{x_n}$ satisfying $\|b-a\|\leqslant\varepsilon_n$. We have $b\in(P_{\beta+1}-P_\gamma)^{-1}(X\setminus(\|(P_{\beta+1}-P_\gamma)x_n\|/2)B_X)$, i.e., $\|(P_{\beta+1}-P_\gamma)b\|>\|(P_{\beta+1}-P_\gamma)x_n\|/2$. We obtain

$$\begin{aligned} \|P_\lambda x_n-P_\alpha x_n\| &= \big\|(P_{\beta+1}-P_\gamma)x_n\big\|\\ &< 2\big\|(P_{\beta+1}-P_\gamma)b\big\|\\ &\leqslant 2\big\|(P_{\beta+1}-P_\gamma)a\big\|+4M\varepsilon_n\end{aligned}$$

$$\begin{aligned}
&\leqslant 2\|P_\lambda a - P_{\beta+1}a\| + 2\|P_\lambda a - P_\gamma a\| + 4M\varepsilon_n \\
&< 4(\varepsilon/8) + 4M\varepsilon_{n_0} \\
&\leqslant \varepsilon.
\end{aligned}$$

The inductive step for a limit ordinal λ is finished. □

We say that a class $\mathcal{C}$ of Banach spaces is a *$\mathcal{P}$-class* if, for every non-separable $X \in \mathcal{C}$, there exists a PRI $\{P_\alpha\}_{\omega \leqslant \alpha \leqslant \mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for every $\alpha < \mu$, where μ is the first ordinal with cardinality $\operatorname{dens}(X)$.

There are several classes which are known to be $\mathcal{P}$-classes (see, e.g., [6]).

Theorem 4.2. *Let $\mathcal{C}$ be a $\mathcal{P}$-class. Then every space in $\mathcal{C}$ is strongly binormal. In particular, every space in $\mathcal{C}$ is binormal.*

Proof. We prove by induction on the density of X that every $X \in \mathcal{C}$ is strongly binormal. If $\operatorname{dens}(X) \leqslant \aleph_0$, then X is separable, and thus strongly binormal by Lemma 3.3. Let $X \in \mathcal{C}$ satisfy $\operatorname{dens}(X) > \aleph_0$ and let every $Y \in \mathcal{C}$ with $\operatorname{dens}(Y) < \operatorname{dens}(X)$ be strongly binormal. Let μ be the first ordinal with cardinality $\operatorname{dens}(X)$. There is a PRI $\{P_\alpha\}_{\omega \leqslant \alpha \leqslant \mu}$ such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for every $\alpha < \mu$. The block $(P_{\alpha+1} - P_\alpha)X$ is strongly binormal for every $\alpha \in [\omega, \mu)$ because $\operatorname{dens}((P_{\alpha+1} - P_\alpha)X) \leqslant \operatorname{card}(\alpha) < \operatorname{dens}(X)$. Now, X is strongly binormal by Proposition 4.1.

The second part of the statement follows from Lemma 3.1. □

5. Examples

Example 5.1. The space $C([0, \mu])$ is binormal for every ordinal μ.

This can be proved directly from Proposition 4.1. We may assume that μ is an initial ordinal and that $\mu \geqslant \omega_1$ (recall that every separable Banach space is strongly binormal by Lemma 3.3). To define a suitable PRI, we take $P_\omega = 0$ and, for $\alpha \in (\omega, \mu]$, the projection

$$P_\alpha f(\nu) = \begin{cases} f(\nu), & 0 \leqslant \nu < \alpha, \\ f(\alpha), & \alpha \leqslant \nu \leqslant \mu \end{cases}$$

(then every block $(P_{\alpha+1} - P_\alpha)C([0, \mu])$ is strongly binormal – for $\alpha > \omega$, it is one-dimensional, for $\alpha = \omega$, it is isometric to $C([0, \omega + 1])$).

Theorem 5.2. *Every Plichko space is binormal. Every dual to an Asplund space is binormal.*

For the definition of a *Plichko space*, see, e.g., [9]. For the definition of an *Asplund space*, see below.

Proof. We use Theorem 4.2. The class of 1-Plichko spaces is a $\mathcal{P}$-class by [9, Theorem 4.14]. Note that every Plichko space can be renormed to be 1-Plichko [9, Theorem 4.16]. The class of duals to Asplund spaces is a $\mathcal{P}$-class by [2]. □

We say that a norm $\|\cdot\|$ is *locally uniformly rotund (LUR)* if $x_n \to x$ whenever $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$. One may expect that every Banach space with a LUR norm is binormal because the norm and weak topologies coincide on the unit sphere. We are going to disprove this conjecture.

Example 5.3. There is a locally compact space T such that the function space $C_0(T)$ is Asplund and admits a LUR norm but it is not binormal.

The presented example is the set

$$T = \left(\bigcup_{n=1}^{\infty} \mathbb{N}^n \right) \cup \mathbb{N}^{\mathbb{N}}$$

endowed with the coarsest topology in which $\{s \in T\colon\ s \subset t\}$ is clopen for every $t \in T$ (we write $s \subset t$ if s is an initial segment of t).

In fact, our space T is a tree. Function spaces on trees were widely studied in the article [5]. The fact that T is a tree is sufficient for $C_0(T)$ to be Asplund. By [5, Theorem 4.1], $C_0(T)$ has a LUR norm.

We denote by $\chi_{(0,t]}$ the characteristic function of the set $\{s \in T\colon\ s \subset t\}$. To show that $C_0(T)$ is not binormal, we put

$$F_n = \left\{ \chi_{(0,t]}\colon\ n \leqslant \operatorname{length}(t) < \infty \right\}, \quad n \in \mathbb{N}.$$

The sets F_n are closed because the functions $\chi_{(0,t]}$ form a discrete set. It is clear that $F_1 \supset F_2 \supset \cdots$ and that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Considering Proposition 2.6, it is sufficient to prove the following claim. Note that the weak and the pointwise topologies coincide on the unit ball of $C_0(T)$ (this can be easily proved from [3, Theorem 12.28] which implies that the linear span of the Dirac measures is dense in the dual of $C_0(T)$).

Claim 5.4. *If $G_n \subset C_0(T)$, $n \in \mathbb{N}$, are open sets in the pointwise topology such that $F_n \subset G_n$, $n \in \mathbb{N}$, then $B_{C_0(T)} \cap \bigcap_{n=1}^{\infty} G_n \neq \emptyset$.*

Proof. We construct a sequence $s_1, s_2, \ldots$ of natural numbers such that

$$(s_1, s_2, \ldots, s_{n+1}) \subset t \quad \Rightarrow \quad \chi_{(0,t]} \in G_n$$

for every $n \in \mathbb{N}$. Choose $s_1 \in \mathbb{N}$ arbitrarily. Assume that $s_1, s_2, \ldots, s_n$ are constructed. We have $\chi_{(0,(s_1,s_2,\ldots,s_n)]} \in F_n \subset G_n$. There are finite $R \subset T$ and $\varepsilon > 0$ such that

$$\forall r \in R\colon \left| f(r) - \chi_{(0,(s_1,s_2,\ldots,s_n)]}(r) \right| < \varepsilon \quad \Rightarrow \quad f \in G_n.$$

It is sufficient to choose s_{n+1} such that $(s_1, s_2, \ldots, s_{n+1}) \not\subset r$ for any $r \in R$. Indeed, if $(s_1, s_2, \ldots, s_{n+1}) \subset t$, then $\chi_{(0,t]}(r) \neq \chi_{(0,(s_1,s_2,\ldots,s_n)]}(r)$ is possible only for r with $(s_1, s_2, \ldots, s_{n+1}) \subset r$, and thus $\chi_{(0,t]}(r) = \chi_{(0,(s_1,s_2,\ldots,s_n)]}(r)$ for every $r \in R$. Hence $\chi_{(0,t]} \in G_n$.

So the construction is done. Now, the function $\chi_{(0,s]}$, where $s = (s_1, s_2, \ldots)$, belongs to G_n for every $n \in \mathbb{N}$. This proves the claim. □

6. Asplund spaces and w^*-binormality

A Banach space E is said to be an *Asplund space* provided every continuous convex function defined on a non-empty open convex subset D of E is Fréchet differentiable at each point of some dense G_δ subset of D.

A topological space (X, τ) is said to be *fragmented by a metric* ϱ if, for every $\varepsilon > 0$ and every non-empty $Y \subset X$, there is a non-empty relatively τ-open subset of Y of ϱ-diameter less than ε.

Further, a topological space (X, τ) is said to be *scattered* if every non-empty subset $Y \subset X$ has an isolated point in Y. In other words, (X, τ) is scattered if and only if it is fragmented by the discrete metric.

A metric ϱ on a topological space (X, τ) is said to be *lower τ-semicontinuous* if the set $\{(x, y) \in X \times X\colon \varrho(x, y) \leqslant r\}$ is closed in $(X, \tau) \times (X, \tau)$ for each $r \geqslant 0$.

We start with a separable reduction for non-fragmentability. The result may be known but we were not able to find a reference for it.

Proposition 6.1. *Let (X, τ) be a compact Hausdorff space and ϱ be a lower τ-semicontinuous metric on X. If (X, τ) is not fragmented by ϱ, then there are an $\varepsilon > 0$ and a countable set $Y \subset X$ such that*

(1) *$\varrho(x_1, x_2) \geqslant \varepsilon$ whenever $x_1, x_2 \in Y$ and $x_1 \neq x_2$,*
(2) *$Y \cap U$ is infinite whenever $U \subset X$ is τ-open and $Y \cap U$ is non-empty.*

Proof. (Cf. proof of [8, Lemma 4.4].) By the implication (d) ⇒ (c) of [8, Theorem 4.1], there are an $\varepsilon > 0$, a τ-compact set $H \subset X$ and a continuous surjective mapping $p : (H, \tau) \to \{0, 1\}^{\mathbb{N}}$ with the inverse images of distinct points of $\{0, 1\}^{\mathbb{N}}$ separated by ϱ-distance at least ε.

By the Zorn lemma, we can take some minimal (in the sense of the inclusion) τ-compact set $K \subset H$ with $p(K) = \{0, 1\}^{\mathbb{N}}$. Let Σ be a countable dense subset of $\{0, 1\}^{\mathbb{N}}$. For every $\sigma \in \Sigma$, we choose some $x(\sigma) \in K \cap p^{-1}(\sigma)$. Let us verify that the choice

$$Y = \left\{ x(\sigma)\colon \sigma \in \Sigma \right\}$$

works. The property (1) is an immediate consequence of the properties of p. Let us verify the property (2). Take a τ-open $U \subset X$ with $Y \cap U$ non-empty. From the minimality of K, we have $p(K \setminus U) \subsetneqq \{0, 1\}^{\mathbb{N}}$. There are infinitely many pairwise distinct points $\sigma_1, \sigma_2, \ldots \in \Sigma$ which are elements of the open set $\{0, 1\}^{\mathbb{N}} \setminus p(K \setminus U)$. Now, the points $x(\sigma_1), x(\sigma_2), \ldots$ are pairwise distinct and they are elements of U. □

Lemma 6.2. *Let (X, τ) be a compact Hausdorff space and ϱ be a lower τ-semicontinuous metric on X. If (X, τ) is not fragmented by ϱ, then there are $F_1 \supset F_2 \supset \cdots$, ϱ-separable and ϱ-closed subsets of X with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, such that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ whenever $G_1, G_2, \ldots$ are τ-open subsets of X with $F_n \subset G_n$, $n \in \mathbb{N}$.*

Proof. Let ε and Y be as in Proposition 6.1. Denote by $y_1, y_2, \ldots$ the elements of Y (in such a way that every element of Y occurs exactly one time in the sequence $y_1, y_2, \ldots$). We claim that the choice

$$F_n = \{y_n, y_{n+1}, \dots\}, \quad n \in \mathbb{N},$$

works. The sets F_n are ϱ-closed due to the property (1) and they are ϱ-separable because they are countable. Clearly, $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Moreover,

$$Y \subset \overline{F_n}^{\tau}, \quad n \in \mathbb{N}.$$

Indeed, the set $Y \setminus \overline{F_n}^{\tau}$, being a subset of $\{y_1, y_2, \dots, y_{n-1}\}$, is finite, and so it is empty by the property (2).

Now, let $G_1, G_2, \dots$ be τ-open subsets of X with $F_n \subset G_n$, $n \in \mathbb{N}$. The sets F_n, $n \in \mathbb{N}$, are dense in $(\overline{Y}^{\tau}, \tau)$, so the sets $G_n \cap \overline{Y}^{\tau}$, $n \in \mathbb{N}$, are dense as well. Using the Baire theorem, we obtain $\bigcap_{n=1}^{\infty} G_n \cap \overline{Y}^{\tau} \neq \emptyset$. This proves the lemma. □

There is a connection between asplundness and w^*-binormality. We are ready to prove it now.

Theorem 6.3. *For a Banach space E, the following assertions are equivalent*:

(i) *For every disjoint separable and closed $A \subset E^*$ and w^*-closed $B \subset E^*$, there are disjoint open $D \subset E^*$ and w^*-open $C \subset E^*$ with $A \subset C$ and $B \subset D$.*
(ii) *If $F_1 \supset F_2 \supset \cdots$ are separable and closed subsets of E^* with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there are $G_1 \supset G_2 \supset \cdots$, w^*-open subsets of E^*, such that $F_n \subset G_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G_n} = \emptyset$.*
(iii) *E is an Asplund space.*

Proof. (i) ⇔ (ii) This follows from Proposition 2.7.

(ii) ⇒ (iii) Assume that E is not Asplund. It means that (B_{E^*}, w^*) is not fragmented by the norm [1, Theorem I.5.2]. By Lemma 6.2, there are $F_1 \supset F_2 \supset \cdots$, separable and closed subsets of B_{E^*} with $\bigcap_{n=1}^{\infty} F_n = \emptyset$, such that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ whenever $G_1, G_2, \dots$ are relatively w^*-open subsets of B_{E^*} with $F_n \subset G_n$, $n \in \mathbb{N}$. This clearly disproves (ii).

(iii) ⇒ (ii) There is a separable closed linear subspace M of E such that

$$\|f - g\| = \sup\bigl\{\bigl|(f - g)(x)\bigr| \colon x \in M,\ \|x\| \leqslant 1\bigr\}, \quad f, g \in F_1.$$

Indeed, we can take $M = \overline{\operatorname{span}}\{x(f, g, k)\colon\ f, g \in P,\ k \in \mathbb{N}\}$ where P is a countable dense subset of F_1 and $x(f, g, k) \in B_E$ is chosen so that $|(f - g)(x(f, g, k))| > \|f - g\| - 1/k$. Denote by r the restriction map $r : E^* \to M^*$, $r(f) = f|_M$. By the choice of M, we have

$$\|f - g\| = \bigl\|r(f) - r(g)\bigr\|, \quad f, g \in F_1.$$

It follows that $r(F_1), r(F_2), \dots$ are closed in M^* and $\bigcap_{n=1}^{\infty} r(F_n) = \emptyset$. As E is Asplund, M^* is separable by [1, Theorem I.5.7]. So M^* is w^*-binormal (Lemma 3.3 and Lemma 3.1). There are $G'_1 \supset G'_2 \supset \cdots$, w^*-open subsets of M^*, such that $r(F_n) \subset G'_n$, $n \in \mathbb{N}$, and $\bigcap_{n=1}^{\infty} \overline{G'_n} = \emptyset$ (Proposition 2.6). Now, the choice

$$G_n = r^{-1}\bigl(G'_n\bigr), \quad n \in \mathbb{N},$$

works, as $\bigcap_{n=1}^{\infty} \overline{r^{-1}(G'_n)} \subset \bigcap_{n=1}^{\infty} r^{-1}(\overline{G'_n}) = r^{-1}(\bigcap_{n=1}^{\infty} \overline{G'_n}) = \emptyset$. □

Corollary 6.4. *If the dual E^* of a Banach space E is w^*-binormal, then E is Asplund.*

Proof. The condition (i) in Theorem 6.3 is evidently weaker than w^*-binormality of E^*. □

One may ask whether the converse implication holds. Before proving that the answer is negative, we mention a positive result suggested by O. Kalenda.

Remark 6.5. It can be shown that E^* is w^*-binormal whenever E is an Asplund and weakly countably determined Banach space. To prove this, we can use the same method by which we proved Theorem 4.2 with the difference that we use the fact that the class of the duals to Asplund WCD spaces forms a $\mathcal{P}$-class with the special property that the projections are continuous with respect to the w^*-topology [1, Theorem VI.4.3].

Example 6.6. The space $C([0, \omega_1])$ is an Asplund space but its dual is not w^*-binormal.

The space $C([0, \omega_1])$ is Asplund because $[0, \omega_1]$ is scattered [3, Theorem 12.29]. To see that $C([0, \omega_1])^*$ is not w^*-binormal, it is sufficient to prove the following lemma. Indeed, the sets $F_1, F_2, \dots$ from the lemma form a counterexample to (ii) in Proposition 2.6 if we identify every point of $[0, \omega_1]$ with the appropriate Dirac measure (note that $[0, \omega_1]$ embeds topologically to $(C([0, \omega_1])^*, w^*)$ by this identification).

Lemma 6.7. *There are* $F_1 \supset F_2 \supset \cdots$*, subsets of* $[0, \omega_1]$ *with* $\bigcap_{n=1}^\infty F_n = \emptyset$*, such that* $\bigcap_{n=1}^\infty G_n \neq \emptyset$ *whenever* $G_1, G_2, \ldots$ *are open subsets of* $[0, \omega_1]$ *with* $F_n \subset G_n$*,* $n \in \mathbb{N}$*.*

Proof. Let us recall a definition first. We say that a set $S \subset [0, \omega_1)$ is *stationary* if $S \cap A \neq \emptyset$ for any $A \subset [0, \omega_1)$, unbounded and closed in $[0, \omega_1)$.

By the Fodor theorem [4], there are pairwise disjoint stationary sets $S_1, S_2, \ldots \subset [0, \omega_1)$. We define

$$F_n = \bigcup_{i=n}^\infty S_i, \quad n \in \mathbb{N}.$$

Suppose that G_n, $n \in \mathbb{N}$, are open sets in $[0, \omega_1]$ for which $F_n \subset G_n$, $n \in \mathbb{N}$. We show that $\bigcap_{n=1}^\infty G_n \neq \emptyset$. Assume the opposite, i.e. that $\bigcap_{n=1}^\infty G_n = \emptyset$. If we denote $A_n = [0, \omega_1) \setminus G_n$, then we obtain $\bigcup_{n=1}^\infty A_n = [0, \omega_1)$. We have that A_n is closed and unbounded for some $n \in \mathbb{N}$. As S_n is stationary, we have $\emptyset \neq S_n \cap A_n \subset F_n \cap A_n \subset G_n \cap A_n = \emptyset$, which is a contradiction. □

Theorem 6.8. *For a compact Hausdorff space* X*, the following assertions are equivalent*:

(i) *If* $F_1 \supset F_2 \supset \cdots$ *are countable subsets of* X *with* $\bigcap_{n=1}^\infty F_n = \emptyset$*, then there are* $G_1 \supset G_2 \supset \cdots$*, open subsets of* X*, such that* $F_n \subset G_n$*,* $n \in \mathbb{N}$*, and* $\bigcap_{n=1}^\infty G_n = \emptyset$*.*

(ii) X *is scattered.*

Proof. (i) $\Rightarrow$ (ii) Assume that X is not scattered. It means that X is not fragmented by the discrete metric. Now, Lemma 6.2 disproves (i).

(ii) $\Rightarrow$ (i) Assume that X is scattered. It means that $C(X)$ is an Asplund space [3, Theorem 12.29]. If we identify every point of X with the appropriate Dirac measure, (i) follows straightforwardly from Theorem 6.3 (note that X embeds topologically to $(C(X)^*, w^*)$ by this identification). □

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