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Integral Representation of Markov Systems and the Existence of Adjoined Functions for Haar Spaces

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Let A be a set of real numbers, and let $Y_n := \{y_0, ..., y_n\}$ be a Čebyšev system on A. Assume, moreover, that if $\inf A$ or $\sup A$ belongs to A, then it is a point of accumulation of A at which all y_i are continuous. We find necessary and sufficient conditions for the existence of a function y_{n+1} such that also $\{y_0, ..., y_n, y_{n+1}\}$ is a Čebyšev system on A. This theorem generalizes earlier results of Zielke and of the author. The proof is based on an integral representation of Markov systems that slightly extends a previous result of Zielke. © 1991 Academic Press, Inc.

1. Introduction and Statement of Results

In what follows, $n \geqslant 0$ is a fixed integer, A denotes a set of real numbers having at least $n+2$ elements, and $F(A)$ denotes the set of real functions on A; if A is an interval, $C(A)$ denotes the set of continuous functions in $F(A)$; if $Z_n := \{z_0, ..., z_n\}$ is a sequence of functions in $F(A)$, by $S(Z_n)$ we denote the linear span of Z_n. Finally, S_n will stand for an $n+1$-dimensional subspace of $F(A)$.

We say that Z_n is a *Čebyšev system* (*weak Čebyšev system*) if $\dim S(Z_n) = n+1$, and for every sequence $\{t_0, ..., t_n\} \subset A$ such that $t_0 < t_1 < \cdots < t_n$, $\det[z_i(t_j);\ i, j = 0, ..., n] > 0$ $(\geqslant 0)$. If Z_k is a (weak) Čebyšev system for $k = 0, ..., n$, we say that Z_n is a (*weak*) *Markov system*, or a *complete* (*weak*) *Čebyšev system*; if $z_0 \equiv 1$, we say that Z_n is *normalized*. The linear span of a (weak) Čebyšev system is called a (*weak*) *Haar space*, and the linear span of a (weak) Markov system is called a (*weak*) *Markov space*. These definitions are consistent with Karlin and Studden [1].

If Z_n is a (weak) Čebyšev system, we say that Z_n has a (*weak*) *Čebyšev extension*, or, simply, a (*weak*) *extension*, if there is a function z_{n+1} such

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that $Z_n \cup \{z_{n+1}\}$ is a (weak) Čebyšev system. We also say that z_{n+1} is (weakly) *adjoined* to $S(Z_n)$.

In this paper, we study the existence of Čebyšev extensions. The existence of weak extensions for weak Čebyšev systems, under very general hypotheses, follows trivially from a representation theorem of Zielke (see Theorem A below). As we show in Theorem 4 below, the existence of Čebyšev extensions and the existence of adjoined functions are equivalent problems.

The problem of existence of adjoined functions was apparently first studied by Laasonen [4], who showed that if S_n is an n-dimensional Haar space of n-times continuously differentiable functions defined on an interval, then it has an adjoined function.

In [5], Rutman asserted that if S_n is a Haar space of right-continuous functions defined on an open interval, then it has an adjoined function. However, he only sketched his proof; this proof is based on an integral representation of Markov systems which both Zielke and this author have shown to be false (cf. [6, 12]). Rutman also claimed that there is a Haar space of continuous functions defined on a closed interval for which no adjoined functions exist (cf. Krein [2, p. 21, footnote 2]). However, no such example seems to have been published, and indeed Krein and Nudel'man [3] attempted to show that the opposite is true: if S_n is a Haar space of continuous functions defined on a closed interval, then it has an adjoined function. However, their proof is based on Rutman's integral representation, and is therefore invalid.

In [12], Zielke essentially showed that if S_n is a Haar space defined on a set having "*property* (D)," then it has an adjoined function (a set A is said to have property (D) if it has no first nor last element, and between any two elements of A is a third element of A), whereas in [7] we showed that if S_n is a Haar space of continuous functions defined on an interval (closed, open, or semiclosed), then it has an adjoined function. Although Zielke's result is stronger, his method cannot be applied to a set that contains one or both of its enpoints, and indeed, in [13] he includes both his proof, and a simplified version of ours. (We believe, however, that this simplified proof is incorrect.)

The purpose of this paper is to combine some of the ideas of [7] with a refinement of Zielke's representation theorem [14, Theorem 3] to obtain necessary and sufficient conditions for the existence of Čebyšev extensions and of adjoined functions that contain the results of [7, 12] as particular cases. But we must first introduce some additional definitions that will be used in the sequel.

A finite-dimensional subspace S of $F(A)$ is called *endpoint nondegenerate* (END) provided that for every c in A the restrictions of the elements of S to $A_1 := A \cap (-\infty, c)$ and to $A_2 := A \cap (c, \infty)$ form subspaces S_1 of $F(A_1)$

and S_2 of $F(A_2)$ that have the same dimension as S. (This term was coined by D. J. Newman in 1980 to describe a concept introduced by Zwick (see [15]). It was also used by Zielke in [14], where it is referred to simply as "nondegeneracy.") We say that $Z_n \subset F(A)$ is an END system, if the elements of Z_n are linearly independent in $F(A)$, and $S(Z_n)$ is END.

Let $f \in F(a, b)$, and $c \in (a, b)$. We say that f is *not constant* at c if for every $\varepsilon > 0$ there are points $x_1, x_2 \in (a, b)$, $c - \varepsilon < x_1 < c < x_2 < c + \varepsilon$, such that $f(x_1) \neq f(x_2)$. (In particular, if $f(x)$ is increasing on (a, b), we have $f(x_1) < f(x_2)$.)

Let $n \geqslant 1$ and let $W_n := \{w_1, ..., w_n\} \subset F(a, b)$, $h \in F(A)$, and $h(A) \subset (a, b)$. We shall say that W_n satisfies *property* (M) with respect to h, provided that, for every choice of points $x_0 < x_1 < \cdots < x_n$ in $h(A)$, there is a double sequence $\{t_{i,j}; i = 0, ..., n, j = 0, ..., n-i\}$ such that:

(a) $x_j = t_{0,j}$; $j = 0, ..., n$.

(b) $t_{i,j} < t_{i+1,j} < t_{i,j+1}$; $i = 0, ..., n-1$, $j = 0, ..., n-i-1$.

(c) For $i = 1, ..., n$, $w_i(x)$ is not constant on $\{t_{i,j}; j = 0, ..., n-i\}$.

If these conditions are satisfied for a specific set of points $x_0 < \cdots < x_n$ in $h(A)$, we say that W_n satisfies property (M) with respect to h at $\{x_0, ..., x_n\}$. We shall also say that W_n satisfies *property* (N) with respect to h, if for every choice of points $x_0 < \cdots < x_{n+1}$ in $h(A)$ there is a double sequence $\{t_{i,j}; i = 0, ..., n+1, j = 0, ..., n-i+1\}$ such that:

(a) $x_j = t_{0,j}$; $j = 0, ..., n+1$.

(b) $t_{i,j} < t_{i+1,j} < t_{i,j+1}$; $i = 0, ..., n$, $j = 0, ..., n-i$.

(c) For $i = 1, ..., n$, $w_i(x)$ is not constant on $\{t_{i,j}; j = 0, ..., n-i+1\}$.

If $Z_n \subset F(A)$ we say that (h, c, W_n, U_n) is a *representation* for Z_n on A, provided that $h(x)$ is a strictly increasing function in $F(A)$, $c \in h(A)$, $h(c) = c$, the functions $w_i(x)$, $i = 1, ..., n$, are increasing and continuous in $j(h) := (\inf h(A), \sup h(A))$, $U_n := \{u_0, ..., u_n\}$, where $u_0 \in F(A)$ is positive, $\{u_0, ..., u_i\}$ is a basis of $S(Z_i)$, $i = 0, ..., n$, and for every x in A, and $i = 1, ..., n$,

$$u_i(x) = u_0(x) \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_{i-1}} dw_i(t_i) \cdots dw_1(t_1).$$

Note that if Z_n is normalized then u_0 must be a constant function. Finally, if (h, c, W_n, U_n) is a representation for some basis Z_n^* of $S(Z_n)$, we say that it is a *quasi-representation* for Z_n.

We can now state:

THEOREM A. *If $Z_n \subset F(A)$ is an END normalized weak Markov system then it has a representation.*

This theorem is essentially [14, Theorem 3], although there are two differences: First Zielke does not mention that if (h, c, W_n, U_n) is a representation for Z_n then the functions $w_i(x)$ are nonconstant. That this must be so is obvious: If w_k is constant it is clear that $z_k = 0$; thus the elements of Z_n are not linearly independent, which is a contradiction. Second, in the statement of Zielke's theorem, no mention is made that $\{u_0, ..., u_k\}$, $k = 0, ..., n$, is a basis of $S(Z_k)$ (it is only asserted that U_n is a basis of $S(Z_n)$). That this stronger statement is true can be inferred by inspection of the proof of the theorem. Another (unpublished) proof of Theorem A was obtained by the author combining the Lemma of [8] with a new embedding property of weak Markov systems [10]. This proof was noted in [14, Remark (6)]. Theorem A also follows from [11, Theorem 1].

Using Theorem A we shall prove

THEOREM 1. *Assume that A has neither a first nor a last point. Then $Z_n \subset F(A)$ is a Markov system if and only if it has a representation (h, c, W_n, U_n) such that W_n satisfies property* (M) *with respect to h.*

If w_k is constant on an interval I, it is readily seen that u_k is proportional to u_{k-1} on $h^{-1}(I \cap h(A))$, and the elements of U_n are therefore linearly dependent on $h^{-1}(I \cap h(A))$. We thus have

COROLLARY 1. *Let A have property* (D). *Then $Z_n \subset F(A)$ is a Markov system if and only if for every representation (h, c, W_n, U_n) of Z_n, the elements of W_n are strictly increasing in (h, A).*

Corollary 1 is essentially due to Zielke (cf. [14, Corollary 3]).

By the *endpoints* of A we mean $\sup A$ and $\inf A$.

As a consequence of Theorem 1 we also have:

THEOREM 2. *Let $Z_n \subset F(A)$ be a Markov system on A, and $B := A \setminus \{\inf A, \sup A\}$. Assume, moreover, that if an endpoint of A belongs to A, then it is a point of accumulation of A at which $z_0, ..., z_n$ are continuous. Then Z_n has a Čebyšev extension if and only if there is a representation (h, c, W_n, U_n) for Z_n on B that satisfies property* (N) *with respect to h.*

Let the set B be defined as in Theorem 2. We also have:

THEOREM 3. *Let $Z_n \subset F(A)$ be a Čebyšev system on A. Assume, moreover, that if an endpoint of A is in A, then it is a point of accumulation of A, and all the functions in Z_n are continuous at that endpoint. Then $S(Z_n)$ has an adjoined function if and only if there is a quasi-representation (h, c, W_n, U_n) for Z_n on B that satisfies property* (N) *with respect to h.*

Note that every Haar space defined on a set that has no first nor last element has a Markov basis (cf., e.g., [9]). In view of this result it is clear that Theorem 3 is a straightforward consequence of Theorem 2 and the following proposition:

THEOREM 4. *Let S_n be a Haar space. Then the following statements are equivalent*:

(a) *S_n has an adjoined function.*

(b) *Every Čebyšev system $Z_n \subset S_n$ has an extension.*

The proof of Theorem 4 readily follows from, e.g., [7, Lemma 2], and will therefore be omitted.

A set A is said to have *property* (B) provided that between any two elements of A is a third element of A. As a consequence of Theorem 3 we shall prove the following proposition, which contains the main results of [7, 12] as particular cases.

THEOREM 5. *Let A have property* (B), *and let $Z_n \subset F(A)$ be a Čebyšev system on A. Assume, moreover, that if an endpoint of A is in A, then it is a point of accumulation of A, and all the functions in Z_n are continuous at that endpoint. Then $S(Z_n)$ has an adjoined function in A.*

2. PROOFS

Theorem 1 is a straightforward consequence of Theorem A and the following auxiliary proposition, of some independent interest:

LEMMA. *Let $W_n := \{w_1, ..., w_n\}$ be a sequence of increasing and continuous functions defined on an open interval (a, b), let $c \in (a, b)$, $u_0 \equiv 1$, and for $k = 1, ..., n$, let $u_k(x) := \int_c^x \int_c^{t_1} \cdots \int_c^{t_{k-1}} dw_k(t_k) \cdots dw_1(t_1)$. Assume that $a < x_0 < \cdots < x_n < b$; then* $\det[u_i(x_j);\ i, j = 0, ..., n] > 0$ *if and only if W_n satisfies property* (M) *with respect to the identity function at $\{x_0, ..., x_n\}$.*

Proof of Lemma. We proceed by induction on n. Since $u_1(x) = w_1(x) - w_1(c)$, the assertion is trivially true for $n = 1$.

To prove the inductive step we proceed as follows: Let $v_0 :\equiv 1$ and, for $k = 2, ..., v_{k-1}(x) := \int_c^x \int_c^{t_1} \cdots \int_c^{t_{k-2}} dw_k(t_{k-1}) \cdots dw_2(t_1)$ if $n > 2$, or $v_1(x) := \int_c^x dw_2(t)$ if $n = 2$. Since $u_k(x) = \int_c^x v_{k-1}(t)\, dw_1(t)$, subtracting from each column the preceding one, we readily deduce that $\det[u_i(x_j);\ i, j = 0, ..., n] = \int_{x_0}^{x_1} \cdots \int_{x_{n-1}}^{x_n} \det[v_i(t_j);\ i, j = 0, ..., n-1]\, dw_1(t_{n-1}) \cdots dw_1(t_0)$. Since the functions $w_i(x)$ are continuous, and $\det[v_i(t_j);\ i, j = 0, ..., n-1] \geq 0$ for any choice of points $a < t_0 < \cdots < t_{n-1} < b$, it is clear that $\det[u_i(x_j);\ i, j = 0, ..., n] > 0$ if there are points t_j, $x_j < t_j < x_{j+1}$; $j = 0, ..., n-1$, such that $\det[v_i(t_j);\ i, j = 0, ..., n-1] > 0$ and $w_1(t)$ is not constant in a neighborhood of t_j, for $j = 0, ..., n-1$. Also the converse is true. To see this we argue as follows: Let $I := [x_0, x_1] \times [x_1, x_2] \times \cdots \times [x_{n-1}, x_n]$, $\mathbf{t} := (t_0, t_1, ..., t_{n-1})$, and $f(\mathbf{t}) := \det[v_i(t_j);\ i, j = 0, ..., n-1]$. Assume that for every $\mathbf{t} \in I$ either $f(\mathbf{t}) = 0$ or $w_1(x)$ is constant in a neighborhood of some component t_j of $\mathbf{t}$. If A is the set of points $\mathbf{t}$ in I for which $f(\mathbf{t}) > 0$, it is clear that

$$0 \leqslant \int_{x_0}^{x_1} \int_{x_1}^{x_2} \cdots \int_{x_{n-1}}^{x_n} f(t_0, ..., t_{n-1})\, dw_1(t_{n-1}) \cdots dw_1(t_0)$$

$$= \int_A f(t_0, ..., t_{n-1})\, dw_1(t_{n-1}) \cdots dw_1(t_0).$$

Let $(t_0, ..., t_{n-1}) \in A$. Then there is an $\varepsilon > 0$ and some j, $0 \leqslant j \leqslant n-1$, such that $w_1(t)$ is constant on $[t_j - \varepsilon, t_j + \varepsilon]$. If $J(\mathbf{t}, \varepsilon) := [t_0 - \varepsilon, t_0 + \varepsilon] \times [t_1 - \varepsilon, t_1 + \varepsilon] \times \cdots \times [t_{n-1} - \varepsilon, t_{n+1} + \varepsilon]$ and $I(\mathbf{t}, \varepsilon) := I \cap J(t, \varepsilon)$, it is clear that

$$\int_{I(t,\varepsilon)} f(\mathbf{t})\, dw_1(t_{n-1}) \cdots dw_1(t_0) = 0.$$

The sets $I(t, \varepsilon)$ form a covering of A, and therefore have a denumerable subcovering, say $\{I(m);\ m = 1, 2, 3, ...\}$. Since

$$0 \leqslant \int_A f(\mathbf{t})\, dw_1(t_{n-1}) \cdots dw_1(t_0)$$

$$\leqslant \sum \int_{I(m)} f(\mathbf{t})\, dw_1(t_{n-1}) \cdots dw_1(t_0) = 0,$$

we have shown that $\det[u_i(x_j);\ i, j = 0, ..., n] = 0$. The proof of the Lemma now readily follows by the inductive hypotheses. Q.E.D.

Proof of Theorem 2. To prove the necessity, assume that z_{n+1} is an extension to Z_n. Then $Z_{n+1} := Z_n \cup \{z_{n+1}\}$ is a Markov system on B, and Theorem 1 yields the existence of a representation (h, c, W_{n+1}, U_{n+1}) for Z_{n+1} on B such that $W_{n+1} := \{w_1, ..., w_{n+1}\}$ satisfies property (M) with respect to h. Thus a fortiori $W_n := \{w_1, ..., w_n\}$ satisfies property (N) with respect to h.

To prove the sufficiency, let (h, c, W_n, U_n) be a representation for Z_n in B such that W_n satisfies property (N) with respect to h, let $w^*_{n+1}(t) :=$ arctan t, $W^*_{n+1} := \{w_1, ..., w_n, w^*_{n+1}\}$, and $u^*_{n+1}(x) := u_0(x) \int_c^{h(x)} \int_c^{t_1} \cdots \int_c^{t_n} dw^*_{n+1}(t_{n+1})\, dw_n(t_n) \cdots dw_1(t_1)$. Since $w^*_{n+1}(t)$ is strictly increasing, it is readily seen that W^*_{n+1} satisfies property (M) with respect to h. Applying the Lemma, we therefore conclude that u^*_{n+1} is adjoined to $S(U_n)$ on B.

Assume now that $b := \sup(A) \in A$. Since $w^*_{n+1}(t)$ is bounded, we have $u^*_{n+1}(x) \leqslant [w^*_{n+1}(h(x)) - w^*_{n+1}(c)]u_n(x) \leqslant K$ for every x such that $h(x) > c$; thus, $u^*_{n+1}(b) := \lim_{x \to b} u^*_{n+1}(x)$ exists, and the continuity of the elements of $S(U_n)$ implies that $U^*_{n+1} := U_n \cup \{u_{n+1}\}$ is a weak Čebyšev system on $B \cup \{b\}$.

We claim that U^*_{n+1} is a Čebyšev system on $B \cup \{b\}$. Suppose the contrary; then there is a $u \in U^*_{n+1} \setminus \{0\}$ with $n+2$ zeros $x_0, ..., x_{n+1} \in B \cup \{b\}$, say $x_0 < \cdots < x_{n+1}$, and so $x_{n+1} = b$. Let $q \in A \cap (x_n, x_{n+1})$ be fixed, and without loss of generality, assume that $u(q) > 0$. Let $\{p_k\}$ be an increasing sequence in B with $\lim_{k \to \infty} p_k = b$. So for sufficiently large k, we have $q < p_k < b$ and $u(q) > u(p_k)$. Thus, using the terminology of [13, Chap. 8], $x_0, ..., x_n, q, p_k$ form a weak oscillation of u of length $n+3$, in contradiction to Lemma 8.7a in [13].

Analogously, if $a := \inf(A) \in A$, then $u^*_{n+1}(a) := \lim_{x \to a} u^*_{n+1}(x)$ exists, and U^*_{n+1} is a weak Čebyšev system on A. A trivial modification of the argument for $B \cup \{b\}$ now yields that U^*_{n+1} is a Čebyšev system on A.

Q.E.D.

Proof of Theorem 5. Let $B := (\inf(A), \sup(A)) \cap A$. From, e.g., [9], we know that Z_n is a Markov space on B. Let $U_n := \{u_0, ..., u_n\}$ be a Markov basis of Z_n on B. Applying [14, Corollary 3] we conclude that U_n has a representation (h, c, W_n, V_n) such that the functions in W_n are strictly increasing in $(\inf h(B), \sup h(B))$. It is therefore clear that this representation satisfies property (N), and therefore Theorem 3 yields the existence of an adjoined function v for $S(V_n)$, whence the conclusion readily follows.

Q.E.D.

3. Example

Let $I := (0, 5)$, $A := (0, 1] \cup \{2, 3\} \cup [4, 5)$,

$$w_1(t) := \begin{cases} t, & 0 < t < 2.25 \\ 2.25, & 2.25 < t \leqslant 2.75 \\ t - 0.5, & 2.75 < t < 5 \end{cases} \qquad w_2(t) := \begin{cases} 4t, & 0 < t \leqslant 1 \\ 4, & 1 < t \leqslant 2.25 \\ 4t - 5, & 2.25 < t \leqslant 2.75 \\ 6, & 2.75 < t \leqslant 4 \\ 4t - 10, & 4 < t < 5 \end{cases}$$

$u_0 :\equiv 1$, $u_1(x) := \int_1^x dw_1(t)$, $u_2(x) := \int_1^x \int_1^t dw_2(s)\, dw_1(t)$, and $U_2 := \{u_0, u_1, u_2\}$. Since for every choice of points $x_0 < x_1 < x_2$ in A there are points t_0, t_1, $x_0 < t_0 < x_1 < t_1 < x_2$, such that $w_1(t)$ is increasing at t_0 and t_1, and $w_2(t_0) < w_2(t_1)$, it is clear that W_2 satisfies property (M) with respect to the identity function. Thus, from Theorem 1 we deduce that U_2

is a Markov system on A. Note, however, that since $w_2(1)=w_2(2)$, W_2 is not strictly increasing on A. It is also easy to see that W_2 does not satisfy property (N) (choose, for example, $x_0=1$, $x_1=2$, $x_2=3$, and $x_3=4$). We shall now show that U_2 has another representation on A, for which property (N) is satisfied.

A straightforward computation shows that

$$u_1(x)=\begin{cases} x-1, & 0<x\leqslant 2.25\\ 1.25, & 2.25<x\leqslant 2.75\\ x-1.5, & 2.75<x<5\end{cases}$$

and

$$u_2(x)=\begin{cases} 2(x-1)^2, & 0<x\leqslant 1\\ 0, & 1<x\leqslant 2.75\\ 2x-5.5, & 2.75<x\leqslant 4\\ 2x^2-14x+26.5, & 4<x<5.\end{cases}$$

Let $v_0:\equiv 1$,

$$v_1(x):=\begin{cases} x-1, & 0<x\leqslant 2\\ 0.5x, & 2<x\leqslant 3\\ x-1.5, & 3<x<5\end{cases}$$

and

$$v_2(x:=\begin{cases} 2(x-1)^2, & 0<x\leqslant 1\\ 0, & 1<x\leqslant 2\\ 0.5(x-2), & 2<x\leqslant 3\\ 2x-5.5, & 3<x\leqslant 4\\ 2x^2-14x+26.5, & 4<x<5.\end{cases}$$

The functions v_i have been obtained by considering the restrictions of the u_i to A, and extending these restrictions to (0, 5) by linear interpolation. It is therefore clear that $V_2:=\{v_0, v_1, v_2\}$ is a normalized weak Markov system on (0, 5). It is also clear that V_2 is END.

Repeating the procedure outlined in the proof of [11, Theorem 1] we see that V_2 can be represented on (0, 5) as

$$v_1(x)=\int_1^{h(x)} dp_1(t), \qquad v_2(x)=\int_1^{h(x)}\int_1^t dp_2(s)\,dp_1(t),$$

where

$$h(x) := \begin{cases} x, & 0 < x \leqslant 2 \\ x+1, & 2 < x \leqslant 3 \\ x+2, & 3 < x < 5 \end{cases} \qquad p_1(x) := \begin{cases} x-1, & 0 < x \leqslant 2 \\ 1, & 2 < x \leqslant 3 \\ 0.5(x-1), & 3 < x \leqslant 4 \\ 1.5, & 4 < x \leqslant 5 \\ x-3.5, & 5 < x < 7 \end{cases}$$

and

$$p_2(x) := \begin{cases} 4(x-1), & 0 < x \leqslant 1 \\ 0, & 1 < x \leqslant 2 \\ x-2, & 2 < x \leqslant 3 \\ 1, & 3 < x \leqslant 4 \\ x-3, & 4 < x \leqslant 5 \\ 2, & 5 < x \leqslant 6 \\ 4x-22, & 6 < x < 7. \end{cases}$$

(This assertion can, of course, be verified directly.) It is readily seen that $P_2 := \{p_1, p_2\}$ satisfies property (N) with respect to h. We have therefore shown that a Markov system may have a representation for which property (N) is not satisfied, and a different representation for which property (N) is satisfied.

From Theorem 3 we deduce that $S(U_n)$ has an adjoined function on A. Since A does not satisfy property (B), this example shows that although the conditions of Theorem 5 are sufficient, they are not necessary.

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