JOURNAL OF APPROXIMATION THEORY 35, 225–242 (1982)

Copositive Rational Approximation*

N. PH. SEIF

Department of Mathematics and Physics,
Faculty of Engineering, Cairo University, Giza, Egypt

AND

G. D. TAYLOR

Department of Mathematics, Colorado State University,
Fort Collins, Colorado 80523, U.S.A.

Communicated by Richard S. Varga

Received June 23, 1981; received September 29, 1981

1. INTRODUCTION

The purpose of this paper is to develop a theory for best uniform copositive rational approximation of continuous functions. In Section 2 the basic definitions and notations needed for the problem are presented. Existence and characterization of best copositive rational approximants on a closed interval are discussed in Section 3 and uniqueness and strong uniqueness are developed in Section 4. The continuity of the best copositive rational approximation operator is discussed in Section 5 and finally, in Section 6, the interval $[a, b]$ is replaced by a finite subset of it and some discretization results are given. This paper generalizes the work of [3].

2. BASIC DEFINITIONS AND NOTATIONS

Let m and n be fixed positive integers, let Π_m denote the class of all real algebraic polynomials of degree $\leqslant m$, and fix $f \in C[a, b]$. Define

$$R_n^m[a, b] = \{r = p/q : p \in \Pi_m, q \in \Pi_n, q(x) > 0, \forall x \in [a, b]\}$$

* Supported in part by Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Contract F-49620-79-C-0124 and National Science Foundation under Grant MCS-80-17056.

0021-9045/82/070225–18$02.00/0

and

$$R_f[a, b] = \{r \in R_n^m[a, b]: r(x)f(x) \geqslant 0, \forall x \in [a, b]\},$$

the set of copositive rationals from $R_n^m[a, b]$ with respect to f.

If $r^* \in R_f[a, b]$ has the property that

$$\|f - r^*\| = \inf_{r \in R_f[a,b]} \|f - r\|,$$

where

$$\|h\| = \sup\{|h(x)|: x \in [a, b]\},$$

then r^* is a best copositive approximation to f from $R_f[a, b]$. For $\hat{r} \in R_f[a, b]$ define

$$S_{\hat{r}} = \{p + \hat{r}q: p \in \Pi_m, q \in \Pi_n\}.$$

We note that $S_{\hat{r}}$ is a Haar subspace in $C[a, b]$ of dimension $N = 1 + \max\{m + \partial\hat{q}, n + \partial\hat{p}\}$, where $\hat{r} = \hat{p}/\hat{q}$ and $\partial\hat{p}, \partial\hat{q}$ denote the degree of $\hat{p}$ and $\hat{q}$, respectively [1, p. 162]. Next, define

$$L^0 = \{x \in [a, b]: f(x) < 0\}, \qquad L = \overline{L^0},$$
$$U^0 = \{x \in [a, b]: f(x) > 0\}, \qquad U = \overline{U^0}, S = L \cap U,$$

where the overbar denotes the closure operator in the standard topology in the reals.

If S contains more than m points then $R_f[a, b]$ consists of just the zero function. Thus assume that S contains $k \leqslant m$ points. We say that f changes sign at $t \in (a, b)$ if and only if $t \in S$. On the other hand, f changes sign on the interval $[c, d] \subset (a, b)$ with $c < d$ if and only if $t \in [c, d]$ implies that $f(t) = 0$ with $c \in U$ and $d \in L$ (or $c \in L$ and $d \in U$). If f does not change sign on any interval and S contains less than $m + 1$ points, then f is admissible. *In what follows we shall assume that f is admissible.*

For $f \in C[a, b] \sim R_n^m[a, b]$ ($\sim$ denotes set subtraction) and for fixed $r \in R_f[a, b]$, $x \in [a, b]$ is said to be a positive extreme point for $f - r$ provided $f(x) - r(x) = \|f - r\|$ or $x \in U \sim S$ and $r(x) = 0$. Likewise, $x \in [a, b]$ is said to be a negative extreme point for $f - r$ provided $f(x) - r(x) = -\|f - r\|$ or $x \in L \sim S$ and $r(x) = 0$. Let X_r denote the set of all positive and negative extreme points for $f - r$. Note that X_r is a compact subset of $[a, b]$. Now define σ on X_r by

$$\sigma(x) = 1 \qquad \text{if } x \text{ is a positive extreme point,}$$
$$\sigma(x) = -1 \qquad \text{if } x \text{ is a negative extreme point.}$$

Also, define $\mathrm{sg}(f(x))$ for f at each $x \in [a, b]$ as follows:

$$\begin{aligned}
&\mathrm{sg}(f(x)) = 0 && \text{if } x \in S, \\
&\mathrm{sg}(f(x)) = \mathrm{sgn}(f(x)) && \text{if } f(x) \neq 0, \\
&\mathrm{sg}(f(x)) = 1 && \text{if } f(x) = 0 \text{ and } x \notin S \text{ and } \exists \rho > 0 \ni \\
& && (x-\rho, x+\rho) \cap L = \phi \text{ and } (x-\rho, x+\rho) \cap U \neq \phi, \\
&\mathrm{sg}(f(x)) = -1 && \text{if } f(x) = 0 \text{ and } x \notin S \text{ and } \exists \rho > 0 \ni \\
& && (x-\rho, x+\rho) \cap L \neq \phi \text{ and } (x-\rho, x+\rho) \cap U = \phi.
\end{aligned}$$

3. Existence and Characterization

For copositive rational approximation the following existence theorem holds.

THEOREM 3.1. *Given $f \in C[a, b]$, then there exists $r^* \in R_f[a, b]$ such that*

$$\|f - r^*\| = \inf_{r \in R_f[a,b]} \|f - r\|.$$

We do not present the proof of this theorem as it is the same as that for the usual unconstrained rational approximation [1] with the additional observation that the copositive property is inherited by the limit rational function.

Next, we shall show that best copositive approximations can be characterized by alternation and a Kolmogorov criterion. Unlike the classical theory, only partial results concerning a zero in the convex hull characterization are known. We start by defining the concept of an alternant and present two lemmas which are used to prove the alternation theorem.

Let $x_i, y_i \in X_r$ be such that $x_i < y_i$, $(x_i, y_i) \cap X_r = \phi$, $(x_i, y_i) \cap S = \{z_{i+1}, ..., z_{i+\nu_i}\}$, $\nu_i \geqslant 0$ for $i = 1, ..., \mu$, and $y_i \leqslant x_{i+1}$ for $i = 1, ..., \mu - 1$. We shall say that $f - r$ alternates once between x_i and y_i if $\sigma(x_i) = (-1)^{\nu_i + 1} \sigma(y_i)$. Whereas $f - r$ alternates twice between x_i and y_i, if $\mathrm{sg}(f(x_i))\, \sigma(x_i) = -1$, $\sigma(x_i) = (-1)^{\nu_i} \sigma(y_i)$ and there exists at least one $z_j \in (x_i, y_i) \cap S$ at which $r'(z_j) = 0$. In addition, $f - r$ is said to alternate once in each of the following cases:

(i) On (a, y) if $y \in X_r$, $[a, y) \cap X_r = \phi$, $(a, y) \cap S = \{z_1, ..., z_\nu\}$, $\nu \geqslant 1$, $\mathrm{sg}(f(y))\, \sigma(y) = -1$ and r has at least $\nu + 1$ zeros in $[a, y] \cap (L \cup U)$ counting multiplicities up to order 2. When this occurs, we write $x_1 = a$, $y_1 = y$, abusing our notation that $x_i \in X_r$.

(ii) On (x, b) if $x \in X_r$, $(x, b] \cap X_r = \phi$, $(x, b) \cap S = \{z_{k-\nu+1}, \ldots, z_k\}$, $\nu \geqslant 1$, $\operatorname{sg}(f(x))\, \sigma(x) = -1$ and r has at least $\nu + 1$ zeros in $[x, b] \cap (L \cup U)$ counting multiplicities up to order 2. Here again we write $x_\mu = x$, $y_\mu = b$ so that $y_\mu \notin X_r$ in this special case.

We say that the set of intervals $\{(x_i, y_i)\}_{i=1}^{\mu}$ is an alternant of length l for $f - r$ if $f - r$ alternates ω_i times on (x_i, y_i), where $\omega_i = 1$ or 2 as defined above and $\sum_{i=1}^{\mu} \omega_i = l$.

LEMMA 3.2. *If* $x, y \in X_{r^*}$, $x < y$, $(x, y) \cap X_{r^*} = \phi$ *and* $(x, y) \cap S = \{z_1, \ldots, z_\nu\}$, $\nu \geqslant 0$ ($\nu = 0$ *implies that* $(x, y) \cap S = \phi$), *then*

(i) $\operatorname{sg}(f(y)) = (-1)^\nu \operatorname{sg}(f(x))$,

(ii) $\operatorname{sg}(f(x))\, \sigma(x) = -1$ *and* $\sigma(x) = (-1)^\nu \sigma(y)$ *imply that* $\operatorname{sg}(f(y))\, \sigma(y) = -1$,

(iii) $\operatorname{sg}(f(x))\, \sigma(x) = -1$ *implies that* $|f(x)| < |r^*(x)|$.

Proof. (i) We first note that $\operatorname{sg}(f(x)) \neq 0$ on $\Gamma = [x, y] \sim S$ by definition. Furthermore, we claim that $\operatorname{sg}(f(x))$ is constant on each connected subset of Γ. To see this it suffices to consider $[x, z_1)$. Thus, assume that there exists $t_0 \in (x, z_1)$ such that $\operatorname{sg}(f(t_0)) = -\operatorname{sg}(f(x))$. Without loss of generality we shall assume that $\operatorname{sg}(f(x)) = 1$, then there exists $\rho_1, \rho_2 > 0$ for which

$$(x - \rho_1, x + \rho_1) \cap U \neq \phi, \qquad (x - \rho_1, x + \rho_1) \cap L = \phi,$$
$$(t_0 - \rho_2, t_0 + \rho_2) \cap U = \phi, \qquad (t_0 - \rho_2, t_0 + \rho_2) \cap L \neq \phi.$$

Let $l_1 = \inf\{t \in [x, t_0 + \rho_2]: t \in L\}$, then $l_1 > x$ (since $l_1 \in L$ and $x \notin L$). Also l_1 must be an element of U (if not then f changes sign on an interval, namely, $[u_1, l_1]$, where $u_1 = \sup\{t \in [a, b]: t \leqslant l_1, t \in U\}$). Hence $l_1 \in L \cap U = S$, which is a contradiction since $l_1 \in (x, z_1)$. Thus, $\operatorname{sg}(f(t)) = \operatorname{sg}(f(x))$ for all $t \in (x, z_1)$. Finally, observe that $\operatorname{sg}(f(x))$ changes sign at each point of S. Indeed, each point of S is a cluster point of L^0 and U^0 by definition and the above argument shows that the points of S locally separate L^0 and U^0. From this (i) follows.

(ii) If $\operatorname{sg}(f(x))\, \sigma(x) = -1$ and $\sigma(x) = (-1)^\nu \sigma(y)$, it follows from (i) that

$$\operatorname{sg}(f(y))\, \sigma(y) = (-1)^\nu \operatorname{sg}(f(x)) \cdot (-1)^\nu \sigma(x) = \operatorname{sg}(f(x))\, \sigma(x) = -1.$$

(iii) We show that $\operatorname{sg}(f(x))\, \sigma(x) = -1$ implies that $|f(x)| < |r^*(x)|$. We consider the case of $\operatorname{sg}(f(x)) = 1$ and $\sigma(x) = -1$. Then either $f(x) > 0$ or $f(x) = 0$ and there exists $\rho > 0$ such that $(x - \rho, x + \rho) \cap U \neq \phi$, $(x - \rho, x + \rho) \cap L = \phi$ and either $f(x) - r^*(x) = -\|f - r^*\|$ or $x \in L \sim S$,

$r^*(x)=0$. We notice that if $f(x)>0$ or there exists $\rho>0$ such that $(x-\rho, x+\rho) \cap U \neq \phi$, $(x-\rho, x+\rho) \cap L=\phi$, x cannot be an element of $L \sim S$. Therefore, in both cases we have

$$f(x)-r^*(x)=-\|f-r^*\| \Rightarrow 0 \leqslant f(x)<r^*(x).$$

Similarly, if $\operatorname{sg}(f(x))=-1$ and $\sigma(x)=1$, we can show that

$$r^*(x)<f(x) \leqslant 0. \quad \blacksquare$$

LEMMA 3.3. *Assume that* $x_i, y_i \in X_{r^*}$, $(x_i, y_i) \cap S=\{z_{i+1}, \ldots, z_{i+v}\}$ *and* $f-r^*$ *alternates* ω_i *times between* x_i *and* y_i. *Let* $r \in R_f[a, b]$ *satisfy* $\|f-r\|<\|f-r^*\|$, *then*:

(i) *If* $r^*(x_i)=r(x_i)=0$ (*or* $r^*(y_i)=r(y_i)=0$), *then* r^*-r *has at least* $v+\omega_i+1$ *zeros in* $[x_i, y_i]$.

(ii) *If* $r^*(x_i) \neq r(x_i)$, $r^*(y_i) \neq r(y_i)$ *and* $\omega_i=1$, *then* r^*-r *has at least* $v+\omega_i$ *zeros in* (x_i, y_i).

(iii) *If* $\omega_i=2$, *then* r^*-r *has at least* $v+\omega_i$ *zeros in* (x_i, y_i).

Proof. (i) In this case ω_i must equal 1 by Lemma 3.2 since $r^*(x_i)=0$ implies $|f(x_i)| \nless |r^*(x_i)|$ which implies that $\operatorname{sg}(f(x_i)) \sigma(x_i) \neq -1$ and neither r^* nor r can change sign at x_i (as $x_i \in L \cup U \sim S$). Therefore $r^{*\prime}(x_i)=r'(x_i)=0$ so that r^*-r has at least $v+2=v+\omega_i+1$ zeros in $[x_i, y_i]$. (The same argument establishes this result when $r^*(y_i)=r(y_i)$.)

(ii) Suppose that $\omega_i=1$ and consider the case where $\sigma(x_i)=-1$ and v is odd. In this case we must have $\sigma(y_i)=-1$, $r^*(x_i)>r(x_i)$ and $r^*(y_i)>r(y_i)$. Now, if r^*-r has only $z_{i+1}, \ldots, z_{i+v}$ as simple zeros then $(r^*-r)(x_i)>0$ and v odd implies that $(r^*-r)(y_i)<0$. Thus r^*-r must have at least one of $z_{i+1}, \ldots, z_{i+v}$ as a zero of order at least 2, or another zero in (x_i, y_i) different from $z_{i+1}, \ldots, z_{i+v}$. Hence r^*-r has at least $v+\omega_i$ $(=v+1)$ zeros in (x_i, y_i). The other cases follow by similar arguments.

For the two special cases where $\omega_1=1$ on (a, y) with $[a, y) \cap X_{r^*}=\phi$ (that is, $x_1=a$ and $y_1=y$) or $\omega_\mu=1$ on (x, b) with $(x, b] \cap X_{r^*}=\phi$ (that is, $x_\mu=x$ and $y_\mu=b$), consider the case of $\omega_1=1$ on (a, y) with $[a, y) \cap X_{r^*}=\phi$ (the other case follows by a similar argument). In this case $\operatorname{sg}(f(y)) \sigma(y)=-1$ and r^* has at least $v+1$ zeros in $[a, y] \cap(L \cup U)$, where $(a, y] \cap S=\{z_1, \ldots, z_v\}$ with $v \geqslant 1$. Now $\operatorname{sg}(f(y)) \sigma(y)=-1$ implies that $|r^*(y)|>|r(y)|$ (since, for example, if $\operatorname{sg}(f(y))=1$ then $\sigma(y)=-1$ only with $f(y)-r^*(y)=-\|f-r^*\|$).

Also, r^* has at least $v+1$ zeros in the set $[a, y] \cap(L \cup U)$ counting multiplicities up to order 2, which implies that $r^{*\prime}(z_j)=0$ for some $j \in\{1, \ldots, v\}$ as $[a, y) \cap X_{r^*}=\phi$ allows $r^*(x)=0$ for $x \in[a, y] \cap(L \cup U)$ only if $x \in S$. If r^*-r has a zero in (z_j, y) other than $z_{j+1}, \ldots, z_v$ we are done. On the other hand, if $r^* - r$ vanishes in (z_j, y) only at $z_{j+1}, \ldots, z_\nu$ and each of these is a simple zero then we have $|r^*(t)| > |r(t)|$ in every interval of the form (z_{l-1}, z_l), $l = j+1, \ldots, \nu$. Looking at (z_j, z_{j+1}) this inequality implies that $r'(z_j) = 0$ and hence $r^* - r$ has at least $\nu + 1$ zeros in (a, y).

(iii) Suppose that $\omega_i = 2$. Here we must have

$$\operatorname{sg}(f(x_i))\, \sigma(x_i) = \operatorname{sg}(f(y_i))\, \sigma(y_i) = -1,$$

and

$$|f(x_i) - r^*(x_i)| = |f(y_i) - r^*(y_i)| = \|f - r^*\|$$

so that

$$|r^*(x_i)| > |r(x_i)| \qquad \text{and} \qquad |r^*(y_i)| > |r(y_i)|.$$

Let z_j be the first element of $(x_i, y_i) \cap S$ for which $r^{*\prime}(z_j) = 0$. Now, suppose that $r^* - r$ vanishes in (x_i, z_j) only on $\{z_{i+1}, \ldots, z_{j-1}\}$ (set is empty if $j = i+1$) and all the zeros are simple. Then, $|r^*(t)| > |r(t)|$ must hold in $(z_j - \rho, z_j)$ for some $\rho > 0$. Thus, $r'(z_j) = 0$ as $r, r^* \in C^2[a, b]$. But r and r^* change sign at z_j and hence $r''(z_j) = r^{*\prime\prime}(z_j) = 0$; that is, $r^* - r$ has z_j as a zero of order at least 3 and the result follows. Now assume that $r^* - r$ has simple zeros at $z_{i+1}, \ldots, z_{j-1}$ and one additional zero in $(x_i, z_j) \sim \{z_{i+1}, \ldots, z_{j-1}\}$. This implies that $|r(t)| > |r^*(t)|$ holds on $(z_{j-1} - \varepsilon, z_j)$ for some $\varepsilon > 0$. Thus, if $r^* - r$ has only $z_{j+1}, \ldots, z_{i+\nu}$ as simple zeros and no other zeros in (z_j, y_i) then we must have $|r(y_i)| > |r^*(y_i)|$, which is a contradiction. Hence $r^* - r$ must have an additional zero in (z_j, y_i) proving that $r^* - r$ has at least $\nu + \omega_i$ zeros in (x_i, y_i). The proof of Lemma 3.3 is now complete. ∎

THEOREM 3.4 (THE ALTERNATION THEOREM). *Let $f \in C[a, b] \sim R_n^m[a, b]$ be an admissible function and $S = \{z_1, \ldots, z_k\}$, $k \leqslant m$ as described earlier. Then $r^* \in R_f[a, b]$ is a best approximation to f if and only if there exists a set of open intervals $\{(x_i, y_i)\}_{i=1}^{\mu}$ which is an alternant of length $N - k$ for $f - r^*$, where*

$$N = 1 + \max\{n + \partial p^*, m + \partial q^*\}, r^* = p^*/q^*.$$

Proof. ($\Leftarrow$) Suppose that $\{(x_i, y_i)\}_{i=1}^{\mu}$ is an alternant of length $N - k$ for $f - r^*$ and there exists $r \in R_f[a, b]$ for which $\|f - r\| < \|f - r^*\|$. Then, using Lemma 3.3, we show that $r^* - r$ has at least N zeros (counting multiplicities up to order 3). To that end, let $\{1, \ldots, \mu\} = I_A \cup I_B$ where $i \in I_A$

if and only if (i) of Lemma 3.3 holds true on (x_i, y_i) and $i \in I_B$ otherwise, $i = 1, \ldots, \mu$. Now, assume that there are

$$\eta \text{ elements of } S \text{ in } \bigcup_{i \in I_A} (x_i, y_i),$$

$$\delta \text{ elements of } S \text{ in } \bigcup_{i \in I_B} (x_i, y_i),$$

and

$$k - \eta - \delta \text{ elements of } S \text{ in the rest of } [a, b].$$

It is now easy to observe that $r^* - r$ has at least

$$\eta + \sum_{i \in I_A} \omega_i + \operatorname{card} I_A \text{ zeros in } \bigcup_{i \in I_A} (x_i, y_i),$$

$$\delta + \sum_{i \in I_B} \omega_i \text{ zeros in } \bigcup_{i \in I_B} (x_i, y_i),$$

and

$$k - \eta - \delta \text{ zeros in the rest of } [a, b].$$

In addition,

$$\eta + \sum_{i \in I_A} \omega_i + \operatorname{card} I_A + \delta + \sum_{i \in I_B} \omega_i + k - \eta - \delta$$

$$= k + \sum_{i=1}^{\mu} \omega_i + \operatorname{card} I_A = k + N - k + \operatorname{card} I_A \geqslant N.$$

This shows that $r^* - r$ has at least N zeros in $[a, b]$, which implies that $r^* \equiv r$ (since $r^* - r = (p^* q - p q^*)/q^* q$ with the degree of the numerator $\leqslant N - 1$).

($\Rightarrow$) Suppose that $f \notin R_n^m[a, b]$ and $r^* \in R_f[a, b]$ is a best copositive approximation to f. Assume that $\{(x_i, y_i)\}_{i=1}^{\mu}$ is an alternant for $f - r^*$ with $\sum_{i=1}^{\mu} \omega_i = l < N - k$ where l is maximal. We shall construct a new function $r \in R_f[a, b]$ for which $\|f - r\| < \|f - r^*\|$, thus contradicting the assumption that $l < N - k$. We assume that $S \neq \phi$. The assumption that l is maximal requires that for each $i = 1, \ldots, \mu - 1$ there are no alternations in $[y_i, x_{i+1}]$. Specifically, for each $x \in (y_i, x_{i+1}] \cap X_{r^*}$ with $[y_i, x] \cap S = \{z_{i+1}, \ldots, z_{i+\nu}\}$, $\nu \geqslant 0$ we must have $\sigma(x) = (-1)^{\nu} \sigma(y_i)$. Furthermore, if there exists $z_j \in [y_i, x] \cap S$ with $r^{*\prime}(z_j) = 0$ then we must also have $|r^*(y_i)| \leqslant |f(y_i)|$ (since $|f(y_i)| < |r^*(y_i)|$ implies that $\operatorname{sg}(f(y_i))\, \sigma(y_i) = -1$, which in turn implies that $f - r^*$ alternates twice between y_i and x) and hence $|r^*(x)| \leqslant |f(x)|$.

We begin by constructing a set of $l+k$ distinct points in (a, b) and a function $p + r^*q \in S_{r^*}$ that vanishes at these points. Then we can find a rational function belonging to $R_f[a, b]$ that gives the required contradiction.

Consider the interval (x_i, y_i) for $i = 1,..., \mu$. If $\omega_i = 1$ define a point $s_i \in (x_i, y_i)$ as follows:

First, consider the case where $(x_i, y_i) \cap S = \phi$. If $r^*(t) \neq 0$ $\forall t \in (x_i, y_i)$, set $s_i = (x_i + y_i)/2$. On the other hand, if $r^*(t) = 0$ for some $t \in (x_i, y_i)$, set $t_i' = \min\{t \in (x_i, y_i): r^*(t) = 0\}$ and $t_i'' = \max\{t \in (x_i, y_i): r^*(t) = 0\}$. Now, if $x_i \in X_{r^*}$ and $\mathrm{sg}(f(x_i))\, \sigma(x_i) = 1$, set $s_i = (t_i'' + y_i)/2$; if $x_i \in X_{r^*}$ and $\mathrm{sg}(f(x_i))\, \sigma(x_i) = -1$, set $s_i = (t_i' + x_i)/2$. Next, consider the case where $(x_i, y_i) \cap S = \{z_{i+1},..., z_{i+\nu}\}$. Define t_i' and t_i'' as above and note that $t_i' \leqslant z_{i+1}$, $t_i'' \geqslant z_{i+\nu}$. If $\mathrm{sg}(f(x_i))\, \sigma(x_i) = 1$ set $s_i = (t_i'' + y_i)/2$; if $\mathrm{sg}(f(x_i))\, \sigma(x_i) = -1$, set $s_i = (t_i' + x_i)/2$; if $x_1 = a$, $[a, y_1) \cap X_{r^*} = \phi$, set $s_1 = (t_1'' + y_1)/2$. Observe that if $|f(x_i)| < |r^*(x_i)|$ and there exists $z_j \in (x_i, y_i) \cap S$ for which $r^{*\prime}(z_j) = 0$, we must have $|f(y_i)| > |r^*(y_i)|$ since $\omega_i = 1$.

Finally, consider the case where $\omega_i = 2$. In this case $(x_i, y_i) \cap S = \{z_{i+1},..., z_{i+\nu}\}$, $\nu \geqslant 1$, $r^{*\prime}(z_j) = 0$ for at least one z_j, $i+1 \leqslant j \leqslant i+\nu$, $|f(x_i)| < |r^*(x_i)|$ and $|f(y_i)| < |r^*(y_i)|$. With t_i' and t_i'' defined as before, set $s_i' = (t_i' + x_i)/2$ and $s_i'' = (t_i'' + y_i)/2$.

Let T denote the set of all the points $\{s_i\} \cup \{s_i'\} \cup \{s_i''\}$ constructed above and set $Z = T \cup S$. Note that Z consists of precisely $l + k < N$ distinct points. Since S_{r^*} is a Haar subspace of dimension N on an interval larger than $[a, b]$, there exist $p \in \Pi_m$ and $q \in \Pi_n$ such that $p + r^*q$ has simple zeros only at the $l + k$ points of Z. We shall show that there exists $\varepsilon > 0$ such that $r_\varepsilon = r^* - \varepsilon(p + r^*q)/(q^* + \varepsilon q)$ is copositive with f and $\|f - r_\varepsilon\| < \|f - r^*\|$ (notice that $r_\varepsilon = (p^* - \varepsilon p)/(q^* + \varepsilon q)$). Suppose that $v(y) = (p + r^*q)(y)$ satisfies $\mathrm{sgn}\, v(y_1) = -\sigma(y_1)$ (this can be easily done by multiplying v by -1 if necessary). Since v has simple zeros at the $l + k$ points of Z and only at these points, it is easy to conclude that $\mathrm{sgn}\, v(x_i) = -\sigma(x_i)$ and $\mathrm{sgn}\, v(y_i) = -\sigma(y_i)$ for $i = 1,..., \mu$ provided that $x_1 \in X_{r^*}$, $y_\mu \in X_{r^*}$.

Now we consider the interval $[y_i, x_{i+1}]$ for fixed i, $i = 1,..., \mu - 1$ and we first show that there exists $\bar{\varepsilon} > 0$ such that for all ε, $0 < \varepsilon \leqslant \bar{\varepsilon}$, we have $\max_{x \in [y_i, x_{i+1}]} |f(x) - r_\varepsilon(x)| < \|f - r^*\|$. Since $\mathrm{sgn}\, v(y_i) = -\sigma(y_i)$ and $f - r^*$ does not alternate on $[y_i, x_{i+1}]$, we must have $\mathrm{sgn}\, v(x) = -\sigma(x)$ for all $x \in [y_i, x_{i+1}] \cap X_{r^*}$. Thus, if $[y_i, x_{i+1}] \cap S = \{z_{i+1},..., z_{i+\nu}\}$ and we set $t_0 = y_i$, $t_j = z_{i+j}$, $j = 1,..., \nu$, $t_{\nu+1} = x_{i+1}$, then for any $t \in [t_j, t_{j+1}]$, $j = 0, 1,..., \nu$ we have $\sigma(y_i)\, v(t)(-1)^j \leqslant 0$ and for any $x \in [t_j, t_{j+1}] \cap X_{r^*}$, $\sigma(x) = (-1)^j \sigma(y_i)$. Fix j, $0 \leqslant j \leqslant \nu$, and without loss of generality assume that $\sigma(y_i) = 1$ and j is even. Then, for all $t \in [t_j, t_{j+1}]$, $-\|f - r^*\| < (f - r^*)(t) \leqslant \|f - r^*\|$. Thus, by the continuity of $f - r^*$ and the compactness of $[t_j, t_{j+1}]$, there exists $\zeta > 0$ such that

$$(f - r^*)(t) \geqslant -\|f - r^*\| + \zeta, \qquad \forall t \in [t_j, t_{j+1}].$$

Now let $\varepsilon' > 0$ be such that $\varepsilon' |q(t)| < q^*(t)$ for all $t \in [t_j, t_{j+1}]$, then for every ε such that $0 < \varepsilon \leqslant \varepsilon'$, $q^* + \varepsilon q$ is positive on $[t_j, t_{j+1}]$ and r_ε converges uniformly to r^* on $[t_j, t_{j+1}]$ as $\varepsilon \to 0$. Thus, noting that $v(t) \leqslant 0$ for all $t \in [t_j, t_{j+1}]$ since j is even and $\sigma(y_i) = 1$, there exists ε_j with $0 < \varepsilon_j \leqslant \varepsilon'$ such that for every ε with $0 < \varepsilon \leqslant \varepsilon_j$ and $t \in [t_j, t_{j+1}]$,

$$(f - r_\varepsilon)(t) = (f - r^*)(t) + \frac{\varepsilon v(t)}{(q^* + \varepsilon q)(t)} \geqslant -\|f - r^*\| + \frac{\zeta}{2} > -\|f - r^*\|.$$

Also, since $v(t) < 0$ on (t_j, t_{j+1}), then for $\varepsilon > 0$ and $t \in (t_j, t_{j+1})$,

$$(f - r_\varepsilon)(t) = (f - r^*)(t) + \frac{\varepsilon v(t)}{(q^* + \varepsilon q)(t)} < \|f - r^*\|.$$

But $(f - r_\varepsilon)(t_l) = 0$ for any l such that $0 < l < v + 1$, $v(t_0) < 0$, and $(-1)^{v+1} v(t_{v+1}) > 0$ imply that

$$(f - r_\varepsilon)(t) < \|f - r^*\| \qquad \text{for any } t \in [t_j, t_{j+1}], j \text{ even}.$$

Thus, we have

$$-\|f - r^*\| < (f - r_\varepsilon)(t) < \|f - r^*\|, \qquad \forall t \in [t_j, t_{j+1}], j \text{ even},$$

which finally shows that

$$\max_{t \in [t_j, t_{j+1}]} |(f - r_\varepsilon)(t)| < \|f - r^*\|.$$

A similar argument works for odd j such that $1 \leqslant j \leqslant v$. Define $\bar{\varepsilon} = \min_{0 \leqslant j \leqslant v} \varepsilon_j$, then we have

$$\max_{x \in [y_i, x_{i+1}]} |(f - r_\varepsilon)(x)| < \|f - r^*\|, \qquad \forall \varepsilon \ni 0 < \varepsilon \leqslant \bar{\varepsilon}.$$

Next we show that there exists $\bar{\bar{\varepsilon}} > 0$ such that for each ε, $0 < \varepsilon \leqslant \bar{\bar{\varepsilon}}$, r_ε is copositive with f on $[y_i, x_{i+1}]$. Note that both f and v change sign in $[y_i, x_{i+1}]$ at the points of $[y_i, x_{i+1}] \cap S$. Thus either v or $-v$ is copositive with f on $[y_i, x_{i+1}]$. First, consider the case where there exists $z_j \in [y_i, x_{i+1}] \cap S$ with $r^{*\prime}(z_j) = 0$. In this case $|r^*(y_i)| \leqslant |f(y_i)|$ and we claim that f is copositive with $-v$. Indeed, suppose that $\sigma(y_i) = -1$, then either $f(y_i) - r^*(y_i) = -\|f - r^*\|$, which implies that $f(y_i) < r^*(y_i)$ and hence $|r^*(y_i)| \leqslant |f(y_i)|$ implies that $f(y_i) < 0$, or $r^*(y_i) = 0$ and $y_i \in L$, which shows that f and $-v$ are copositive since $\operatorname{sgn} v(y_i) = -\sigma(y_i) = 1$. The case when $\sigma(y_i) = 1$ follows in the same manner. Thus, in this case, for any $\bar{\bar{\varepsilon}} > 0$ with $\bar{\bar{\varepsilon}} |q(x)| < q^*(x)$ for all $x \in [y_i, x_{i+1}]$, we have r_ε and f are copositive (since $r_\varepsilon = r^* - \varepsilon v / (q^* + \varepsilon q)$) on $[y_i, x_{i+1}]$ for all ε with

$0 < \varepsilon \leqslant \bar{\bar{\varepsilon}}$. Second, consider the case where $[y_i, x_{i+1}] \cap S = \{z_{i+1}, ..., z_{i+v}\}$, $v \geqslant 0$ and $r^{*\prime}(z_j) \neq 0$ for all z_j. If f and $-v$ are copositive we are done. Thus, assume that f and v are copositive on $[y_i, x_{i+1}]$. Suppose that $x \in (L \cup U) \cap ([y_i, x_{i+1}] \sim S)$. Then we claim that $r^*(x) \neq 0$. Indeed, if, for example, $x \in L \cap ([y_i, x_{i+1}] \sim S)$ and $r^*(x) = 0$, then $x \in X_{r^*}$ and $\sigma(x) = -1$. But $x \in L \sim S$ implies that $v(x) < 0$ since v is copositive with f, which contradicts $\operatorname{sgn}(v(x)) = -\sigma(x)$. Thus r^* and v vanish only at the points of S in $\Gamma = [y_i, x_{i+1}] \cap (L \cup U)$ and they both change sign at these points. Also, $\operatorname{sgn} r^*(x) = \operatorname{sgn} v(x)$ for each $x \in \Gamma$ (since f, r^* and v are copositive). Now, at each $z_j \in \Gamma \cap S$ we have $r^{*\prime}(z_j) \neq 0$, thus there exists $\theta_j \leqslant \frac{1}{2} \min\{z_{i+1} - z_i : i = 1, ..., k-1\}$, $\theta_j > 0$ such that $r^{*\prime}(x) \neq 0$ in $I_j = [z_j - \theta_j, z_j + \theta_j]$. By the mean value theorem, for each $x \in I_j$, there exists ξ_x and δ_x between x and z_j such that $r^*(x) = r^{*\prime}(\xi_x)(x - z_j)$ and $v(x) = v'(\delta_x)(x - z_j)$. Hence, we can select $\varepsilon_j > 0$ such that $0 < \varepsilon \leqslant \varepsilon_j$ implies $|r^*(x)| \geqslant \varepsilon\, |v(x)|$ for all $x \in I_j$ (by choosing $\varepsilon_j = \min_{t \in I_j} |r^{*\prime}(t)/v'(t)|$). Repeat this argument for each $z_j \in \Gamma \cap S$ and let $\varepsilon' = \min_j \varepsilon_j$. Thus $|r^*(x)| \geqslant \varepsilon' \, |v(x)|$ for all $x \in \bigcup_{j=1}^{v} I_j$. Now $\Gamma - \bigcup_{j=1}^{v} (z_j - \theta_j, z_j + \theta_j)$ is a compact subset of $[a, b]$ on which r^* does not vanish,. hence there exists $\varepsilon'' > 0$ such that $|r^*(x)| \geqslant \varepsilon'' \, |v(x)|$ for all $x \in \Gamma - \bigcup_{j=1}^{v} (z_j - \theta_j, z_j + \theta_j)$. Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \, |q(t)| < q^*(t)$ for all $t \in [y_i, x_{i+1}]$ and let $m = \min_{t \in [y_i, x_{i+1}]} \{q^*(t) + \varepsilon_0 q(t), q^*(t)\}$. Note that $q^*(t) + \varepsilon q(t) \geqslant m$ for all $t \in [y_i, x_{i+1}]$ and $0 < \varepsilon \leqslant \varepsilon_0$. Define $\bar{\bar{\varepsilon}}$ by $\bar{\bar{\varepsilon}} = \min\{\varepsilon_0, \varepsilon', \varepsilon'', m\varepsilon', m\varepsilon''\}$. Then for any $t \in [y_i, x_{i+1}]$ we have that $|r^*(t)| \geqslant \varepsilon(|v(t)|)/(q^* + \varepsilon q)(t)$ for any ε satisfying $0 < \varepsilon \leqslant \bar{\bar{\varepsilon}}$. From this it follows that r_ε is copositive with f on $[y_i, x_{i+1}]$ for every ε with $0 < \varepsilon \leqslant \bar{\bar{\varepsilon}}$.

Next, we consider an interval of the form $[x_i, y_i]$, i fixed, $1 \leqslant i \leqslant \mu$. Select $\delta > 0$ sufficiently small such that $f - r^*$ does not alternate on either $[x_i, x_i + \delta]$ or $[y_i - \delta, y_i]$. Since $\operatorname{sgn} v(x_i) = -\sigma(x_i)$, $\operatorname{sgn} v(y_i) = -\sigma(y_i)$, by using continuity and compactness as before, we can show that there exists $\varepsilon' > 0$ such that $0 < \varepsilon \leqslant \varepsilon'$ implies

$$\max_{x \in [x_i, x_i + \delta] \cup [y_i - \delta, y_i]} |f(x) - r_\varepsilon(x)| < \|f - r^*\|.$$

Also, for $\varepsilon > 0$ sufficiently small,

$$\max_{x \in [x_i + \delta, y_i - \delta]} |f(x) - r_\varepsilon(x)| < \|f - r^*\|$$

since $(x_i, y_i) \cap X_{r^*} = \phi$, which implies that $|f(x) - r^*(x)| < \|f - r^*\|$ for all $x \in [x_i + \delta, y_i - \delta]$. (Note that this is also true if $x_1 = a$ and $[a, y_1) \cap X_{r^*} = \phi$ or $y_\mu = b$ and $(x_\mu, b] \cap X_{r^*} = \phi$.)

Now it remains to prove that for $\varepsilon > 0$ sufficiently small, r_ε is copositive with f on $[x_i, y_i]$. First we note that if $\omega_i = 1$ and $\operatorname{sg}(f(x_i))\, \sigma(x_i) = 1$ so that

$\mathrm{sg}(f(x_i)) \operatorname{sgn} v(x_i) = -1$ then the facts that $s_i > z_j$ for each $z_j \in [x_i, y_i] \cap S$ and f, v change sign only at the points of Z imply that f and $-v$ are copositive on $[x_i, s_i]$. Thus, r_ε is copositive with f on $[x_i, s_i]$ for every $\varepsilon > 0$ satisfying $\varepsilon |q(t)| < q^*(t)$ for all $t \in [x_i, y_i]$. Now, consider the interval (s_i, y_i) and note that $s_i = (t_i'' + y_i)/2$ implies that $r^*(t) \neq 0$ for any $t \in [s_i, y_i)$. Also, $\sigma(x_i) = (-1)^{\nu+1} \sigma(y_i)$ and $\mathrm{sg}(f(x_i)) = (-1)^\nu \mathrm{sg}(f(y_i))$ (by Lemma 3.2(i)) imply that $\mathrm{sg}(f(y_i))\, \sigma(y_i) = -1$, hence $|r^*(y_i)| > |f(y_i)|$ (by Lemma 3.2(iii)) (that is, $r^*(y_i) \neq 0$). Thus $r^*(t) \neq 0$ for all $t \in [s_i, y_i]$ and by continuity and compactness arguments as before we can find $\varepsilon_1 > 0$ such that $0 < \varepsilon \leqslant \varepsilon_1$ implies r_ε and f are copositive on $[s_i, y_i]$ and hence on $[x_i, y_i]$. Similar arguments hold in the case of $\omega_i = 1$ and $\mathrm{sg}(f(x_i))\, \sigma(x_i) = -1$ or $\omega_1 = 1$, $x_1 = a$ and $\mathrm{sg}(f(y_1))\, \sigma(y_1) = -1$, or $\omega_\mu = 1$, $y_\mu = b$ and $\mathrm{sg}(f(x_\mu))\, \sigma(x_\mu) = -1$. Now assume that $\omega_i = 2$. In this case there exists $z_j \in [x_i, y_i] \cap S$ such that $r^{*\prime}(z_j) = 0$ and v vanishes at $s_i' < z_{i+1} < \cdots < z_{i+\nu} < s_i''$ and only at these points in $[x_i, y_i]$. Also $\mathrm{sg}(f(x_i))\, \sigma(x_i) = \mathrm{sg}(f(y_i))\, \sigma(y_i) = -1$. Hence $\mathrm{sg}(f(x_i)) \operatorname{sgn} v(x_i) = \mathrm{sg}(f(y_i)) \operatorname{sgn} v(y_i) = 1$ and f and $-v$ are copositive on $[s_i', s_i'']$. For the intervals $[x_i, s_i']$ and $[s_i'', y_i]$, r^* is never zero (by choice of s_i' and s_i'' and the fact that $|f(x_i)| < |r^*(x_i)|$ and $|f(y_i)| < |r^*(y_i)|$). Using the same argument as given before, f and r_ε are copositive on these intervals for a proper choice of $\varepsilon > 0$. Thus r_ε is copositive with f on $[x_i, y_i]$.

Finally, consider the case where $y_\mu < b$. Here, using the same argument given for the $[y_i, x_{i+1}]$ case we can show that $\max_{x \in [y_\mu, b]} |f(x) - r_\varepsilon(x)| < \|f - r^*\|$ for $\varepsilon > 0$ sufficiently small. Moreover, if $\mathrm{sg}(f(y_\mu))\, \sigma(y_\mu) = 1$ then $\mathrm{sg}(f(y_\mu)) \operatorname{sgn} v(y_\mu) = -1$ implies that f and $-v$ are copositive on $[y_\mu, b]$ and hence r_ε is copositive with f on $[y_\mu, b]$ for any $\varepsilon > 0$. On the other hand, if $\mathrm{sg}(f(y_\mu))\, \sigma(y_\mu) = -1$, then f and v are copositive and r^* vanishes in $[y_\mu, b]$ only at $[y_\mu, b] \cap S$ and each of these is a simple zero. Then, by the same argument used for the $[y_i, x_{i+1}]$ case with $r^{*\prime}(z_j) \neq 0$ for all $z_j \in [y_i, x_{i+1}] \cap S$, we can show that for $\varepsilon > 0$ sufficiently small, r_ε is copositive with f on $[y_\mu, b]$. Similarly, we can show that r_ε is copositive with f on $[a, x_1]$ if $x_1 > a$.

This covers all possible cases and shows that for a proper choice of $\varepsilon > 0$ sufficiently small, r_ε is copositive with f on $[a, b]$ and $\|f - r_\varepsilon\| < \|f - r^*\|$, which is a contradiction, and the proof of the alternation theorem is now complete. ∎

The second type of characterization for a best approximation, r^*, is a modified Kolmogorov-type characterization. Define

$$\bar{S}_{r^*} = \{p + r^* q : p \in \Pi_m,\ q \in \Pi_n,\ p \text{ copositive with } f\},$$

$$S_1 = \{x \in [a, b] : |f(x) - r^*(x)| = \|f - r^*\|\},$$

$$S_2 = \{x \in [a, b]: r^*(x) = 0, x \in (L \cup U \sim S)\},$$

$$\sigma(x) = \operatorname{sgn}(f(x) - r^*(x)) \text{ for } x \notin S_1 \cup S_2$$

($\sigma(x)$ is defined as before for $x \in S_1 \cup S_2$). Thus, we have

THEOREM 3.5 (KOLMOGOROV CRITERION [5]). *Let $f \in C[a, b] \sim R_n^m[a, b]$ and $r^* \in R_f[a, b]$, then r^* is a best approximation to f from $R_f[a, b]$ if and only if for each $h \in \bar{S}_{r^*}$, $\min_{x \in S_1 \cup S_2} \sigma(x)\, h(x) \leqslant 0$.*

The proof of this result follows via the usual arguments. Note that, comparing this Kolmogorov criterion for copositive rational approximation with that of rational approximation with interpolation [4], one might expect that $S'_{r^*} = \{p + r^*q\colon\ p \in \Pi_m,\ q \in \Pi_n,\ p(x) = 0,\ \forall x \in S\}$ could be used instead of $\bar{S}_{r^*} = \{p + r^*q\colon p \in \Pi_m, q \in \Pi_n, p$ copositive with $f\}$. Indeed, the Kolmogorov criterion holding for S'_{r^*} is a sufficient condition for r^* to be a best copositive rational approximation. However, with r^* being a best approximation to f from $R_f[a, b]$, the condition $\min_{x \in S_1 \cup S_2} \sigma(x)\, h(x) \leqslant 0$ for each $h \in S'_{r^*}$ need not hold whenever there exist $x, y \in X_{r^*}$ such that $x < y$, $(x, y) \cap X_{r^*} = \phi$, and $f - r^*$ alternates twice on (x, y). This can be shown by considering simple examples. Thus, the copositiveness assumption in $\bar{S}_{r^*}$ is essential for Theorem 3.5.

As the final type of characterization for a best approximation r^* we consider the possibility of an "origin in the convex hull" type of characterization. Here, the results are not as complete as that for the standard case. Once again, we simply state these theorems without proofs as the standard arguments suffice.

THEOREM 3.6 (SUFFICIENCY [5]). *If the origin of Euclidean l-space, $\underline{O}_l$, belongs to the convex hull of the set $\{\sigma(x)\hat{x}: x \in X_{r^*}\}$, $\hat{x} = (\phi_1(x),..., \phi_l(x))$ where $\{\phi_1,..., \phi_l\}$ is a basis for $\Pi_{m-k}\omega(x) + r^*(x)\,\Pi_n$, $\omega(x) = (x - z_1) \cdots (x - z_k)$, then r^* is a best approximation to f from $R_f[a, b]$.*

It is worth mentioning that the conclusion in Theorem 3.6 can be made stronger by replacing $R_f[a, b]$ by $\tilde{R}_n^m = \{r \in R_n^m[a, b]\colon r(z_i) = 0,\ i = 1,..., k\}$ without any vital change in the proof. On the other hand, the converse of Theorem 3.6 does not hold true and to prove necessity, more restrictions are needed that make the origin belong to the convex hull of a smaller set as shown in the following theorem.

THEOREM 3.7 (NECESSITY [5]). *Suppose that r^* is a best approximation to f from $R_f[a, b]$, then $\underline{O}_l$ belongs to the convex hull of $\{\sigma(x)\hat{x}: x \in X_{r^*}\}$, $\hat{x} = (\phi_1(x),..., \phi_l(x))$ where $\{\phi_1,..., \phi_l\}$ is a basis for $\tilde{S}_{r^*} = \{p + r^*q\colon\ p \in \Pi_m$, $q \in \Pi_n$, p has same zeros as p^* and of at least the same orders$\}$.*

4. Uniqueness and Strong Uniqueness of Best Approximations

We start this section with a lemma which plays an essential role in the uniqueness and strong uniqueness for best copositive rational approximation.

LEMMA 4.1. *Let $r^* = p^*/q^* \in R_f[a, b]$ be a best approximation to $f \in C[a, b] \sim R_n^m[a, b]$. If there exists $r = p/q \in R_f[a, b]$ such that $\sigma(x)[p(x) - r^*(x) q(x)] \geqslant 0$ for all $x \in X_{r^*}$, then $p - r^*q \equiv 0$.*

Proof. Suppose that there exists $r = p/q \in R_f[a, b]$ such that $\sigma(x)[p(x) - r^*(x) q(x)] \geqslant 0$ for all $x \in X_{r^*}$. Further, assume for a proof by contradiction that $p - r^*q \not\equiv 0$. Under this assumption, we show that there exists an $h \in \bar{S}_{r^*}$ with $\sigma(x) h(x) > 0$ for all $x \in X_{r^*}$ which is a contradiction to the Kolmogorov criterion given in Theorem 3.5. To that end define

$$\tilde{p} = p + p^*, \qquad \tilde{q} = q + q^*, \qquad \tilde{r} = \tilde{p}/\tilde{q},$$

and note that

$$\sigma(x)[\tilde{p}(x) - r^*(x) \tilde{q}(x)] \geqslant 0, \qquad \forall x \in X_{r^*}$$

(since $\tilde{p} - r^*\tilde{q} = (1/q^*)[(p + p^*) q^* - p^*(q + q^*)] = p - r^*q$). Now, our object is to find $p_1 \in \Pi_m$, $q_1 \in \Pi_n$ such that

(i) $\tilde{p} + \lambda p_1$ is copositive with f,

and

(ii) $\sigma(x)[(\tilde{p} + \lambda p_1) - r^*(\tilde{q} + \lambda q_1)](x) > 0$, $\forall x \in X_{r^*}$, where λ is a positive constant.

First, observe that

$$\begin{aligned}(\tilde{p} + \lambda p_1) - r^*(\tilde{q} + \lambda q_1) &= \frac{1}{q^*} [(p + p^* + \lambda p_1) q^* - p^*(q + q^* + \lambda q_1)] \\ &= \frac{1}{q^*} [pq^* + \lambda p_1 q^* - p^*q - \lambda p^* q_1] \\ &= (p - r^*q) + \lambda(p_1 - r^*q_1).\end{aligned}$$

Thus,

$$\begin{aligned}&\sigma(x)[(\tilde{p} + \lambda p_1) - r^*(\tilde{q} + \lambda q_1)](x) \\ &\quad = \sigma(x)(p - r^*q)(x) + \lambda\sigma(x)(p_1 - r^*q_1)(x).\end{aligned}$$

Hence for $\lambda > 0$ chosen sufficiently small, conditions (i) and (ii) stated above become equivalent to:

Find $p_1 \in \Pi_m$ and $q_1 \in \Pi_n$ such that

(i′) $p_1(z) = 0$ for all $z \in S$ and p_1 is locally copositive with f at every $z \in S$ at which $\tilde{p}'(z) = 0$ (we say that g is locally copositive with f at $x \in [a, b]$ if there exists $\delta > 0$ such that g is copositive with f on $[x - \delta, x + \delta] \cap [a, b]$).

(ii′) $\sigma(x)(p_1 - r^*q_1)(x) > 0$ at every $x \in X_{r^*}$ at which $(p - r^*q)(x) = 0$, say $(p_1 - r^*q_1)(x) = \sigma(x)$ at these points.

It is worth noting that if $x \in \tilde{S}_2 = \{x \in X_{r^*}\colon \tilde{r}(x) = 0,\ x \in (U \cup L) \sim S\}$ then (ii′) implies that p_1 is locally copositive with f at x. This is because $x \in \tilde{S}_2$ implies that $\tilde{p}(x) = p(x) + p^*(x) = 0$ so that $p(x) = p^*(x) = 0$ implying that $(p - r^*q)(x) = 0$ and hence $p_1(x) = \sigma(x)$ by (ii′).

Now, if (i′) is to be satisfied then we may write $(p_1 - r^*q_1)(x)$ in the form

$$(p_1 - r^*q_1)(x) = (x - z_1) \cdots (x - z_k)[(p_2 - \tilde{r}q_1)(x)],$$

where $p_2 = p_1/\prod_{i=1}^{k}(x - z_i)$, $\hat{r} = \hat{p}/q^*$ with $\hat{p} = p^*/\prod_{i=1}^{k}(x - z_i)$. Thus $p_2 - \hat{r}q_1 \in \Pi_{m-k} + \hat{r}\Pi_n$, which is a Haar subspace of $C[a, b]$ of dimension

$$\begin{aligned} d &= 1 + \max\{m - k + \partial q^*, n + \partial \hat{p}\} \\ &= 1 + \max\{m - k + \partial q^*, n - k + \partial p^*\} \\ &= 1 + \max\{m + \partial q^*, n + \partial p^*\} - k = N - k. \end{aligned}$$

Assume that $\hat{z}_1, \ldots, \hat{z}_l$ are the elements of S at which $\tilde{p}'(z) = 0$ and $\hat{x}_1, \ldots, \hat{x}_t$ are the zeros of $p - r^*q$ in X_{r^*}. Thus, our problem reduces to finding nonzero polynomials $p_2 \in \Pi_{m-k}$ and $q_1 \in \Pi_n$ such that

$$p_2(\hat{z}_j) = \pm 1, \qquad j = 1, \ldots, l,$$

and

$$\prod_{i=1}^{k} (\hat{x}_j - z_i)[(p_2 - \hat{r}q_1)(\hat{x}_j)] = \sigma(\hat{x}_j), \qquad j = 1, \ldots, t,$$

which can be done if $l + t \leqslant d$ (by definition of Haar subspace of dimension d). Now, if $l + t \leqslant d$, then there exists $p_2 - \hat{r}q_1$ with the above properties implying that there exists an $h = \tilde{p} + \lambda p_1 - r^*(\tilde{q} + \lambda q_1) \in \tilde{S}_{r^*}$ such that

$\sigma(x)\,h(x) > 0$ for all $x \in X_{r^*}$, a contradiction to Theorem 3.5. On the other hand, if $l+t > d$ we claim that $p - r^*q$ has more than N zeros in $[a, b]$ (since $\tilde{p}'(\hat{z}_j) = 0$ implies that $p - r^*q$ has a triple zero at every $\hat{z}_j$, $j = 1, \ldots, l$; and $X_{r^*} \cap S = \phi$ implies that $p - r^*q$ has at least

$$3l + k - l + t = k + l + t + l > k + d + l = N + l$$

zeros in $[a, b]$). But $p - r^*q \in S_{r^*}$, a Haar subspace of $C[a, b]$ of dimension N. Thus, $p - r^*q \equiv 0$, again a contradiction, and the proof is now complete. ∎

The following corollaries follow immediately.

COROLLARY 4.2 (UNIQUENESS). *Let $f \in C[a, b] \sim R_n^m[a, b]$. Then f has a unique best approximation from $R_f[a, b]$.*

COROLLARY 4.3 (MODIFICATION OF THEOREM 3.5). *Let r^* be the best approximation to f from $R_f[a, b]$. Then for every $h \in \bar{S}_{r^*}$ with $\|h\| = 1$, $\min_{x \in X_{r^*}} \sigma(x)\,h(x) < 0$.*

Before stating the strong unicity theorem for copositive rational approximation we need to define the concept of normality which plays an essential role in the strong uniqueness result for the standard rational approximation theory. The proof of the theorem is not presented here as it is essentially the same as for standard rationals [1].

DEFINITION 4.4. Let $f \in C[a, b]$ and $r^* = p^*/q^*$ be the best approximation to f from $R_f[a, b]$. f is said to be copositive normal if either $\partial p^* = m$ or $\partial q^* = n$.

THEOREM 4.5 (STRONG UNIQUENESS). *Let $r^* \in R_f[a, b]$ be the best approximation to f from $R_f[a, b]$ with $(p^*, q^*) = 1$, where $f \in C[a, b]$. If f is copositive normal then there exists a constant $\gamma = \gamma(f) > 0$ such that for all $r \in R_f[a, b]$*

$$\|f - r\| \geqslant \|f - r^*\| + \gamma\,\|r - r^*\|.$$

The condition of copositive normality stated in Theorem 4.5 is essential for strong uniqueness to hold true. This is shown by the following theorem in which we assume, without loss of generality, that $[a, b] = [0, 1]$. The proof is omitted for brevity and it can be found in [5].

THEOREM 4.6. *Let $f \in C[0, 1] \sim R_n^m[0, 1]$ be non-copositive normal and $r^* = p^*/q^*$ be the best copositive approximation to f from $R_f[0, 1]$, then the strong uniqueness theorem does not hold for f.*

5. Continuity of the Best Copositive Rational Approximation Operator

As in the standard rational approximation theory, the strong uniqueness result implies Lipschitz continuity of the best approximation operator at copositive normal points. However, we need to restrict the domain of this operator to a subset of the continuous functions defined on $[a, b]$ and copositive with f in order for this result to be true.

Let $f \in C[a, b]$. For any $g \in C[a, b]$, let $\tau(g)$ be the best copositive rational approximation to g from $R_g[a, b]$. Define

$$C_f[a, b] = \{ g \in C[a, b] : g(x) f(x) \geqslant 0, \forall x \in [a, b] \}$$

and

$$\tilde{C}_f[a, b] = \{ g \in C_f[a, b] : g(x) = 0, \forall x \in [a, b] \sim (L \cup U) \}.$$

Then we have

Theorem 5.1. *If $f \in R_f[a, b]$ or f is copositive normal then τ is continuous in the sense that there exists a $\beta > 0$ such that $g \in \tilde{C}_f[a, b]$ implies that*

$$\| \tau(g) - \tau(f) \| \leqslant \beta \| g - f \|.$$

In the above theorem, the restriction of the domain of the operator to $\tilde{C}_f[a, b]$ is essential. An example to illustrate this fact is given in [5].

6. Discretization

In this section the set $[a, b]$ is replaced by finite subsets of it and a brief summary of some results is presented. We assume that X_j, $j = 1, 2, \ldots$, is a finite subset of $[a, b]$ containing j points, $X_j \subset X_{j+1}$ for all j and that X_j becomes dense in $[a, b]$ as $j \to \infty$ (that is, each point of $[a, b]$ is either a limit point for $\bigcup_{j=1}^{\infty} X_j$ or belongs to X_j for $j \geqslant j_0$). In this case, there may exist no $r_j^* \in R_f(X_j)$ such that r_j^* is a best copositive rational approximation to f on X_j, where $R_f(X_j) = \{ r = p/q : p \in \Pi_m, q \in \Pi_n, q > 0 \text{ on } X_j \text{ and } r(x) f(x) \geqslant 0 \text{ on } X_j \}$. However, we have

Theorem 6.1 [5]. *Let $f \in C[a, b]$ and r^* be the best rational copositive approximation to f from $R_f[a, b]$. If $e_j = \inf_{r \in R_f(X_j)} \| f - r \|_{X_j}$, $e = \| f - r^* \|_{[a,b]}$ then $e_j \to e$ as $j \to \infty$.*

A more general result about approximating on finite sets that are becoming dense in an interval is given in the following theorem.

THEOREM 6.2 [5]. *Let r^* be the best approximation to $f \in C[a,b]$ from $R_f[a,b]$. Let $\{f_j\}$ be a sequence of functions defined and copositive with f on X_j and $\{r_j\}$ be a sequence of rationals such that $r_j = p_j/q_j \in R_f(X_j)$, $\|p_j\| + \|q_j\| = 1$ and $\|f_j - r_j\|_{X_j} \leqslant e_j + 1/j$, where $e_j = \inf_{r \in R_{f_j}(X_j)} \|f_j - r\|_{X_j}$. Suppose that*

(i) *$X_j \subset X_{j+1}$ and X_j becomes dense in $[a,b]$ as $j \to \infty$, and*

(ii) *$f_j \to f$ as $j \to \infty$ (i.e., $M_j = \|f_j - f\|_{X_j} \to 0$ as $j \to \infty$).*

Then

(a) *$\|f_j - f\|_{X_j} \to \|f - r^*\|$ and*

(b) *$r_j \to r^*$ in measure.*

Moreover,

(c) *if f is copositive normal then $r_j \to r^*$ uniformly.*

For characterization of best copositive rational approximations in this setting we refer to restricted range work by K. A. Taylor [6] and Loeb *et al.* [2]. This theory contains the theory of best copositive rational approximation on X whenever X is a finite subset of $[a,b]$. Here we state two of their theorems modified to our notation. As before the set S_{r^*} is defined by

$$S_{r^*} = \{p + r^*q : p \in \Pi_m \text{ and } q \in \Pi_n\}.$$

THEOREM 6.3 (KOLMOGOROV CRITERION [6]). *Let $f \in C(X)$ and $\inf_{r \in R_f(X)} \|f - r\| \equiv \xi > 0$. Then r^* is not a best copositive rational approximation to f if and only if there exists an $h \in S_{r^*}$ such that $\sigma(x)\, h(x) > 0$ for all $x \in X_{r^*}$.*

THEOREM 6.4 (origin in the convex hull characterization [2]). *Let $f \in C(X)$ and $r^* \in R_f(X)$, then r^* is a best approximation to f from $R_f(X)$ if and only if the origin of Euclidean N-space lies in the convex hull of $\{\sigma(x)\hat{x} : x \in X_{r^*}\}$, where $\hat{x} = (\phi_1(x), \ldots, \phi_N(x))$, $\{\phi_1, \ldots, \phi_N\}$ is a basis for S_{r^*}.*

Remark. It is worth noting that the property of the origin in the convex hull holds as a necessary and sufficient criterion for the best copositive rational approximation when approximating on a finite set. However, this property does not hold true on $[a,b]$. More work needs to be done to give an accurate mathematical illustration for this fact.

References

1. E. W. Cheney, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966.
2. H. L. Loeb, D. G. Moursund, and G. D. Taylor, Uniform rational weighted approximations having restricted ranges, *J. Approx. Theory* **1** (1968), 401–411.
3. E. Passow and G. D. Taylor, An alternation theory for copositive approximation, *J. Approx. Theory* **19** (1977), 123–134.
4. A. L. Perrie, Uniform rational approximation with osculatory interpolation, *J. Comput. System Sci.* **4** (1970), 509–522.
5. N. Ph. Seif, "Copositive Rational Approximation and an Exchange Algorithm for Convex Optimization," Ph.D. thesis, Colorado State University, Fort Collins, 1981.
6. K. A. Taylor, "Contributions to the Theory of Restricted Polynomial and Rational Approximation," Ph.D. thesis, Michigan State University, East Lansing, 1970.