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Note

Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs

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Abstract

We prove the nonexistence of a distance-regular graph with intersection array $\{74, 54, 15; 1, 9, 60\}$ and of distance-regular graphs with intersection arrays

$$\{4r^3 + 8r^2 + 6r + 1, 2r(r + 1)(2r + 1), 2r^2 + 2r + 1; 1, 2r(r + 1), (2r + 1)(2r^2 + 2r + 1)\}$$

with r an integer and $r \geqslant 1$. Both cases serve to illustrate a technique which can help in determining structural properties for distance-regular graphs and association schemes with a sufficient number of vanishing Krein parameters.

Keywords: Krein parameters; Association scheme; Distance-regular graph; Strongly regular graph; Triple intersection numbers

1. Introduction

In [3] Cameron, Goethals and Seidel prove that a strongly regular graph for which one of the Krein parameters q_{11}^1 or q_{22}^2 is equal to zero must have subconstituents that are also strongly regular. A similar result by Taylor and Levingston [8] for antipodal distance-regular graphs of diameter 3, states that $q_{33}^3 = 0$ implies that the first subconstituent of every vertex of the graph must be strongly regular, cf. Godsil [5]. (We follow the standard convention of numbering the eigenvalues $\theta_0, \dots, \theta_d$ of a graph in such a way that $\theta_0 > \theta_1 > \cdots > \theta_d$. The Krein parameters q_{ij}^k use the same numbering.) Jurišić and Koolen [6] proved that the same result must hold for

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antipodal distance-regular graphs of diameter 4 for which $q_{11}^4 = 0$. Furthermore, the latter result is just a particular instance of a property of tight graphs which were introduced by Jurišić, Koolen and Terwilliger [7] and satisfy the same property.

In all these cases, the fact that a certain Krein parameter is zero provides additional information concerning the structure of the graph. In particular, one obtains extra relations among the triple intersection numbers (defined below), such as in the case of 3-tuple regular graphs discussed by Cameron and van Lint [4].

In this paper, we discuss a method which allows one to draw similar conclusions for general association schemes and distance-regular graphs, provided that a sufficient number of Krein parameters are zero. As illustrations of this technique, we shall prove the following two results:

Theorem 1. *A distance-regular graph with intersection array* $\{74, 54, 15; 1, 9, 60\}$ *does not exist.*

Theorem 2. *Let r be an integer greater than* 0. *Then, a distance-regular graph with intersection array*

$$\begin{aligned}&\{4r^3 + 8r^2 + 6r + 1, 2r(r + 1)(2r + 1), 2r^2 + 2r + 1;\\ &\quad 1, 2r(r + 1), (2r + 1)\bigl(2r^2 + 2r + 1\bigr)\bigr\}\end{aligned} \tag{1}$$

does not exist.

The parameter sets pass all standard feasibility conditions.

To the best of our knowledge the nonexistence of distance-regular graphs with these parameters has not been previously established, except in the case of $r = 1$, with intersection array $\{19, 12, 5; 1, 4, 15\}$, where nonexistence was established first by A. Neumaier, see [2, Section 5.5A] (lecture held at CWI, Amsterdam, September 9, 1989).

Our methods bear some similarity to those of P. Terwilliger in [9,10] where he derives certain structural properties of Q-polynomial association schemes.

2. Preliminaries

Let us first review some of the theory of association schemes. We follow the notation and terminology of Chapter 2 of [1], to which we also refer for definitions and further information.

Consider a (symmetric) association scheme Ω over a finite set of vertices X with relations $R_0, R_1, \ldots, R_d$ and intersection numbers p_{ij}^k, $0 \leqslant i, j, k \leqslant d$. (We shall use the notation aR_ib as an alternative to $(a, b) \in R_i$.) Write $n \stackrel{\text{def}}{=} |X|$ for the order of the association scheme and $n_i \stackrel{\text{def}}{=} p_{ii}^0$ for the valency of the relation R_i.

The $\{0, 1\}$-adjacency matrix of relation R_i shall be denoted by A_i. The Bose–Mesner algebra generated by the A_i, $i = 0, 1, \ldots, d$, has a basis of minimal idempotents $E_0, \ldots, E_d$. We write $E_j(a, b)$ for the entry of E_j on the row–column position that corresponds to two vertices $a, b \in X$.

Let P_{ij} denote the ith eigenvalue of A_j (occurring with multiplicity f_i) where we order the corresponding eigenspaces in such a way that $P_{01} > P_{11} > \cdots > P_{d1}$. The $(d + 1) \times (d + 1)$ matrix P is called the *eigenmatrix* of Ω. The *dual eigenmatrix* Q is the matrix which is obtained from the expansion

$$nE_j = \sum_{i=0}^{d} Q_{ij} A_i \tag{2}$$

of minimal idempotents in terms of adjacency matrices. These matrices are known to satisfy $PQ = nI$ (where I denotes the identity matrix). Note that this equation implies $E_i(a,x) = Q_{\ell i}/n$, where R_ℓ is the unique relation of Ω such that $aR_\ell x$.

We may also compute a set of dual intersection numbers q_{ij}^k (also called *Krein parameters*) which have a similar relation with the dual eigenmatrices Q_{ij} as have the intersection numbers p_{ij}^k with the eigenmatrices P_{ij}. We have

$$nf_k q_{ij}^k = \sum_{\ell=0}^{d} n_l Q_{\ell i} Q_{\ell j} Q_{\ell k}, \tag{3}$$

by [1, Theorem 2.3.2]. We shall be particularly interested in the case $q_{ij}^k = 0$. In that case the relation (3) implies that also $q_{ik}^j = 0$ and $q_{jk}^i = 0$.

It is important to note that the values of n_i, f_i, P_{ij}, Q_{ij} and q_{ij}^k can all be computed from the parameters p_{ij}^k without the use of extra structure information for the particular association scheme.

3. Triple intersection numbers and Krein parameters

We introduce some further notation. Let $a_1, \ldots, a_k \in X$, $r_1, \ldots, r_k \in \{0, \ldots, d\}$. Then $\left[\begin{smallmatrix} a_1 & \ldots & a_k \\ r_1 & \ldots & r_k \end{smallmatrix}\right]$ will denote the number of vertices $x \in X$ such that $xR_{r_i}a_i$ for all $i = 1, \ldots, k$. Note that this symbol is invariant under permutations of its columns.

The definition of an association scheme immediately proves

$$\begin{bmatrix} a \\ i \end{bmatrix} = n_i, \qquad \begin{bmatrix} a & b \\ i & j \end{bmatrix} = p_{ij}^\ell, \quad \text{whenever } aR_\ell b, \tag{4}$$

for given $a, b \in X$, $i, j \in \{0, \ldots, d\}$.

In this paper we shall be mainly interested in the case $k = 3$. From now on we shall fix three vertices $a, b, c \in X$ and write $[i\ j\ k]$ for $\left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right]$ when no confusion may arise. Let A, B, C be such that $aR_C bR_A cR_B a$. Although in general it is not possible to find a formula like (4) for $[i\ j\ k]$, in particular cases some properties may still be derived.

For example, when $i = 0$, we easily find that there is exactly one vertex x such that $xR_0 a$, namely a, and hence $[0\ j\ k]$ is either 0 or 1. A similar argument can be used when $j = 0$ or $k = 0$. More specifically

$$\begin{aligned}
[0\ j\ k] &= \begin{cases} 1, & \text{when } j = C,\ k = B, \\ 0, & \text{otherwise,} \end{cases} \\
[i\ 0\ k] &= \begin{cases} 1, & \text{when } i = C,\ k = A, \\ 0, & \text{otherwise,} \end{cases} \\
[i\ j\ 0] &= \begin{cases} 1, & \text{when } i = B,\ j = A, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{5}$$

If we fix i, j and count all vertices x such that $xR_i a$ and $xR_j b$, irrespective of the relation between x and c, we obtain a total of p_{ij}^C vertices. This fact (and its counterparts when j, k or k, i are fixed) can be expressed as follows:

$$\begin{aligned}
[i\ j\ 0] + [i\ j\ 1] + \cdots + [i\ j\ d] &= p_{ij}^C, \\
[0\ j\ k] + [1\ j\ k] + \cdots + [d\ j\ k] &= p_{jk}^A, \\
[i\ 0\ k] + [i\ 1\ k] + \cdots + [i\ d\ k] &= p_{ik}^B.
\end{aligned} \tag{6}$$

(We remind the reader that each of the symbols $[i\ j\ k] = \left[\begin{smallmatrix} a & b & c \\ i & j & k \end{smallmatrix}\right]$ in the equations above is defined with respect to the *same* fixed triple a, b, c of vertices.)

Identities (5)–(6) can be interpreted as a system of equations in the $(d+1)^3$ unknowns $[i\ j\ k]$. Although in general there are fewer equations than unknowns, in some special cases this system can still be solved, or at least provide some additional structural information about the association scheme. In subsequent sections we shall use this fact to prove that association schemes with certain sets of intersection numbers cannot exist.

For instance, when the association scheme corresponds to a distance-regular graph, and even more so when it is antipodal or bipartite, a lot of the intersection numbers p_{ij}^k are zero. It then may occur that the right-hand side of one of Eq. (6) is zero. In that case also each term on the left-hand side of this equation is zero, for every $[i\ j\ k]$ must be a nonnegative integer. In the next section we shall consider this in more detail for the special case of a distance-regular graph of diameter 3 and a triple a, b, c that forms a triangle. It will turn out that in that case, each of the 64 unknowns can be expressed in terms of only two parameters.

More can be achieved when also a Krein parameter q_{ij}^k happens to be zero:

Theorem 3. *Consider a d-class association scheme Ω over a set of vertices X with dual eigenmatrix Q and Krein parameters q_{ij}^k for $0 \leqslant i, j, k \leqslant d$. For $a, b, c \in X$ define*

$$S_{ijk}(a,b,c) \stackrel{\text{def}}{=} \sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{tk} \begin{bmatrix} a & b & c \\ r & s & t \end{bmatrix}. \tag{7}$$

Then $S_{ijk}(a,b,c) = 0$ whenever $q_{ij}^k = q_{ik}^j = q_{jk}^i = 0$.

Proof. The well-known *Krein conditions* [1, Theorem 2.3.2] state that $q_{ij}^k \geqslant 0$ with equality if and only if

$$\sum_{x \in X} E_i(a,x) E_j(b,x) E_k(c,x) = 0, \quad \text{for all } a, b, c \in X.$$

We regroup the elements x in this sum according to their relations with a, b and c. If $x R_r a$, $x R_s b$ and $x R_t c$, then x contributes a value to the sum equal to $Q_{ri} Q_{sj} Q_{tk}/n^3$ and there are $\left[\begin{smallmatrix} a & b & c \\ r & s & t \end{smallmatrix}\right]$ vertices x of this kind. The sum therefore simplifies to $S_{ijk}(a,b,c)/n^3$, proving the theorem. □

We shall write S_{ijk} instead of $S_{ijk}(a,b,c)$ when a, b, c are clear from context.

Note that the identity $S_{ijk} = 0$ can be interpreted as yet another equation in the same set of unknowns as before, and in general (when $i, j, k \neq 0$) it turns out to be independent of Eqs. (5)–(6).

4. The special case of a distance-regular graph of diameter 3 and a triangle of reference points

Theorems 1 and 2 both concern distance-regular graphs of diameter 3. To prove these theorems we shall consider the special case where the reference vertices a, b and c form a triangle, i.e., $A = B = C = 1$ or equivalently $d(a,b) = d(b,c) = d(c,a) = 1$.

In this case we have the additional identities $p_{ij}^1 = 0$ whenever $|i-j| > 1$ (this is the triangle inequality). As a consequence of (6) we immediately see that $[i\ j\ k] = 0$ as soon as $|i-j| > 1$, $|j-k| > 1$ or $|k-i| > 1$. In other words, the 'unknowns' with nontrivial values are of the form $[i\ i\ i]$, $[i\ i\ i-1]$ or $[i\ i\ i+1]$ (or permutations of these).

Moreover, consider the following special cases of (6), with $0 \leqslant i < 3$:

$$\begin{aligned}
[i\ i+1\ i] + [i\ i+1\ i+1] &= p^1_{i,i+1},\\
[i\ i\ i+1] + [i\ i+1\ i+1] &= p^1_{i,i+1},\\
[i\ i\ i+1] + [i+1\ i\ i+1] &= p^1_{i,i+1}.
\end{aligned}$$

It follows that $[i\ i\ i+1] = [i\ i+1\ i]$ (and by a similar argument $[i\ i+1\ i] = [i+1\ i\ i]$) and $[i\ i+1\ i+1] = [i+1\ i\ i+1]$ $(=[i+1\ i+1\ i])$. As a result, the 64 variables of the system of Eqs. (5)–(6) have already been reduced to a more manageable number. We also know that $[0\ 0\ 0] = [0\ 0\ 1] = [0\ 1\ 0] = [1\ 0\ 0] = 0$ and $[0\ 1\ 1] = [1\ 0\ 1] = [1\ 1\ 0] = 1$ by (5).

Using all this information we obtain the following expressions for the remaining variables in terms of only two parameters $E = [1\ 1\ 1]$ and $F = [3\ 3\ 3]$:

$$\begin{aligned}
[0\ 1\ 1] = [1\ 0\ 1] = [1\ 1\ 0] &= 1,\\
[1\ 1\ 1] &= E,\\
[2\ 1\ 1] = [1\ 2\ 1] = [1\ 1\ 2] &= p^1_{11} - [0\ 1\ 1] - [1\ 1\ 1] = p^1_{11} - 1 - E,\\
[2\ 2\ 1] = [2\ 1\ 2] = [1\ 2\ 2] &= p^1_{12} - [2\ 1\ 1] = p^1_{12} - p^1_{11} + 1 + E,\\
[3\ 3\ 3] &= F,\\
[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] &= p^1_{33} - [3\ 3\ 3] = p^1_{33} - F,\\
[3\ 2\ 2] = [2\ 3\ 2] = [2\ 2\ 3] &= p^1_{23} - [3\ 3\ 2] = p^1_{23} - p^1_{33} + F,\\
[2\ 2\ 2] &= p^1_{22} - [2\ 2\ 1] - [2\ 2\ 3] = p^1_{33} - p^1_{23} + p^1_{22} - p^1_{12} + p^1_{11}\\
&\quad -1 - E - F.
\end{aligned} \tag{8}$$

(Any variable $[i\ j\ k]$ not listed here has value 0.)

5. Proof of Theorem 1

Consider a distance-regular graph with intersection array $\{74, 54, 15; 1, 9, 60\}$. The corresponding intersection parameters p^1_{ij} and eigenmatrix P are as follows (standard techniques to compute these values are described in [1, Section 4.1]):

$$\left(p^1_{ij}\right) = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 19 & 54 & 0\\ 0 & 54 & 300 & 90\\ 0 & 0 & 90 & 21 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 74 & 444 & 111\\ 1 & 20 & -6 & -15\\ 1 & -1 & -6 & 6\\ 1 & -10 & 24 & -15 \end{pmatrix}.$$

The graph has diameter 3 and 630 vertices. The distance graphs Γ_2 and Γ_3 are feasible strongly regular graphs (their parameters are respectively

$$\mathrm{SRG}(630, 111, 12, 21) \quad \text{and} \quad \mathrm{SRG}(630, 444, 318, 300),$$

while the nontrivial eigenvalues and their multiplicities are respectively $6^{444}, -15^{185}$ and $24^{111}, -6^{518}$).

As $p^1_{11} = 19 > 0$ we know that at least one triangle abc exists, so we may apply the results of the previous section. Eq. (8) yields

$$\begin{aligned}
[0\ 1\ 1] = [1\ 0\ 1] = [1\ 1\ 0] &= 1,\\
[1\ 1\ 1] &= E,\\
[2\ 1\ 1] = [1\ 2\ 1] = [1\ 1\ 2] &= 18 - E,\\
[2\ 2\ 1] = [2\ 1\ 2] = [1\ 2\ 2] &= 36 + E,\\
[3\ 3\ 3] &= F,\\
[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] &= 21 - F,\\
[3\ 2\ 2] = [2\ 3\ 2] = [2\ 2\ 3] &= 69 + F,\\
[2\ 2\ 2] &= 195 - E - F.
\end{aligned}$$

(Any variable $[i\ j\ k]$ not listed here has value 0.)

We want to determine the relation between E and F using the additional equalities (3). For this purpose we need to find out which of the Krein parameters are zero, and hence we must compute the dual eigenmatrix Q.

It can easily be verified that $P^2 = 630I$, and therefore $P = Q$ (a scheme with this property is called *formally self-dual*). As an immediate consequence we find that $p^k_{ij} = q^k_{ij}$ for all $0 \leqslant i, j, k \leqslant 3$, and in particular that $q^3_{11} = 0$.

We may now apply Theorem 3 to obtain

$$S_{113} = \sum Q_{r1} Q_{s1} Q_{t3} [r\ s\ t] = 0.$$

For computational purposes it is more convenient to consider $S_{113}/3$ because Q_{t3} is an integral multiple of 3 for every t. (The last column of $P = Q$ is divisible by 3.)

Expanding $S_{113}/3$ yields

$$\begin{aligned}
S_{113}/3 = {} & -7400[0\ 1\ 1] - 7400[1\ 0\ 1] + 14800[1\ 1\ 0] - 2000[1\ 1\ 1]\\
& + 100[2\ 1\ 1] + 100[1\ 2\ 1] + 800[1\ 1\ 2]\\
& - 5[2\ 2\ 1] - 40[2\ 1\ 2] - 40[1\ 2\ 2] + 2[2\ 2\ 2]\\
& + 20[3\ 2\ 2] + 20[2\ 3\ 2] - 5[2\ 2\ 3]\\
& + 200[3\ 3\ 2] - 50[3\ 2\ 3] - 50[2\ 3\ 3] - 500[3\ 3\ 3]\\
= {} & 0 - 2000E + 1000(18 - E) - 85(36 + E)\\
& + 2(195 - E - F) + 35(69 + F) + 100(21 - F) - 500F\\
= {} & 19845 - 3087E - 567F = 63(315 - 49E - 9F).
\end{aligned}$$

It follows that $49E + 9F = 315$. Reducing this equation modulo 9 proves that E must be a multiple of 9, and as $49 \cdot 9 = 441 > 315$, it follows that $E = 0$, $F = 35$ is the only solution in nonnegative integers. But $F = [3\ 3\ 3] = 35$ contradicts $[3\ 3\ 3] \leqslant p^1_{33} = 21$.

6. Proof of Theorem 2, for the case $r > 1$

Consider a distance-regular graph Γ with an intersection array given by (1).

A graph with these parameters has diameter 3 and order $n = 4(2r + 1)(r + 1)^3$.

The valencies $n_0, \ldots, n_3$ are as follows:

$$n_0 = 1, \qquad n_1 = 4r^3 + 8r^2 + 6r + 1, \qquad n_2 = (2r + 1)n_1, \qquad n_3 = n_1.$$

From the intersection array we may compute the intersection numbers p^1_{ij}:

$$(p^1_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2r(r+2) & 2r(r+1)(2r+1) & 0 \\ 0 & 2r(r+1)(2r+1) & 2r(2r+1)(2r^2+2r+1) & (2r+1)(2r^2+2r+1) \\ 0 & 0 & (2r+1)(2r^2+2r+1) & 2r(r+1) \end{pmatrix}.$$

As before, because $p^1_{11} = 2r(r+2) > 0$, at least one triangle exists, and Eqs. (8) are valid. Substituting the appropriate values for p^1_{ij}, we obtain

$$\begin{aligned}
[0\ 1\ 1] = [1\ 0\ 1] = [1\ 1\ 0] &= 1, \\
[1\ 1\ 1] &= E, \\
[2\ 1\ 1] = [1\ 2\ 1] = [1\ 1\ 2] &= 2r^2 + 4r - 1 - E, \\
[2\ 2\ 1] = [2\ 1\ 2] = [1\ 2\ 2] &= 4r^3 + 4r^2 - 2r + 1 + E, \\
[3\ 3\ 3] &= F, \\
[3\ 3\ 2] &= 2r(r+1) - F, \\
[3\ 2\ 2] = [2\ 3\ 2] = [2\ 2\ 3] &= 4r^3 + 4r^2 + 2r + 1 + F, \\
[2\ 2\ 2] &= 2(r+1)(2r-1)\big(2r^2+1\big) - E - F.
\end{aligned} \tag{9}$$

The eigenmatrix P has the following value (we again refer to [1] for information on how to compute this):

$$P = \begin{pmatrix} 1 & 4r^3+8r^2+6r+1 & (2r+1)(4r^3+8r^2+6r+1) & 4r^3+8r^2+6r+1 \\ 1 & 2r^2+4r+1 & -2r-1 & -2r^2-2r-1 \\ 1 & -1 & -2r-1 & 2r+1 \\ 1 & -2r^2-2r-1 & (2r+1)^2 & -2r^2-2r-1 \end{pmatrix}.$$

As with the graph on 630 vertices of Theorem 1, also here the distance graphs Γ_2 and Γ_3 are feasible strongly regular graphs.

It can be verified that $P^2 = nI$. Therefore $P = Q$ and the scheme is again formally self-dual, so as an immediate consequence we find that $q^k_{ij} = p^k_{ij}$ for all $i, j, k = 1$. In particular $q^1_{13} = q^1_{31} = q^3_{11} = 0$ and hence, by Theorem 3, $S_{113} = S_{131} = S_{311} = 0$. Handing these equations, together with (9), to a computer algebra system, we obtain

$$(r+1)^2E + r^2F = (r+1)\big(2r^2+2r-1\big). \tag{10}$$

We now have sufficient information to prove Theorem 2 in case $r > 1$. Note that in (10) both $E = [1\ 1\ 1]$ and $F = [3\ 3\ 3]$ must be nonnegative integers. As an immediate consequence we see that F must be an integral multiple of $r+1$, say $F = (r+1)f$, with f a nonnegative integer, and then

$$(r+1)E + r^2 f = 2r^2 + 2r - 1.$$

Reducing this modulo $r+1$, we find $f = -1 \bmod r+1$, say $f = g(r+1) - 1$ for integral $g > 0$. It follows that

$$E + r^2 g = 3r - 1. \tag{11}$$

When $r \geqslant 3$ then $r^2 g \geqslant r^2 \geqslant 3r$ and hence this has no solution for $E > 0$.

When $r = 2$ there is exactly one solution $E = 1$, $g = 1$. The equality $E = 1$ proves that every triangle of Γ is contained in exactly one 4-clique, or equivalently, that every edge in the first component of Γ lies in exactly one triangle. When $r = 2$ the first component of Γ is a regular graph of order $4r^3 + 8r^2 + 6r + 1 = 77$ and valency $2r(r + 2) = 16$. Counting the number of triangles in this graph now yields $77 \cdot 16/6$, which is not an integer, proving that Γ cannot exist also when $r = 2$.

As was mentioned in the introduction, the nonexistence of Γ for the case $r = 1$ was already established by A. Neumaier. In the next section we shall give a new proof of this result, first to demonstrate that our techniques can also be useful when the reference triple a, b, c is not a triangle (and hence that the same techniques can also be applied to triangle-free distance-regular graphs), and second for the sake of completeness as the Neumaier result is not easily accessible.

7. Proof of Theorem 2, for the case $r = 1$

The special case of $r = 1$ is a graph Γ with intersection array $\{19, 12, 5; 1, 4, 15\}$. Below we list some of its parameters:

$$\left(p^1_{ij}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 6 & 12 & 0 \\ 0 & 12 & 30 & 15 \\ 0 & 0 & 15 & 4 \end{pmatrix}, \qquad \left(p^2_{ij}\right) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 4 & 10 & 5 \\ 1 & 10 & 36 & 10 \\ 0 & 5 & 10 & 4 \end{pmatrix},$$

$$P = Q = \begin{pmatrix} 1 & 19 & 57 & 19 \\ 1 & 7 & -3 & -5 \\ 1 & -1 & -3 & 3 \\ 1 & -5 & 9 & -5 \end{pmatrix}.$$

We shall prove that the first subconstituent Δ of Γ must be strongly regular with parameters $v = 19$, $k = 6$, $\lambda = 1$ and $\mu = 2$. Because a strongly regular graph with these parameters cannot exist, neither can Γ.

To prove that $\lambda = 1$, i.e., that every edge of Δ belongs to exactly one triangle, we may use a similar argument as in the previous section. Indeed, when $r = 1$ Eq. (11) has the solutions $E = 0$, $g = 2$ (and then $F = 6$) and $E = 1$, $g = 1$ (and then $F = 2$). The first case however would yield $[3\ 3\ 2] = -2 < 0$, a contradiction, proving $E = 1$.

To prove that $\mu = 2$, we apply the techniques of the previous sections to a different starting configuration of vertices a, b, c. For the remainder of this section we shall choose a, b, c such that $d(a, b) = d(a, c) = 1$ and $d(b, c) = 2$, i.e., such that bac induces a path of length 2 in Γ.

As before, all values of $[i\ j\ k]$ can be expressed in terms of $E = [1\ 1\ 1]$ and $F = [3\ 3\ 3]$. Because the configuration of vertices a, b and c lacks the symmetry of the earlier case, we need to do some more work. In particular, we can no longer assume that $[i\ j\ k]$ remains the same when i, j and k are permuted.

We obtain the following expressions for the variables in terms of E and F (as before, any symbol that does not occur in this list has value 0):

$$\begin{aligned} &[0\ 1\ 1] &&= 1, \\ &[1\ 0\ 2] = [1\ 2\ 0] &&= 1, \\ &[1\ 1\ 1] &&= E, \end{aligned}$$

$$\begin{aligned}
[1\ 1\ 2]=[1\ 2\ 1] &= p^1_{11}-[1\ 1\ 1] &&= 6-E,\\
[2\ 1\ 1] &= p^2_{11}-[0\ 1\ 1]-[1\ 1\ 1] &&= 3-E,\\
[1\ 2\ 2] &= p^1_{12}-[1\ 2\ 0]-[1\ 2\ 1] &&= 5+E,\\
[2\ 1\ 2]=[2\ 2\ 1] &= p^2_{12}-[1\ 1\ 2] &&= 4+E,\\
[2\ 1\ 3]=[2\ 3\ 1] &= p^2_{13} &&= 5,\\
[3\ 3\ 3] & &&= F,\\
[3\ 2\ 3]=[3\ 3\ 2] &= p^1_{33}-[3\ 3\ 3] &&= 4-F,\\
[2\ 3\ 3] &= p^2_{33}-[3\ 3\ 3] &&= 4-F,\\
[3\ 2\ 2] &= p^1_{32}-[3\ 2\ 3] &&= 11+F,\\
[2\ 3\ 2]=[2\ 2\ 3] &= p^2_{32}-[3\ 3\ 2] &&= 6+F,\\
[2\ 2\ 2] &= p^1_{22}-[2\ 2\ 3]-[2\ 2\ 1] &&= 20-E-F,\\
&= p^2_{22}-[3\ 2\ 2]-[1\ 2\ 2] &&= 20-E-F.
\end{aligned}$$

Expanding S_{113} in this case yields

$$\begin{aligned}
S_{113} &= -665[0\ 1\ 1]+399[1\ 0\ 2]-133[1\ 2\ 0]-245[1\ 1\ 1]\\
&\quad +147[1\ 1\ 2]+35[1\ 2\ 1]+35[2\ 1\ 1]\\
&\quad -21[1\ 2\ 2]-21[2\ 1\ 2]-5[2\ 2\ 1]+3[2\ 2\ 2]\\
&\quad +35[2\ 1\ 3]-25[2\ 3\ 1]\\
&\quad +15[3\ 2\ 2]+15[2\ 3\ 2]-5[2\ 2\ 3]\\
&\quad +75[3\ 3\ 2]-25[3\ 2\ 3]-25[2\ 3\ 3]-125[3\ 3\ 3]\\
&= -399-245E+182(6-E)+35(3-E)-21(5+E)-26(4+E)\\
&\quad +3(20-E-F)+50+15(11+F)+10(6+F)+50(4-F)\\
&\quad -25(4-F)-125F\\
&= 1024-512E-128F=128(8-4E-F).
\end{aligned}$$

From $S_{113}=0$ it follows that $4E+F=8$ and hence that $[1\ 1\ 1]=E\leqslant 2$.

Choose $a\in\Gamma$ and let b,c belong to the neighbourhood $\Delta=\Gamma(a)$ of a. From the above we know that b,c can have at most 2 common neighbours in Δ. We shall count in two ways the number of edges dc in Δ for which d is a neighbour of b and c is not (and $b\neq c$). As Δ is regular of degree 6 and the edge ad lies in exactly one triangle of Δ, we find that for every of the 6 neighbours d of b there are exactly 4 vertices c that satisfy the conditions. This yields 24 edges.

There are exactly $19-6-1=12$ vertices c of Δ not adjacent to b (and different from b), hence on average for each such c there are $24/12=2$ edges of the required type. However, by the above, there can be *at most* two edges of the required type through each c, hence there must always be *exactly* two such edges. It follows that Δ must be strongly regular with $\mu=2$.

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