JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **203,** 270–277 (1996)
ARTICLE NO. 0379

# Box Splines with Rational Directions and Linear Diophantine Equations

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Received January 24, 1995

A necessary and sufficient condition for the linear independence of integer translates of Box splines with rational directions is presented in terms of intrinsic properties of the defining matrices. We also give a necessary and sufficient condition for the space of linear dependence relations to be finite dimensional. A method to compute the approximation order of these Box spline spaces is obtained. All these conditions can be tested by finite steps of computations based on elementary properties of the matrices. The method of proofs is from linear diophantine equations. © 1996 Academic Press, Inc.

## 1. INTRODUCTION AND MAIN RESULTS

Let $\Xi$ be an $s \times n$ matrix of nonzero columns called directions. The Box spline $M_\Xi$ associated with $\Xi$ is the compactly supported distribution defined by

$$\langle M_\Xi, \psi \rangle := \int_{[0,1)^n} \psi(\Xi x)\, dx, \qquad \psi \in C^\infty(\mathbf{R}^s). \tag{1.1}$$

Its Fourier–Laplace transform is given by

$$\hat{M}_\Xi(\omega) = \prod_{\xi \in \Xi} \frac{1 - e^{-i\xi^T \omega}}{i\xi^T \omega}, \qquad \omega \in \mathbf{C}^s. \tag{1.2}$$

*The author is supported by the Alexander von Humboldt Foundation and NSERC Canada under Grants OGP 121336 and A7687. E-mail address: dxzhou@approx.math.ualberta.ca.



0022-247X/96 $18.00

The first purpose of this paper is to characterize the linear independence of integer translates of Box splines with rational directions. Define the set of linear dependence relations among the integer translates of $M_\Xi$ as

$$K_\Xi := \Big\{ f \in \mathbf{C}^{\mathbf{Z}^s} : \sum_{\alpha \in \mathbf{Z}^s} f(\alpha) M_\Xi(\cdot - \alpha) = 0 \Big\}. \tag{1.3}$$

Then we say that the integer translates of $M_\Xi$ are linearly independent if $K_\Xi$ contains only the zero sequence. In case $\Xi \subset \mathbf{Z}^{s\times n}$, necessary and sufficient conditions for $K_\Xi$ to be null or finite dimensional were given in [1, 3, 5, 8]; see also the monograph of de Boor, Höllig, and Riemenschneider [2].

Initiated by his joint investigation with Jetter on cardinal interpolation on submodules of $\mathbf{Z}^s$ in [6], Riemenschneider posed in his survey [10] the problem of characterizing the linear independence of integer translates of a Box spline with rational directions in terms of elementary properties of the defining matrix $\Xi \in \mathbf{Q}^{s\times n}$. Since then some partial results concerning this problem have been obtained in [9, 12]. The main results in the first part of this paper, Theorems 1 and 2, give an answer to this problem.

We first note that any $s \times n$ rational matrix can be written as $(1/P)\Xi$ with $P \in \mathbf{N}$ and $\Xi \in \mathbf{Z}^{s\times n}$. So in what follows we always take such a form for the defining matrix. For an $l \times m$ integer matrix $A$ we also think of it as the multiset of its column vectors and denote $\#A, d_A$ as its cardinality and the greatest common divisor (g.c.d.) of all $(\min\{l,m\}) \times (\min\{l,m\})$ minors of $A$, respectively. For $P,\ s \in \mathbf{N}$, we denote

$$\mathscr{E}_{P,s} := \big\{ \gamma = (\gamma_1, \ldots, \gamma_s) : 0 \le \gamma_j \le P-1 \text{ for } 1 \le j \le s \big\} \tag{1.4}$$

and for $0 \le k = \sum_{j=1}^{s} \gamma_j P^{j-1} \le P^s - 1$ with $0 \le \gamma_j \le P-1$ define

$$\gamma(k) = (\gamma_1, \ldots, \gamma_s) \in \mathscr{E}_{P,s}.$$

The following concept of correlation sets plays an important role in the statements and proofs of our main results.

Definition. For $\Xi \in \mathbf{Z}^{s\times n}$, we define the following sets:

$$\widetilde{\mathbf{1\text{-}1}}(\Xi) := \{ X \subset \Xi : \operatorname{rank} X = \#X > 0, d_X > 1 \}; \tag{1.5}$$

$$\mathscr{F}(\Xi) := \Big\{ (X, b) : X \in \widetilde{\mathbf{1\text{-}1}}(\Xi), b \in \mathbf{Z}^{\#X}, d_{X^T} \ne d_{[X^T, b]} \Big\}; \tag{1.6}$$

$$\tilde{\mathscr{F}}(\Xi) := \big\{ (X, b) : (X, b) \in \mathscr{F}(\Xi), b \in X^T(-1, 0]^s \big\}; \tag{1.7}$$

$$\mathrm{Cor}(\Xi, P) := \left\{\left\{(X_j, b_j)\right\}_{j=0}^{P^s-1} : (X_j, b_j) \in \mathscr{F}(\Xi) \text{ for } 0 \le j \le P^s - 1\right\}; \tag{1.8}$$

$$\widetilde{\mathrm{Cor}}(\Xi, P) := \left\{\left\{(X_j, b_j)\right\}_{j=0}^{P^s-1} : (X_j, b_j) \in \tilde{\mathscr{F}}(\Xi) \text{ for } 0 \le j \le P^s - 1\right\}. \tag{1.9}$$

We call $\mathrm{Cor}(\Xi, P)$ and $\widetilde{\mathrm{Cor}}(\Xi, P)$ the correlation set and the fundamental correlation set of $\{\Xi, P\}$, respectively.

For $Y = \{(X_j, b_j)\}_{j=0}^{P^s-1} \in \mathrm{Cor}(\Xi, P)$, called a correlation of $\{\Xi, P\}$, we define its correlation matrix $X_Y$ and correlation vector $b_Y$ as

$$X_Y = (X_0, X_1, \ldots, X_{P^s-1})^T, \tag{1.10}$$

$$b_Y = -P\left(b_0^T, \ldots, b_{P^s-1}^T\right)^T - \left(\gamma(0)^T X_0, \ldots, \gamma(P^s-1)^T X_{P^s-1}\right)^T. \tag{1.11}$$

With these notations and definitions we can now state our main results on linear independence and linear dependence relations as follows.

THEOREM 1. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *and* $P \in \mathbf{N}$. *Then the integer translates of* $M_{(1/P)\Xi}$ *are linearly independent if and only if for any* $Y \in \widetilde{\mathrm{Cor}}(\Xi, P)$,

$$\mathrm{rank}(X_Y) < \mathrm{rank}[X_Y, b_Y]. \tag{1.12}$$

THEOREM 2. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *and* $P \in \mathbf{N}$. *Then* $\dim K_{(1/P)\Xi} < \infty$ *if and only if for any* $Y \in \widetilde{\mathrm{Cor}}(\Xi, P)$, $\mathrm{rank}(X_Y) = \mathrm{rank}[X_Y, b_Y]$ *implies* $\mathrm{rank}(X_Y) = s$.

Let us mention that $\tilde{\mathscr{F}}(\Xi)$, hence $\widetilde{\mathrm{Cor}}(\Xi, P)$, is a finite set. Therefore the conditions in Theorems 1 and 2 can be tested by finite steps of computations based on elementary properties of the defining matrix.

The second purpose of this paper is to give a method to compute the approximation order of spaces of Box splines with rational directions. An interesting characterization was presented by Ron and Sivakumar in [13], which states that in our case of rational directions, the approximation order of the Box spline space $S(M_{(1/P)\Xi})$ is the number

$$\min\left\{\#K_{\alpha,P}(\Xi) : \alpha \in \mathbf{Z}^s \setminus \{0\}\right\}, \tag{1.13}$$

where

$$K_{\alpha,P}(\Xi) := \left\{\xi \in \Xi : \xi^T\alpha \in P\mathbf{Z} \setminus \{0\}\right\}. \tag{1.14}$$

Note that (1.13) concerns taking the minimum over an infinite set. We give a method here to calculate this approximation order by finite steps of elementary computations based on the defining matrix.

To this end, we use the following notations from [2, 14]:

$$\mathbf{1\text{-}1}(\Xi) := \{X \subset \Xi : \operatorname{rank} X = \#X > 0\}; \tag{1.15}$$

$$\Xi_\gamma^c := \{\xi \in \Xi : \xi^T\gamma \in P\mathbf{Z}\}, \qquad \gamma \in \mathscr{E}_{P,s}. \tag{1.16}$$

Then our main result on approximation order can be stated as follows.

THEOREM 3. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *with* $\operatorname{rank}\Xi = s$ *and* $P \in \mathbf{N}$. *Then the approximation order of the Box spline space* $S(M_{(1/P)\Xi})$ *is given by*

$$\min\Big\{\min\{\#(\Xi \setminus Y) : Y \subset \Xi, \operatorname{rank} Y < s\}, \min_{\gamma \in \mathscr{E}_{P,s}\setminus\{0\}} \Big\{\#\big(\Xi_\gamma^c \setminus \operatorname{span} X\big) :$$
$$X \in \mathbf{1\text{-}1}(\Xi), X \subset \Xi_\gamma^c, d_{X^T} = d_{[X^T,(1/P)X^T\gamma]}\Big\}\Big\}. \tag{1.17}$$

The proofs of our main results depend mainly upon linear diophantine equations which were first introduced to the investigation of multivariate splines by Dahmen and Micchelli [4]. In [7] Jia used a solvability condition for linear diophantine equations to solve the problem of linear independence of discrete Box splines. It is the nice paper of Jia [7] that leads the author to solve the problems in this paper. For our purpose we need the following solvability condition for linear diophantine equations which can be found, e.g., in [7, Theorem 3.2].

LEMMA. *Let* $A$ *be an* $l \times m$ *integer matrix with full row rank and* $b \in \mathbf{Z}^l$. *Then the following system of linear diophantine equations*

$$Ay = b \tag{1.18}$$

*has an integer solution for* $y$ *if and only if* $d_A = d_{[A,b]}$.

## 2. CHARACTERIZATIONS AND PROOFS FOR LINEAR INDEPENDENCE

Let $\Xi \in \mathbf{Z}^{s\times n}$ and $P \in \mathbf{N}$. The following sets are essential for the proofs of the main results on linear independence,

$$N_{\Xi,P} := \Big\{\omega \in \mathbf{C}^s : \widetilde{M_{(1/P)\Xi}}(\omega + 2\pi\alpha) = 0 \text{ for any } \alpha \in \mathbf{Z}^s\Big\}; \tag{2.1}$$

$$\tilde{N}_{\Xi,P} := \Big\{\omega \in \mathbf{C}^s : \omega \in N_{\Xi,P}, \operatorname{Re}\omega \in [0, 2\pi)^s\Big\}, \tag{2.2}$$

here for $\omega = (\omega_1, \ldots, \omega_s)^T \in \mathbf{C}^s$, we denote $\operatorname{Re}\omega = (\operatorname{Re}\omega_1, \ldots, \operatorname{Re}\omega_s)^T$.

When $P = 1$, the set $N_{\Xi,1}$ can be characterized by $\mathscr{F}(\Xi)$ and the Lemma as follows.

THEOREM 4. *Let* $\Xi \in \mathbf{Z}^{s\times n}$. *Then*

$$N_{\Xi,1} = \bigcup_{(X,b)\in\mathscr{F}(\Xi)} \{2\pi\omega \in \mathbf{C}^s : X^T\omega = -b\}. \tag{2.3}$$

*Proof.* From the Fourier–Laplace transform (1.2) we see that for $\omega \in \mathbf{C}^s$, $2\pi\omega \in N_{\Xi,1}$ if and only if for any $\alpha \in \mathbf{Z}^s$ there is some $\xi_\alpha \in \Xi$ such that $\xi_\alpha^T(\omega + \alpha) \in \mathbf{Z} \setminus \{0\}$, i.e., $\xi_\alpha^T\omega \in \mathbf{Z}$ and $\xi_\alpha^T(\omega + \alpha) \neq 0$. If $2\pi\omega \in N_{\Xi,1}$, we choose a greatest linearly independent subset $X_\omega$ of the set $\{\xi_\alpha : \alpha \in \mathbf{Z}^s\} \subset \{\xi \in \Xi : \xi^T\omega \in \mathbf{Z}\}$, then for any $\alpha \in \mathbf{Z}^s$ there exists some $\xi \in X_\omega$ such that $\xi^T(\omega + \alpha) \neq 0$, i.e., the following system of linear diophantine equations

$$X_\omega^T y = -X_\omega^T\omega$$

has no integer solutions for $y$. Let $X = X_\omega$ and $b = -X_\omega^T\omega \in \mathbf{Z}^{\#X_\omega}$, by the Lemma we have $d_{X^T} > d_{[X^T,b]} \geq 1$, from which it follows that $X \in \widetilde{\mathbf{1\text{-}1}}(\Xi)$. Hence $(X, b) \in \mathscr{F}(\Xi)$ and $2\pi\omega \in \{2\pi\omega \in \mathbf{C}^s : X^T\omega = -b\}$.

On the other hand, if $\omega \in \mathbf{C}^s$ and there exists $(X, b) \in \mathscr{F}(\Xi)$ such that $X^T\omega = -b$, then by the Lemma for any $\alpha \in \mathbf{Z}^s$, $X^T\alpha \neq b$, hence $X^T(\omega + \alpha) \neq 0$, i.e., for any $\alpha \in \mathbf{Z}^s$ we have some $\xi \in X \subset \Xi$ such that $\xi^T(\omega + \alpha) \in \mathbf{Z} \setminus \{0\}$ which implies $\widehat{M_\Xi}(2\pi\omega + 2\pi\alpha) = 0$. Therefore, $2\pi\omega \in N_{\Xi,1}$. The proof of Theorem 4 is complete.

From Theorem 4, the following characterization for $\tilde{N}_{\Xi,1}$ is easy.

THEOREM 5. *Let* $\Xi \in \mathbf{Z}^{s\times n}$. *Then*

$$\begin{aligned} \tilde{N}_{\Xi,1} &= \bigcup_{(X,b)\in\tilde{\mathscr{F}}(\Xi)} \left\{2\pi\omega \in \mathbf{C}^s : X^T\omega = -b, \operatorname{Re}\omega \in [0,1)^s\right\} \\ &\subset \bigcup_{(X,b)\in\tilde{\mathscr{F}}(\Xi)} \{2\pi\omega \in \mathbf{C}^s : X^T\omega = -b\}. \end{aligned} \tag{2.4}$$

When $P \in \mathbf{N}$ is general, we know from (1.2) that for $\omega \in \mathbf{C}^s$, $2\pi\omega \in N_{\Xi,P}$ if and only if for any $\alpha = \gamma + P\beta \in \mathbf{Z}^s = \mathscr{E}_{P,s} + P\mathbf{Z}^s$, i.e., any $\gamma \in \mathscr{E}_{P,s}$ and $\beta \in \mathbf{Z}^s$, there exists some $\xi \in \Xi$ such that $(1/P)\xi^T(\omega + \alpha) = \xi^T((\omega + \gamma)/P + \beta) \in \mathbf{Z} \setminus \{0\}$, which is equivalent to the statement that for any $\gamma \in \mathscr{E}_{P,s}$, $2\pi((\omega + \gamma)/P) \in N_{\Xi,1}$. Thus we conclude the following characterization for $N_{\Xi,P}$.

THEOREM 6. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *and* $P \in \mathbf{N}$. *Then*

$$N_{\Xi,P} = \left\{2\pi\omega \in \mathbf{C}^s : 2\pi\frac{\omega+\gamma}{P} \in N_{\Xi,1} \text{ for any } \gamma \in \mathscr{E}_{P,s}\right\}. \tag{2.5}$$

Combining Theorems 4 and 6, for $\omega \in \mathbf{C}^s$, $2\pi\omega \in N_{\Xi,P}$ if and only if there exists a correlation $Y = \{(X_j, b_j)\}_{j=0}^{P^s-1} \in \mathrm{Cor}(\Xi, P)$ such that $X_j^T((\omega+\gamma(j))/P) = -b_j$, i.e., $X_Y\omega = b_Y$. Therefore, we have.

THEOREM 7. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *and* $P \in \mathbf{N}$. *Then*

$$N_{\Xi,P} = \bigcup_{Y\in \mathrm{Cor}(\Xi,P)} \{2\pi\omega \in \mathbf{C}^s : X_Y\omega = b_Y\}. \tag{2.6}$$

We notice that for $\omega \in [0,1)^s$ and $\gamma \in \mathscr{E}_{P,s}$,$(\omega+\gamma)/P \in [0,1)^s$. Then in the same way, from Theorems 5, 6, and 7, the characterization of $N_{\Xi,P}$ follows.

THEOREM 8. *Let* $\Xi \in \mathbf{Z}^{s\times n}$ *and* $P \in \mathbf{N}$. *Then*

$$\tilde{N}_{\Xi,P} \subset \bigcup_{Y\in \widetilde{\mathrm{Cor}}(\Xi,P)} \{2\pi\omega \in \mathbf{C}^s : X_Y\omega = b_Y\} \subset N_{\Xi,P}. \tag{2.7}$$

After presenting the above characterization and observing that $N_{\Xi,P} = \tilde{N}_{\Xi,P} + 2\pi\mathbf{Z}^s$, the proof of Theorem 1 follows from [11], while the proof of Theorem 2 is obtained directly from [3].

It is worth mentioning that unlike the case with integer directions, linear independence of integer translates of Box splines with rational directions is not equivalent to local linear independence as shown by the following simple example: $s = 1$, $\Xi = \{3\}$, $P = 2$. Then $\{M_{(1/P)\Xi}(\cdot - \alpha) : \alpha \in \mathbf{Z}\}$ are linearly independent, while on $(0, 1/2)$, $M_{(1/P)\Xi}(\cdot) - M_{(1/P)\Xi}(\cdot + 1) = 0$.

## 3. APPROXIMATION ORDER OF BOX SPLINE SPACES

In this section we prove the main result on approximation order.

*Proof of Theorem* 3. Let $\gamma \in \mathscr{E}_{P,s}$ and $\beta \in \mathbf{Z}^s$, then

$$\begin{aligned} K_{\gamma+P\beta,P}(\Xi) &= \left\{\xi \in \Xi_\gamma^c : \xi^T(\gamma + P\beta) \neq 0\right\} \\ &= \Xi_\gamma^c \setminus \left\{\xi \in \Xi_\gamma^c : \xi^T\beta = -\frac{1}{P}\xi^T\gamma\right\}. \end{aligned} \tag{3.1}$$

If $X \in \mathbf{1\text{-}1}(\Xi)$, $X \subset \Xi_\gamma^c$, and, $d_{X^T} = d_{[X^T,(1/P)X^T\gamma]}$ when $\gamma \neq 0$; rank $X < s$ when $\gamma = 0$. Then, by the Lemma, there exists some $\beta \in \mathbf{Z}^s$ when $\gamma \neq 0$; $\beta \in \mathbf{Z}^s \setminus \{0\}$ when $\gamma = 0$, such that

$$X^T\beta = -\frac{1}{P}X^T\gamma,$$

which implies $K_{\gamma+P\beta,P}(\Xi) \subset \Xi_\gamma^c \setminus \operatorname{span} X$. Hence for $\gamma \in \mathscr{E}_{P,s} \setminus \{0\}$,

$$\min\Big\{\#K_{\gamma+P\beta,P}(\Xi) : \beta \in \mathbf{Z}^s\Big\} \leq \min\Big\{\#\big(\Xi_\gamma^c \setminus \operatorname{span} X\big) : X \in \mathbf{1\text{-}1}(\Xi),$$
$$X \subset \Xi_\gamma^c, d_{X^T} = d_{[X^T,(1/P)X^T\gamma]}\Big\},$$

while for $\gamma = 0$,

$$\min\Big\{\#K_{P\beta,P}(\Xi) : \beta \in \mathbf{Z}^s \setminus \{0\}\Big\} \leq \min\{\#(\Xi \setminus Y) : Y \subset \Xi, \operatorname{rank} Y < s\}.$$

Thus, $\min\{\#K_{\alpha,P}(\Xi) : \alpha \in \mathbf{Z}^s \setminus \{0\}\}$ is not more than the number given by (1.17).

On the other hand, for $\gamma \in \mathscr{E}_{P,s}$ and $\beta \in \mathbf{Z}^s$ with $\beta \neq 0$ when $\gamma = 0$, we take a greatest linearly independent subset $X$ of $\{\xi \in \Xi_\gamma^c : \xi^T(\beta + \gamma/P) = 0\}$. Evidently, $X \in \mathbf{1\text{-}1}(\Xi)$, $X \subset \Xi_\gamma^c$.

If $\gamma = 0$, then rank $X < s$ since $\beta \neq 0$ while $X^T\beta = 0$. Hence

$$\Xi \setminus \operatorname{span} X \subset K_{P\beta,P}(\Xi)$$

and

$$\min\{\#(\Xi \setminus Y) : Y \subset \Xi, \operatorname{rank} Y < s\} \leq \min\Big\{\#K_{P\beta,P}(\Xi) : \beta \in \mathbf{Z}^s \setminus \{0\}\Big\}.$$

If $\gamma \neq 0$, then $\beta$ is an integer solution to the following system of linear diophantine equations

$$X^Ty = -\frac{1}{P}X^T\gamma,$$

which implies by the Lemma $d_{X^T} = d_{[X^T,(1/P)X^T_\gamma]}$. Moreover,

$$\Xi_\gamma^c \setminus \operatorname{span} X \subset K_{\gamma+P\beta,P}(\Xi).$$

Hence

$$\min\Big\{\#\big(\Xi_\gamma^c \setminus \operatorname{span} X\big) : X \in \mathbf{1\text{-}1}(\Xi), X \subset \Xi_\gamma^c, d_{X^T}$$
$$= d_{[X^T,(1/P)X^T\gamma]}\Big\} \leq \min\Big\{\#K_{\gamma+P\beta,P}(\Xi) : \beta \in \mathbf{Z}^s\Big\}.$$

Combining the above two cases, we know that the number given by (1.17) is not more than $\min\{\#K_{\alpha,P}(\Xi) : \alpha \in \mathbf{Z}^s \setminus \{0\}\}$.

The proof of Theorem 3 is complete.

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