



ELSEVIER

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Some simple modules for classical groups and p -ranks of orthogonal and Hermitian geometries

Ogul Arslan^a, Peter Sin^{b,*}

^a Department of Mathematics and Statistics, Coastal Carolina University, P.O. Box 261954, Conway, SC 29528-6054, USA

^b Department of Mathematics, University of Florida, P.O. Box 118105, Gainesville, FL 32611, USA

ARTICLE INFO

Article history:

Received 4 September 2009

Communicated by Leonard L. Scott, Jr.

Keywords:

Classical groups
Simple modules
Incidence matrices
Weyl modules
Orthogonal geometry
Hermitian geometry

ABSTRACT

We determine the characters of the simple composition factors and the submodule lattices of certain Weyl modules for classical groups. The results have several applications. The simple modules arise in the study of incidence systems in finite geometries and knowledge of their dimensions yields the p -ranks of these incidence systems.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

In this paper we study some special cases of the general problem of determining the formal characters of irreducible rational representations of a semisimple algebraic group G over an algebraically closed field k of characteristic $p > 0$. The groups we shall consider are classical groups and the simple modules are related to their standard modules. To be more precise, if we number the fundamental weights so that the first one, ω_1 is the highest weight of the standard module, then simple modules to be studied are those having highest weight λ in the following list.

- (B) For G of type B_ℓ ($\ell \geq 2$), $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;
- (D) For G of type D_ℓ ($\ell \geq 3$), $\lambda = r(\omega_1)$, $0 \leq r \leq p - 1$;
- (A) For G of type A_ℓ ($\ell \geq 3$), $\lambda = r(\omega_1 + \omega_\ell)$, $0 \leq r \leq p - 1$;

* Corresponding author.

E-mail address: sin@math.ufl.edu (P. Sin).

- (A') For G of type A_ℓ ($\ell \geq 4$) $\lambda = \omega_2 + \omega_{\ell-1}$; and
- (A'') For G of type A_4 , $\lambda = (p-2)(\omega_2 + \omega_{\ell-1})$ or $(p-1)(\omega_2 + \omega_{\ell-1})$.

For each weight in this list, our purpose is to give a complete description of the composition factors and the submodule structure of the corresponding Weyl module $V(\lambda)$ or, dually, the induced module $H^0(\lambda)$. The formal characters of these modules are given by Weyl's Character Formula.

We should also mention some weights which do not appear in above list but which play the same role in applications. These are the weights $\lambda = r\omega_1$, $0 \leq r \leq p-1$ for groups of type A_ℓ and C_ℓ . The Weyl modules for these weights are isomorphic to symmetric powers of the standard module and are all simple.

1.1. Statements of results

The main tool we use for examining the deeper structure of Weyl modules is the Jantzen Sum Formula [8, II.8.19]. In general, due to the recursive nature of its application, the formula can be expected to yield only partial information about composition factors. In certain cases one can obtain precise results, as demonstrated in the original paper [9]. The weights in the list above give further examples where complete results can be obtained. For these cases, the combinatorics of the Sum Formula can be kept under control and we obtain accurate bounds which, when supplemented with certain properties of good filtrations and classical facts about tensor products, yield the full submodule structure of the Weyl modules.

We can now state our main results.

Theorem 1.1. *Let G be of type B_ℓ , $\ell \geq 2$. Let ω_1 be the highest weight of the standard orthogonal module of dimension $2\ell + 1$. Assume $0 \leq r \leq p-1$. Then the following hold.*

- (a) $H^0(r\omega_1)$ is simple unless (i) $p = 2$ and $r = 1$ or (ii) $p > 2$ and there exists a positive odd integer m such that

$$r + 2\ell - 1 \leq mp \leq 2r + 2\ell - 2.$$

- (b) If (i) holds then the quotient $H^0(\omega_1)/L(\omega_1)$ is the one-dimensional trivial module.
- (c) If (ii) holds then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 1 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

- (d) We have

$$\dim L(r\omega_1) = \begin{cases} 2\ell, & \text{if (i) holds,} \\ \binom{2\ell+r}{2\ell} - \binom{2\ell+r-2}{2\ell} - \binom{mp-r+1}{2\ell} + \binom{mp-r-1}{2\ell}, & \text{if (ii) holds,} \\ \binom{2\ell+r}{2\ell} - \binom{2\ell+r-2}{2\ell}, & \text{otherwise.} \end{cases}$$

Theorem 1.2. *Let G be of type D_ℓ , $\ell \geq 3$. Let ω_1 be the highest weight of the standard orthogonal module of dimension 2ℓ . Assume $0 \leq r \leq p-1$. Then the following hold.*

- (a) Suppose that there exists a positive even integer m such that

$$r + 2\ell - 2 \leq mp \leq 2r + 2\ell - 3.$$

Then m is unique and

$$H^0(r\omega_1)/L(r\omega_1) \cong H^0(r_1\omega_1),$$

where $r_1 = mp - 2\ell + 2 - r$. Furthermore the module $H^0(r_1\omega_1)$ is simple.

- (b) Otherwise, $H^0(r\omega_1)$ is simple.
- (c) We have

$$\dim L(r\omega_1) = \begin{cases} \binom{2\ell+r-1}{2\ell-1} - \binom{2\ell+r-3}{2\ell-1} - \binom{mp-r+1}{2\ell-1} + \binom{mp-r-1}{2\ell-1}, & \text{in (a),} \\ \binom{2\ell+r-1}{2\ell-1} - \binom{2\ell+r-3}{2\ell-1}, & \text{in (b).} \end{cases}$$

For groups of type A_ℓ we number the fundamental weights be so that ω_i is the highest weight of the i -th exterior power of the $(\ell + 1)$ -dimensional standard module.

Theorem 1.3. Let G be of type A_ℓ , $\ell \geq 3$. Assume $0 \leq r \leq p - 1$. Then the following hold.

- (a) Suppose that there exists a positive integer m such that

$$r + \ell \leq mp \leq 2r + \ell - 1.$$

Then m is unique and

$$H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell)) \cong H^0(r_1(\omega_1 + \omega_\ell)),$$

where $r_1 = mp - \ell - r$. Furthermore the module $H^0(r_1(\omega_1 + \omega_\ell))$ is simple.

- (b) Otherwise, $H^0(r(\omega_1 + \omega_\ell))$ is simple.
- (c) We have

$$\dim L(r(\omega_1 + \omega_\ell)) = \begin{cases} \binom{\ell+r}{\ell}^2 - \binom{\ell+r-1}{\ell}^2 - \binom{mp-r}{\ell}^2 + \binom{mp-r-1}{\ell}^2, & \text{in (a),} \\ \binom{\ell+r}{\ell}^2 - \binom{\ell+r-1}{\ell}^2, & \text{in (b).} \end{cases}$$

Theorem 1.4. Let G be of type A_ℓ , $\ell \geq 4$. If $p > 2$ then the following hold.

- (a) If $\ell \equiv 0 \pmod{p}$ then $H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k$.
- (b) If $\ell \equiv 1 \pmod{p}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong H^0(\omega_1 + \omega_\ell)$$

and this module is simple.

- (c) In all other cases $H^0(\omega_2 + \omega_{\ell-1})$ is simple.

If $p = 2$ then the following hold.

- (d) If $\ell \equiv 0 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong k.$$

(e) If $\ell \equiv 1 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong V(\omega_1 + \omega_\ell).$$

(f) If $\ell \equiv 2 \pmod{4}$ then $H^0(\omega_2 + \omega_{\ell-1})$ is simple.

(g) If $\ell \equiv 3 \pmod{4}$ then

$$H^0(\omega_2 + \omega_{\ell-1})/L(\omega_2 + \omega_{\ell-1}) \cong L(\omega_1 + \omega_\ell).$$

We have

(h)

$$\dim L(\omega_2 + \omega_{\ell-1}) = \begin{cases} \frac{1}{4}\ell^4 + \frac{1}{2}\ell^3 - \frac{3}{4}\ell^2 - 2\ell - 2, & \text{in (a) and (d),} \\ \frac{1}{4}(\ell + 2)(\ell^3 - 7\ell - 2), & \text{in (b) and (e),} \\ \frac{1}{4}(\ell - 2)(\ell + 2)(\ell + 1)^2, & \text{in (c) or (f),} \\ \frac{1}{4}\ell(\ell + 3)(\ell^3 + 2\ell^2 - 7\ell - 16), & \text{in (g).} \end{cases}$$

Theorem 1.5. Let G be of type A_4 .

(a) If $p = 2$, then $H^0((p - 1)(\omega_2 + \omega_3))/L((p - 1)(\omega_2 + \omega_3)) \cong k$.

If $p > 2$, then the following hold.

(b) $H^0((p - 1)(\omega_2 + \omega_3))/L((p - 1)(\omega_2 + \omega_3)) \cong L((p - 2)(\omega_2 + \omega_3))$.

(c) $H^0((p - 2)(\omega_2 + \omega_3))/L((p - 2)(\omega_2 + \omega_3)) \cong H^0((p - 2)(\omega_1 + \omega_4))$, which is simple.

(d) If $p > 2$, then

$$\dim L((p - 1)(\omega_2 + \omega_3)) = \frac{p(p + 1)}{32} \binom{2p + 2}{3}^2 - \frac{p(p - 1)}{32} \binom{2p}{3}^2 + \frac{p}{2} \binom{p + 1}{3}^2.$$

(e) If $p = 2$, then $\dim L(\omega_2 + \omega_3) = 74$.

1.2. Some applications

One motive for studying the simple modules in cases (B), (D) and (A) above comes from problems about incidence matrices arising from Hermitian and orthogonal geometries. Let $q = p^f$ and let $V(q)$ be a vector space of dimension n over the field \mathbb{F}_q of q elements. We assume that $V(q)$ carries a nondegenerate quadratic form. Let \widehat{P} be the set of all one-dimensional subspaces and P be the set of singular one-dimensional subspaces. Let \widehat{P}^* be the set of all $(n - 1)$ -dimensional subspaces in V and P^* the set of polar hyperplanes $\langle v \rangle^\perp$ for $\langle v \rangle \in P$. Then we may consider various incidence systems such as $(\widehat{P}, \widehat{P}^*)$, (P, \widehat{P}^*) and (P, P^*) under the incidence relation of inclusion.

Similarly, if $V(q^2)$ is a vector space of dimension n over \mathbb{F}_{q^2} with a nondegenerate Hermitian form then we can consider the incidence relations of inclusion involving the set \widehat{P} of all one-dimensional subspaces, the set P of singular one-dimensional subspaces, the set \widehat{P}^* of all hyperplanes and the set P^* of polar hyperplanes of singular one-dimensional subspaces.

These incidence relations were studied by Blokhuis and Moorhouse [2] and Moorhouse [12]. We can order the sets \widehat{P} and \widehat{P}^* so that the incidence matrix A_1 of (P, \widehat{P}^*) and the incidence matrix A_{11} of (P, P^*) are submatrices of the incidence matrix A of $(\widehat{P}, \widehat{P}^*)$ in the following way.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_1 = (A_{11} \quad A_{12}). \tag{1}$$

We may consider these 0-1 matrices over any commutative ring with 1 and try to compute their invariants. One such invariant is the p -rank, the rank of the matrix regarded as having entries in the field of p elements. The p -rank of A is well known from [11,15,7]. In [2] and [12] the p -ranks of the A_1 were determined, while the problem of finding the p -ranks of the A_{11} was left open.

These p -ranks may be deduced from our results on simple modules; the results for orthogonal spaces of dimension n are consequences of our results for types B_ℓ with $n = 2\ell + 1$, and types D_ℓ with $n = 2\ell$, while the p -ranks for n -dimensional Hermitian spaces come from our results for type A_ℓ with $n = \ell + 1$.

Theorem 1.6. *Suppose we are in the orthogonal case with $\dim V(q) = n \geq 4$. The following hold.*

(a) *Assume $p = 2$. Then*

$$\text{rank}_p A_{11} = \begin{cases} 1 + n^t, & \text{if } n \text{ is even,} \\ 1 + (n - 1)^t, & \text{if } n \text{ is odd.} \end{cases}$$

(b) *Assume $p > 2$. Then the p -rank depends on whether there exists a positive integer u such that*

$$u \equiv n \pmod{2} \quad \text{and} \quad n - 3 \leq up \leq p + n - 5.$$

If u exists then

$$\text{rank}_p A_{11} = 1 + \left[\binom{n + p - 2}{n - 1} - \binom{n + p - 4}{n - 1} - \binom{up + 2}{n - 1} + \binom{up}{n - 1} \right]^t.$$

Otherwise,

$$\text{rank}_p A_{11} = 1 + \left[\binom{n + p - 2}{n - 1} - \binom{n + p - 4}{n - 1} \right]^t.$$

Remark 1.7. When n is even, there are two types of nondegenerate forms, distinguished by the Witt index. However, the p -rank of A_{11} is the same for both types. The same was also true for A_1 , as shown in [2].

Our proof of Theorem 1.6 will not include the case $n = 4$, but the p -rank is well known in that case and fits our formula.

Theorem 1.8. *Suppose we are in the Hermitian case with $\dim V(q^2) = n \geq 4$. There are two cases for the rank depending on the existence of a positive integer u satisfying*

$$n - 2 \leq up \leq p + n - 3.$$

If u exists then

$$\text{rank}_p A_{11} = 1 + \left[\binom{n+p-2}{n-1}^2 - \binom{n+p-3}{n-1}^2 - \binom{up+1}{n-1}^2 + \binom{up}{n-1}^2 \right]^t.$$

Otherwise,

$$\text{rank}_p A_{11} = 1 + \left[\binom{n+p-2}{n-1}^2 - \binom{n+p-3}{n-1}^2 \right]^t.$$

When $n = 4$ or 5 , the totally isotropic subspaces of dimensions one and two form the points and lines of the Hermitian generalized quadrangle. (See [14] for basics on generalized quadrangles.) The p -rank of the incidence relation of points and lines of this generalized quadrangle is still unknown in general. Each generalized quadrangle has a *dual*, obtained by interchanging the roles of the points and lines, while keeping the same incidence relation. By a polar hyperplane in the dual quadrangle, we mean the set of dual points which are collinear (in the dual sense) with a given dual point. Taking duals does not affect the p -rank of the point-line incidence relation. However, the point-hyperplane incidence relations for a quadrangle and its dual have different p -ranks in general. When $n = 4$, the dual Hermitian generalized quadrangle $DH(3, q^2)$ is isomorphic to the generalized quadrangle $Q(5, q)$ arising from the totally isotropic subspaces of a 6-dimensional quadratic module of Witt index 2, so the p -rank of A_{11} in this case is given by Theorem 1.6, with $\ell = 3$. The modules in case (A'') arise from considering the point-hyperplane incidences of the dual of the Hermitian generalized quadrangle for $n = 5$.

Theorem 1.9. *The p -rank of the point-hyperplane incidence matrix A_{11} for the dual Hermitian generalized quadrangle $DH(4, q^2)$ is as follows.*

(a) If $p > 2$ then

$$\text{rank}_p A_{11} = 1 + \left[\frac{p(p+1)}{32} \binom{2p+2}{3}^2 - \frac{p(p-1)}{32} \binom{2p}{3}^2 + \frac{p}{2} \binom{p+1}{3}^2 \right]^t.$$

(b) If $p = 2$ then $\text{rank}_2 A_{11} = 1 + 74^t$.

The reduction of the p -rank problem to simple modules is achieved by reformulating it in terms of representations of the finite classical group of automorphisms of the given form. This is explained in Section 11.

The simple modules in case (A') were brought to our attention by Pham Huu Tiep, who asked about their dimensions as well as those in (B), (D) and (A) in connection with work on heights of characters in blocks [13]. A. Kleshchev has informed us that Theorem 1.4 is included among the results in the PhD thesis (1992, Moscow State University) of A. Adamovich, where the Weyl modules with highest weight equal to the sum of two fundamental weights are treated. To our knowledge there is no published proof, but the statement is reproduced in [10].

2. Semisimple algebraic groups

We shall use the standard reference [8] for notation and background results in our discussion of algebraic groups and their representations. Let G be a simply connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. Let T be a maximal torus in G and let ℓ denote its rank. Let $(X(T), R, Y(T), R^\vee)$ be the root datum determined by G and T and let W be its Weyl group. Here $X(T) = \text{Hom}(T, k^\times)$ is the group of rational characters of T , $R \subset X(T)$ is the set of roots, $Y(T) = \text{Hom}(k^\times, T)$ and $R^\vee \subset Y(T)$ is the set of coroots. There is a natural pairing $X(T) \times$

$Y(T) \rightarrow \mathbf{Z}$ such that for $\lambda \in X(T)$ and $\phi \in Y(T)$, $\langle \lambda, \phi \rangle$ is the integer such that the endomorphism of k^\times given by $\lambda \circ \phi$ is the map $a \mapsto a^{\langle \lambda, \phi \rangle}$. Let R^+ be a positive system in R and $S = \{\alpha_1, \dots, \alpha_\ell\}$ the corresponding set of simple roots. Let B be a Borel subgroup containing T which corresponds to the set $-R^+$ of negative roots. The simply connectedness of G means that $Y(T) = \mathbf{Z}R^\vee$. Then $X := X(T)$ has a basis consisting of the fundamental weights $\omega_1, \dots, \omega_\ell$, defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $1 \leq i, j \leq \ell$. The weights in the set

$$X_+ = \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0, \forall \alpha \in R^+ \}$$

are called *dominant weights*. They are the nonnegative integer combinations of the fundamental weights. As usual, the half-sum of the positive roots is denoted by ρ . The real vector space $E = X \otimes_{\mathbf{Z}} \mathbf{R}$ can be given an inner product $\langle -, - \rangle$ such that the set R becomes a root system in the sense of [3, 2.1]. Then $Y(T)$ can be identified with the sublattice of E which is dual to X and for each root α , its coroot α^\vee is identified with the element $2\alpha / \langle \alpha, \alpha \rangle$.

3. Weyl modules

The set X_+ parametrizes the simple $G(k)$ -modules, but also two other sets of modules, the Weyl modules $V(\lambda)$ and the induced modules $H^0(\lambda)$. The induced modules are also called dual Weyl modules because each $H^0(\lambda)$ is the dual module of $V(\lambda^*)$, where $\lambda^* = -w_0(\lambda)$ and w_0 is the longest element of W . In the case of the types B_ℓ and D_ℓ , we actually have $\lambda^* = \lambda$. The induced modules $H^0(\lambda)$ have the property that they have a unique simple submodule, and this simple module is $L(\lambda)$. The Weyl modules and induced modules can be defined for Chevalley groups over any algebraically closed field. Their weight multiplicities are independent of the field characteristic and are given by Weyl's Character Formula. The modules $V(\lambda)$ and $H^0(\lambda)$ have the same formal characters. In characteristic 0 they are irreducible. In the case of the types B_ℓ and D_ℓ , it is easy to see that $H^0(\omega_1)$ is the standard orthogonal module, which we denote by V . It is not hard to check using Weyl's dimension formula that for $r \geq 1$, $H^0(r\omega_1)$ is the quotient $S^r(V^*) / f S^{r-2}(V^*)$, where $S^m(V^*)$ is the space of homogeneous polynomials of degree r on V and $f \in S^2(V^*)$ is the quadratic form defining the orthogonal group.

3.1. Weyl module dimensions

Here we give the dimensions of the Weyl modules for the weights (B), (D), (A), (A') and (A'') of the Introduction. For type B_ℓ , $\ell \geq 2$, we have

$$\dim H^0(r\omega_1) = \binom{2\ell + r}{2\ell} - \binom{2\ell + r - 2}{2\ell} \tag{2}$$

and for type D_ℓ , $\ell \geq 3$, we have

$$\dim H^0(r\omega_1) = \binom{2\ell + r - 1}{2\ell - 1} - \binom{2\ell + r - 3}{2\ell - 1}. \tag{3}$$

For type A_ℓ , $\ell \geq 3$, Weyl's formula gives

$$\dim H^0(r(\omega_1 + \omega_\ell)) = \binom{r + \ell}{\ell}^2 - \binom{r + \ell - 1}{\ell}^2, \tag{4}$$

and for $\ell \geq 4$,

$$\dim H^0(r(\omega_2 + \omega_{\ell-1})) = \binom{r + \ell - 2}{r + 1} \binom{r + \ell - 3}{r}^2 \frac{(2r + \ell)(2r + \ell - 1)^2(2r + \ell - 2)}{\ell(\ell - 1)^2(\ell - 2)^3}. \tag{5}$$

4. Jantzen Sum Formula

The Weyl module $V(\lambda)$ has a descending filtration, defined by Jantzen, of submodules $V(\lambda)^i, i > 0$. These submodules are a little mysterious; they are eventually all zero, but it can happen that nonzero terms can be equal. One good property is that

$$V(\lambda)^1 = \text{rad } V(\lambda), \quad \text{so } V(\lambda)/V(\lambda)^1 \cong L(\lambda). \tag{6}$$

Another good property is that we have a formula for the character of the (external) direct sum of the $V(\lambda)^i$, the *Jantzen Sum Formula*:

$$\sum_{i>0} \text{Ch}(V(\lambda)^i) = - \sum_{\alpha>0} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} v_p(mp) \chi(\lambda - mp\alpha). \tag{7}$$

The left hand side is the sum of the characters of the $V(\lambda)^i$. The sign of a Weyl group element is its determinant as a transformation of E . The number $v_p(n)$ is the exponent of p in a prime power factorization of n . Next we explain the meaning of $\chi(\mu)$, for $\mu \in X$. If $\mu \in X_+$ then $\chi(\mu)$ is the character of $V(\mu)$ (and $H^0(\mu)$), given by Weyl’s Character Formula. For general μ , there is a unique Weyl conjugate μ' of $\mu + \rho$ in X_+ . Then $\mu' - \rho$ may or may not be in X_+ . If not then $\chi(\mu) = 0$. If so, then $\chi(\mu) = \text{sign}(w)\chi(\mu' - \rho)$, where $w \in W$ satisfies $w(\mu + \rho) = \mu'$.

Since the characters $\chi(\mu)$ for $\mu \in X_+$ are linearly independent, the Sum Formula will tell us whether $V(\lambda)$ is simple or not. In the case that is not simple, the formula gives incomplete information about the composition factors, because the LHS of (7) *overestimates* the actual weight multiplicities. Nevertheless, there are certain situations where this information is very useful.

The right hand side can be computed by the following algorithm: For each positive root α ,

- (i) Compute $\langle \lambda + \rho, \alpha^\vee \rangle$.
- (ii) Compute $\lambda + \rho - mp\alpha$ for $0 < m < \langle \lambda + \rho, \alpha^\vee \rangle$.
- (iii) Find the Weyl group conjugate $w(\lambda + \rho - mp\alpha)$ in X_+ and note the sign of a Weyl group element w .
- (iv) Compute $w(\lambda + \rho - mp\alpha) - \rho$.
- (v) The contribution to the sum (7) is $-\text{sign}(w)v_p(mp)\chi(w(\lambda + \rho - mp\alpha) - \rho)$. This will be zero if $w(\lambda + \rho - mp\alpha) - \rho$ is not in the dominant Weyl chamber, and is otherwise given by Weyl’s Character Formula.

5. Weyl groups and root systems of types B_ℓ, D_ℓ and A_ℓ

For types B_ℓ and D_ℓ let $e_i, i = 1, \dots, \ell$ be an orthonormal basis of $E = \mathbf{R} \otimes_{\mathbf{Z}} X$. Then the inner product of E becomes the usual dot product.

5.1. Type B_ℓ

The root system R can be taken to consist of the vectors $\pm e_i \pm e_j, 1 \leq i < j \leq \ell$, together with the vectors $\pm e_i, 1 \leq i \leq \ell$. The roots $\pm e_i \pm e_j$ are equal to their coroots, but for $\alpha = e_i$ we have $\alpha^\vee = 2e_i$. The Weyl group consists of all possible permutations of the indices and all possible sign changes of the e_i , so has order $\ell!2^\ell$. A fundamental system of roots is

$$S = \{ \alpha_i = e_i - e_{i+1} \ (1 \leq i \leq \ell - 1), \ \alpha_\ell = e_\ell \}.$$

The fundamental weights are

$$\omega_i = (\underbrace{1, 1, \dots, 1}_i, 0, \dots, 0) \quad \text{for } 1 \leq i \leq \ell - 1, \quad \omega_\ell = \left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2} \right)$$

and we have

$$\rho = \left(\ell - \frac{1}{2}, \ell - \frac{3}{2}, \dots, \frac{1}{2} \right).$$

Let $\mu = (a_1, \dots, a_\ell) \in X$. Then

$$\mu \in X_+ \iff a_1 \geq a_2 \geq \dots \geq a_\ell \geq 0.$$

5.2. Type D_ℓ

Here, $R = R^\vee$ may be taken to consist of the vectors $\pm e_i \pm e_j$, $1 \leq i < j \leq \ell$. Roots and coroots are the same in this case. The Weyl group consists of all possible permutations of the indices and sign changes of an even number of the e_i , so has order $\ell!2^{\ell-1}$. A fundamental system of roots is

$$S = \{ \alpha_i = e_i - e_{i+1} \ (1 \leq i \leq \ell - 1), \ \alpha_\ell = e_{\ell-1} + e_\ell \}.$$

The fundamental weights are

$$\omega_i = (\underbrace{1, 1, \dots, 1}_i, 0, \dots, 0) \quad \text{for } 1 \leq i \leq \ell - 2,$$

$$\omega_{\ell-1} = \left(\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2} \right), \quad \omega_\ell = \left(\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2} \right)$$

and we have

$$\rho = (\ell - 1, \ell - 2, \dots, 1, 0).$$

Let $\mu = (a_1, \dots, a_\ell) \in X$. Then

$$\mu \in X_+ \iff a_1 \geq a_2 \geq \dots \geq a_\ell, \quad a_i \geq 0 \text{ for } 1 \leq i \leq \ell - 1, \text{ and } a_{\ell-1} + a_\ell \geq 0.$$

5.3. Type A_ℓ

We let $e_1, \dots, e_{\ell+1}$, be an orthonormal basis of an $(\ell + 1)$ -dimensional Euclidean space and we identify E with the hyperplane perpendicular to $e_1 + e_2 + \dots + e_{\ell+1}$. We may take $R = R^\vee$ to be the set of vectors $e_i - e_j$, $1 \leq i, j \leq \ell + 1$, $i \neq j$. The Weyl group is the set of all permutations of the $\ell + 1$ indices. A fundamental system is

$$S = \{ \alpha_i = e_i - e_{i+1} \ (1 \leq i \leq \ell) \}.$$

The fundamental dominant weights ω_i for $1 \leq i \leq \ell$ are

$$\omega_i = \frac{1}{\ell + 1} (\underbrace{\ell - i + 1, \dots, \ell - i + 1}_i, -i, \dots, -i)$$

and we have

$$\rho = \left(\frac{\ell}{2}, \frac{\ell}{2} - 1, \dots, 1 - \frac{\ell}{2}, -\frac{\ell}{2} \right).$$

Let $\mu = (a_1, \dots, a_\ell) \in X$. Then

$$\mu \in X_+ \iff a_1 \geq a_2 \geq \dots \geq a_\ell.$$

5.4. A shortcut

Suppose R is of type B_ℓ or D_ℓ . After computing $\lambda + \rho - mp\alpha$ in step (ii) we may see that its coordinate vector has two entries with the same absolute values. Then the same will hold for its conjugate in X_+ . But then since the coordinates of ρ are strictly decreasing and nonnegative, we see that subtracting ρ from this conjugate will take us out of X_+ . Similarly, if R is of type A_ℓ , if $\lambda + \rho - mp\alpha$ has two equal coordinates, then subtracting ρ will result in a weight outside X_+ .

Lemma 5.1.

- (a) If R is of type B_ℓ or D_ℓ and $\lambda + \rho - mp\alpha$ has two coordinates with the same absolute value then the pair (α, m) contributes nothing to the final sum.
- (b) If R is of type A_ℓ and $\lambda + \rho - mp\alpha$ has two equal coordinates, then the pair (α, m) contributes nothing to the final sum (7).

6. Computations for B_ℓ

6.1. Evaluation of the Sum Formula

We fix r with $0 \leq r \leq p - 1$ and set $\lambda = r\omega_1 = (r, 0, \dots, 0)$. Here, we determine the contribution of each positive root to the right hand side of (7). We have $\rho = (\ell - \frac{1}{2}, \ell - \frac{3}{2}, \dots, \frac{1}{2})$, so

$$\lambda + \rho = \left(r + \ell - \frac{1}{2}, \ell - \frac{3}{2}, \dots, \underbrace{\ell - i + \frac{1}{2}}_i, \dots, \frac{1}{2} \right).$$

6.1.1. $\alpha = e_i - e_j, 1 < i < j$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = j - i$. So we must consider m with

$$0 < mp < j - i. \tag{8}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\dots, \underbrace{\ell - i + \frac{1}{2} - mp}_i, \dots \right).$$

Since

$$\ell - i + \frac{1}{2} > \ell - i + \frac{1}{2} - mp > \ell - i + \frac{1}{2} - (j - i) = \ell - j + \frac{1}{2},$$

it follows that $v_i = v_s$ for some s with $i < s < j$, so Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

6.1.2. $\alpha = e_i + e_j, 1 < i < j$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = 2\ell - i - j + 1$. So we must consider m with

$$0 < mp < 2\ell - i - j + 1. \tag{9}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\dots, \underbrace{\ell - i + \frac{1}{2} - mp}_i, \dots, \underbrace{\ell - j + \frac{1}{2} - mp}_j, \dots \right).$$

If $\ell - i + 1/2 - mp < 0$ then

$$\left| \ell - i + \frac{1}{2} - mp \right| = mp - \left(\ell - i + \frac{1}{2} \right) < (2\ell - i - j + 1) - \left(\ell - i + \frac{1}{2} \right) = \ell - j + \frac{1}{2},$$

so $|v_i| = |v_s|$ for some $s > j$, and Lemma 5.1 applies. So assume $\ell - i + \frac{1}{2} - mp > 0$. Then $v_i = v_s$ for some $s > i$, unless $\ell - i + \frac{1}{2} - mp = \ell - j + \frac{1}{2}$, i.e. $mp = j - i$. Then

$$|v_j| = \left| \ell - j + \frac{1}{2} - mp \right| < \left| \ell - j + \frac{1}{2} \right| + (j - i) = \ell - i + 1,$$

so $|v_j| = v_s$ for some $s \leq i$ and Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

6.1.3. $\alpha = e_i, 1 < i$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = 2\ell - 2i + 1$. So we must consider m with

$$0 < mp < 2\ell - 2i + 1. \tag{10}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\dots, \underbrace{\ell - i + \frac{1}{2} - mp}_i, \dots \right).$$

Since $|\ell - i + \frac{1}{2} - mp| < \ell - i + \frac{1}{2}$, by (10), we have $|v_i| = v_j$ for some $j > i$, so Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

6.1.4. $\alpha = e_1 - e_j, j > 1$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = r + j - 1$. So we must consider m with

$$0 < mp < r + j - 1. \tag{11}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(r + \ell - \frac{1}{2} - mp, \ell - \frac{3}{2}, \dots, \underbrace{\ell - j + \frac{1}{2} + mp}_j, \dots \right).$$

By (11)

$$\ell - \frac{1}{2} > r + \ell - \frac{1}{2} - mp > r + \ell - \frac{1}{2} - r - j + 1 = \ell - j + \frac{1}{2}.$$

Thus $v_1 = v_s$ for some s with $1 < s < j$ and Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

6.1.5. $\alpha = e_1$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = 2r + 2\ell - 1$. So we must consider m with

$$0 < mp < 2r + 2\ell - 1. \tag{12}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(r + \ell - \frac{1}{2} - mp, \ell - \frac{3}{2}, \dots \right).$$

Then v_1 will have the same size as some $v_i, i > 1$ if $r + \ell - \frac{1}{2} - mp > 0$, and also if $r + \ell - \frac{1}{2} - mp < 0$ but $mp - r - \ell + \frac{1}{2} < \ell - \frac{1}{2}$. So we may assume $mp - r - \ell + \frac{1}{2} \geq \ell - \frac{1}{2}$, hence $mp \geq r + 2\ell - 1$. Thus, the only m we have to consider is the unique m (if it exists) which satisfies

$$r + 2\ell - 1 \leq mp < 2r + 2\ell - 1. \tag{13}$$

Assuming m exists, the conjugate of $\lambda + \rho - mp\alpha$ in X_+ is

$$\left(mp - r - \ell + \frac{1}{2}, \ell - \frac{3}{2}, \dots, \frac{1}{2} \right),$$

obtained from $\lambda + \rho - mp\alpha$ by changing the sign of v_1 , so has sign -1 . Subtracting ρ gives

$$(mp - r - 2\ell + 1, 0, \dots, 0) = (mp - 2\ell + 1 - r)\omega_1.$$

The total contribution to (7) from e_1 is therefore zero unless there exists m satisfying (13), in which case m is unique and the contribution to (7) is equal to $v_p(mp)\chi(r_1\omega_1)$, where $r_1 = mp - r - 2\ell + 1$.

6.1.6. $\alpha = e_1 + e_j, j > 1$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = r + 2\ell - j$. So we must consider m with

$$0 < mp < r + 2\ell - j. \tag{14}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(r + \ell - \frac{1}{2} - mp, \ell - \frac{3}{2}, \dots, \underbrace{\ell - j + \frac{1}{2} - mp}_{j}, \dots \right).$$

If $r + \ell - \frac{1}{2} - mp < 0$ then by (14)

$$\left| r + \ell - \frac{1}{2} - mp \right| = mp - \left(r + \ell - \frac{1}{2} \right) < \ell - j + \frac{1}{2},$$

so $|v_1| = v_s$ for some $s > j$. So we may assume $r + \ell - \frac{1}{2} - mp > 0$. Then v_1 equals some v_s for $s > 1$ unless $r + \ell - \frac{1}{2} - mp = \ell - j + \frac{1}{2}$, i.e.

$$j = mp - r + 1. \tag{15}$$

Next, we consider v_j . Clearly if $v_j > 0$, then $v_j = v_s$ for some $s > j$, so we can assume $v_j < 0$. Next, if $|v_j| = mp - (\ell - j + \frac{1}{2}) \leq \ell - \frac{3}{2}$, we have $|v_j| = v_s$ for some s , by the pigeon-hole principle. Thus, we may also assume that $mp - (\ell - j + \frac{1}{2}) \geq \ell - \frac{1}{2}$, i.e. $mp \geq 2\ell - j$. We have shown that by Lemma 5.1, a pair (m, j) contributes nothing to (7), except when

$$2\ell - j \leq mp < r + 2\ell - j, \quad \text{and } j = mp - r + 1. \tag{16}$$

Thus, if m exists, it is the unique integer satisfying

$$2\ell - 1 + r \leq 2mp < 2r + 2\ell - 1. \tag{17}$$

For this (m, j) , we have

$$\lambda + \rho - mp\alpha = \left(\ell - j + \frac{1}{2}, \dots, \underbrace{\ell - 2mp + r - \frac{1}{2}}_j, \dots \right).$$

The Weyl group element mapping this element into X_+ is the transposition of 1 with j followed by the sign change in the first coordinate. This group element has sign 1. On subtracting ρ from the result we end up with

$$(2mp - 2\ell + 1 - r, 0, \dots, 0) = (2mp - 2\ell + 1 - r)\omega_1.$$

The total contribution to (7) from positive roots of this form is zero unless there exists m satisfying (17) in which case the contribution is $-v_p(mp)\chi(r_1\omega_1)$, where $r_1 = 2mp - 2\ell + 1 - r$.

6.1.7. By considering the above cases, we see that the sum (7) is zero unless there exists an integer M with

$$r + 2\ell - 1 \leq Mp < 2r + 2\ell - 1.$$

In the latter case, let $r_1 = Mp - 2\ell + 1 - r$. Then there is a contribution of $v_p(Mp)\chi(r_1\omega_1)$ from the root e_1 and when M is even there is a contribution of $-v_p(\frac{M}{2}p)\chi(r_1\omega_1)$ from the root $e_1 + e_j$, where $j = \frac{M}{2}p - r + 1$. This means that if M exists the value of (7) for $\lambda = r\omega_1$ is:

$$\sum_{i>0} \text{Ch } V(\lambda)^i = \begin{cases} v_p(Mp)\chi(r_1\omega_1), & \text{if } M \text{ is odd,} \\ \chi(r_1\omega_1), & \text{if } M \text{ is even and } p = 2, \\ 0, & \text{if } M \text{ is even and } p \text{ is odd.} \end{cases}$$

6.2. Weyl module structures for type B_ℓ

The above computations show that $\chi(r\omega_1)$ is equal to either $\text{Ch}L(r\omega_1)$ or to $\text{Ch}L(r\omega_1) + e\chi(r_1\omega_1)$, for a certain r_1 with $0 \leq r_1 < r$ and some positive integer e . We note that if we repeat the process setting $\lambda = r_1\omega_1$, and denote the parameters (corresponding to m and r_1) arising from any nonsimple case by m_1 and r_2 , then we see that $r_2 \equiv r \pmod{p}$. Since we assume $r \leq p - 1$, this is impossible. This shows that $V(r_1\omega_1)$ is irreducible in all cases where $V(r\omega_1)$ is not. Thus, $\text{rad}V(r\omega_1)$ is either zero or else all of its composition factors are isomorphic to a simple Weyl module $V(r_1\omega_1)$. Since there are no self-extensions of simple modules [8, II.2.12] we see that $\text{rad}V(r\omega_1)$ is either zero or a direct sum of e copies of the simple Weyl module $V(r_1\omega_1)$.

Lemma 6.1. $V(r\omega_1)$ is either simple or else $e = 1$.

Proof. We first treat the special case $p = 2$. Then $r = 1$ and $V(\omega_1) \cong V$. In this case, it is well known that V has a one-dimensional G -fixed subspace T with V/T irreducible. Thus the lemma holds in this case. Now assume $p > 2$. Then $V \cong V^*$ as G -modules, whence all tensor powers of V^* are also self-dual. Since $r < p$, $S^r(V^*)$ is naturally a direct summand of the r -th tensor power of V^* , so $S^r(V^*)$ is also self-dual. For $0 \leq r \leq p - 1$, $S^r(V^*)$ has a good filtration [1, 4.1(4)], and the factors are the modules $H^0((r - 2j)\omega_1)$, for $0 \leq j \leq \lfloor \frac{r}{2} \rfloor$, each with multiplicity one, as can be seen from the character equation

$$\text{Ch}(S^r(V^*)) - \text{Ch}(S^{r-2}(V^*)) = \chi(r\omega_1),$$

which follows from the definition of $H^0(r\omega_1)$. Therefore,

$$\begin{aligned} \dim \text{Hom}_G(V(r_1\omega_1), V(r\omega_1)) &= \dim \text{Hom}_G(H^0(r\omega_1), H^0(r_1\omega_1)) \\ &\leq \dim \text{Hom}_G(S^r(V^*), H^0(r_1\omega_1)) \\ &= \dim \text{Hom}_G(V(r_1\omega_1), S^r(V^*)) \\ &\quad \text{(by self-duality of } S^r(V^*)) \\ &= \text{multiplicity of } H^0(r_1\omega_1) \\ &\quad \text{in a good filtration of } S^r(V^*) \\ &\leq 1. \quad \square \end{aligned}$$

The proof of Theorem 1.1 can now be completed. Parts (a), (b), and (c) have been proved in Section 6.1.7 and Section 6.2 and part (d) is obtained by substitution of Weyl module dimensions, given in Section 3.1.

7. Computations for D_ℓ

7.1. Evaluation of the Sum Formula

We fix r with $0 \leq r \leq p - 1$ and set $\lambda = r\omega_1 = (r, 0, \dots, 0)$. Here, we determine the contribution of each positive root to the right hand side of (7). We have $\rho = (\ell - 1, \ell - 2, \dots, 1, 0)$, so

$$\lambda + \rho = (r + \ell - 1, \ell - 2, \dots, \underbrace{\ell - i}_i, \dots, \dots, 1, 0).$$

7.1.1. $\alpha = e_i - e_j, 1 < i < j$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = j - i$. So we must consider m with

$$0 < mp < j - i. \tag{18}$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = (\dots, \underbrace{\ell - i - mp}_i, \dots, \underbrace{\ell - j + mp}_j, \dots).$$

We have

$$\ell - i > \nu_i = \ell - i - mp > \ell - i - (j - i) = \ell - j,$$

so $\nu_i = \nu_s$ for some s with $i < s < j$, and Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

7.1.2. $\alpha = e_i + e_j, 1 < i < j$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = 2\ell - i - j$. So we must consider m with

$$0 < mp < 2\ell - i - j. \tag{19}$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = (\dots, \underbrace{\ell - i - mp}_i, \dots, \underbrace{\ell - j - mp}_j, \dots).$$

If $\ell - i - mp \leq 0$, then by (19)

$$|\ell - i - mp| = mp - \ell + i < 2\ell - i - j - \ell + i = \ell - j,$$

so $|\nu_i| = \nu_s$ for some $s > j$ and Lemma 5.1 applies. So we can assume that $\nu_i = \ell - i - mp > 0$. Then ν_i equals to ν_s for $s > i$ unless $\ell - i - mp = \ell - j$, i.e. $mp = (j - i)$. In that case we have

$$|\nu_j| = |\ell - j - mp| = |(\ell - j) - (j - i)| < (\ell - j) + (j - i) = \ell - i,$$

so $|\nu_j| = \nu_s$ for some $s > i$, and Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

7.1.3. $\alpha = e_1 - e_j, j > 1$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = r + j - 1$. So we must consider m with

$$0 < mp < r + j - 1. \tag{20}$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = (r + \ell - 1 - mp, \ell - 2, \dots, \underbrace{\ell - j + mp}_j, \dots).$$

Then by (20)

$$\ell - 2 \geq r + \ell - 1 - mp > \ell - j,$$

so $v_1 = v_s$ for some s with $1 < s < j$, and Lemma 5.1 applies. The total contribution to (7) from positive roots of this form is zero.

7.1.4. $\alpha = e_1 + e_j, j > 1$

We have $\langle \lambda + \rho, \alpha^\vee \rangle = r + 2\ell - j - 1$. So we must consider m with

$$0 < mp < r + 2\ell - j - 1. \tag{21}$$

Consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = (r + \ell - 1 - mp, \ell - 2, \dots, \underbrace{\ell - j - mp}_j, \dots).$$

If $v_1 = r + \ell - 1 - mp \leq 0$, then by (21)

$$|v_1| = mp - r - \ell + 1 < (r + 2\ell - j - 1) - r - \ell + 1 = \ell - j,$$

so $|v_1| = v_s$ for some $s > j$. We can therefore assume that $v_1 = r + \ell - 1 - mp > 0$. Then we have $v_1 = v_s$ for some $s > 1$ unless $r + \ell - 1 - mp = \ell - j$, i.e.

$$j = mp - r + 1. \tag{22}$$

Next we consider v_j . Clearly if $v_j > 0$, then $v_j = v_s$ for some $s > j$, so we can assume $v_j < 0$.

Suppose $mp \leq 2\ell - j - 2$. Then

$$|v_j| = mp - \ell - j \leq 2\ell - j - 2 - \ell + j = \ell - 2,$$

so $|v_j| = v_s$ for some s , by the pigeon-hole principle. Thus, we may also assume that $mp \geq 2\ell - j - 1$. We have shown that by Lemma 5.1, a pair (m, j) contributes nothing to (7), except when

$$2\ell - j - 1 \leq mp < r + 2\ell - j - 1 \quad \text{and} \quad j = mp - r + 1. \tag{23}$$

Thus, if m exists, it is the unique integer satisfying

$$r + 2\ell - 2 \leq 2mp < 2r + 2\ell - 2. \tag{24}$$

For this (m, j) , we have

$$\lambda + \rho - mp\alpha = (\ell - j, \ell - 2, \dots, \underbrace{r + \ell - 2mp - 1}_j, \dots, 0).$$

The Weyl group element mapping this element into X_+ is the transposition of 1 with j followed by the sign changes in the first coordinate and the uniquely determined s -th coordinate with $v_s = 0$ (s is either ℓ or 1). This group element has sign -1 . On subtracting ρ from the result we end up with

$$(2mp - 2\ell + 2 - r, 0, \dots, 0) = (2mp - 2\ell + 2 - r)\omega_1.$$

The total contribution to (7) from positive roots of this form is zero unless there exists m satisfying (24) in which case the contribution is $v_p(m)\chi(r_1\omega_1)$, where $r_1 = 2mp - 2\ell + 2 - r$.

7.1.5. By considering the above cases, we see that the sum (7) is zero unless there exists an even integer M with

$$r + 2\ell - 2 \leq Mp < 2r + 2\ell - 2.$$

In the latter case, let $r_1 = Mp - 2\ell + 2 - r$. Then there is a contribution of $-v_p(\frac{M}{2}p)\chi(r_1\omega_1)$ from the root $e_1 + e_j$, where $j = \frac{M}{2}p - r + 1$. This means that if M exists the value of (7) for $\lambda = r\omega_1$ is:

$$\sum_{i>0} \text{Ch } V(\lambda)^i = \begin{cases} v_p(\frac{M}{2}p)\chi(r_1\omega_1), & \text{if } M \text{ is even,} \\ 0, & \text{if } M \text{ is odd.} \end{cases}$$

7.2. Weyl Module structure for type D_ℓ

The above computations show that $\chi(r\omega_1)$ is equal to either $\text{Ch } L(r\omega_1)$ or to $\text{Ch } L(r\omega_1) + e\chi(r_1\omega_1)$, for a certain r_1 with $0 \leq r_1 < r$ and some positive integer e . We note that if we repeat the process setting $\lambda = r_1\omega_1$, and denote the parameters (corresponding to m and r_1) arising from any nonsimple case by m_1 and r_2 , then we see that $r_2 \equiv r \pmod p$. Since we assume $r \leq p - 1$, this is impossible. This shows that $V(r_1\omega_1)$ is irreducible in all cases where $V(r\omega_1)$ is not. Thus, $\text{rad } V(r\omega_1)$ is either zero or else all of its composition factors are isomorphic to a simple Weyl module $V(r_1\omega_1)$. Since there are no self-extensions of simple modules [8, II.2.12] we see that $\text{rad } V(r\omega_1)$ is either zero or a direct sum of e copies of the simple Weyl module $V(r_1\omega_1)$.

Lemma 7.1. $V(r\omega_1)$ is either simple or else $e = 1$.

Proof. We have $V \cong V^*$ as G -modules, and we can argue exactly as for the case $p > 2$ of Lemma 6.1. \square

The proof of Theorem 1.2 can now be completed. Parts (a) and (b) have been proved in Section 7.1.5 and Section 7.2 and part (c) is obtained by substitution of Weyl module dimensions, given in Section 3.1.

8. Computations for $A_\ell, \lambda = r(\omega_1 + \omega_\ell)$

Let $\lambda = r(\omega_1 + \omega_\ell), 0 \leq r \leq p - 1$.

8.1. Evaluation of the Sum Formula

Fix $0 \leq r \leq p - 1$ and set $\lambda = r(\omega_1 + \omega_\ell) = r(1, 0, \dots, 0, -1)$. Here, we determine the contribution of each positive root to the right hand side of (7). Recall that $\rho = (\frac{\ell}{2}, \frac{\ell}{2} - 1, \dots, -\frac{\ell}{2} + 1, -\frac{\ell}{2})$. Hence $\lambda + \rho = (\frac{\ell}{2} + r, \frac{\ell}{2} - 1, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 1, -\frac{\ell}{2} - r)$.

8.1.1. $\alpha = e_i - e_j, 1 < i < j < \ell + 1$

We have $\langle \lambda + \rho, \alpha \rangle = j - i$. So we must consider m with

$$0 < mp < \langle \lambda + \rho, \alpha \rangle = j - i.$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\dots, \underbrace{\frac{\ell}{2} - i + 1 - mp}_i, \dots, \underbrace{\frac{\ell}{2} - j + 1 + mp}_j, \dots \right).$$

Then

$$i < mp + i < j$$

and we have $\nu_{mp+i} = \frac{\ell}{2} - (mp + i) + 1 = \nu_i$. By Lemma 5.1, the contribution to the sum (7) from positive roots of this type is zero.

8.1.2. $\alpha = e_1 - e_i$ or $e_i - e_{\ell+1}$, $1 < i < \ell + 1$

We have $\langle \lambda + \rho, \alpha \rangle = r + i - 1$, so we must consider m satisfying

$$0 < mp < r + i - 1.$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + r - mp, \dots, \underbrace{\frac{\ell}{2} - i + 1 + mp}_i, \dots, -\frac{\ell}{2} - r \right).$$

Since $r \leq p - 1$ we also have $mp > r$. So

$$1 < mp - r + 1 < i.$$

Then, $\nu_{mp-r+1} = \frac{\ell}{2} - mp + r = \nu_1$. By Lemma 5.1, the contribution to the sum from this type of positive roots is zero. By symmetry, the contribution from the roots of the form $\alpha = e_i - e_{\ell+1}$ is also zero.

8.1.3. $\alpha = e_1 - e_{\ell+1}$

We have $\langle \lambda + \rho, \alpha \rangle = \ell + 2r$, so we must consider m with

$$0 < mp < 2r + \ell.$$

Consider

$$(\nu_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + r - mp, \frac{\ell}{2} - 1, \dots, 1 - \frac{\ell}{2}, -\frac{\ell}{2} - r + mp \right).$$

Suppose $mp \leq r + \ell - 1$. Then $r + \frac{\ell}{2} - mp \geq 1 - \frac{\ell}{2}$, so $\nu_1 = \nu_s$ for some s with $2 \leq s \leq \ell$ and Lemma 5.1 applies. Thus, in order for all ν_i to be distinct we must have

$$r + \ell - 1 < mp < 2r + \ell. \tag{25}$$

Since $(2r + \ell) - (r + \ell - 1) = r + 1 \leq p$, there is at most one m satisfying (25). If such an m exists, then the transposition $(1, \ell + 1)$ takes $\lambda - mp\alpha$ to

$$\left(mp - \frac{\ell}{2} - r, \frac{\ell}{2} - 1, \dots, 1 - \frac{\ell}{2}, -mp + \frac{\ell}{2} + r \right) \in X_+$$

and when we subtract ρ , the resulting weight is $(mp - \ell - r)(\omega_1 + \omega_\ell)$.

8.2. Contributions combined

By considering the above cases, we see that the sum (7) is zero unless there exists an integer m satisfying the condition (25) and that if m exists, then the value of (7) for $\lambda = r(\omega_1 + \omega_\ell)$ is

$$\sum_{i>0} \text{Ch } V(\lambda)^i = v_p(mp)\chi(r_1(\omega_1 + \omega_\ell)),$$

where $r_1 = mp - \ell - r$. We claim that $V(r_1(\omega_1 + \omega_\ell))$ is simple. If not, then the above analysis applied to r_1 would show that $\text{rad } V(r_1(\omega_1 + \omega_\ell))$ would have the character of $V(r_2(\omega_1 + \omega_\ell))$, where $0 \leq r_2 = m'p - \ell - r_1$, for some integer m' . But then $r_2 \equiv r \pmod{p}$, contradicting the hypothesis that $0 \leq r \leq p - 1$. This proves our claim.

8.3. Structure of $V(\lambda)$

We have shown either $V(\lambda)$ is simple or else its radical has character equal to some positive integer e times the character of the simple Weyl module $V(r_1(\omega_1 + \omega_\ell))$. We now show that in the latter case, we must have $e = 1$.

Lemma 8.1. *Let G be of type A_ℓ , $\ell \geq 3$ and let $0 \leq r \leq p - 1$. Then $H^0(r(\omega_1 + \omega_\ell))$ is either simple or else $H^0(r(\omega_1 + \omega_\ell))/L(r(\omega_1 + \omega_\ell))$ is simple and isomorphic to $H^0(r_1(\omega_1 + \omega_\ell))$, where $r_1 = mp - \ell - r$.*

Proof. The following character formula is classical and a special case of the Littlewood–Richardson rule:

$$\chi(r\omega_1)\chi(r\omega_\ell) = \sum_{s=0}^r \chi(s(\omega_1 + \omega_\ell)). \tag{26}$$

Then, since $H^0(r_1(\omega_1 + \omega_\ell))$ is simple and self-dual, while $H^0(r\omega_1)$ and $H^0(r\omega_\ell)$ are simple and dual to each other, we have

$$\begin{aligned} 1 \leq e &= \dim \text{Hom}_G(H^0(r(\omega_1 + \omega_\ell)), L(r_1(\omega_1 + \omega_\ell))) \\ &\leq \dim \text{Hom}_G(H^0(r\omega_1) \otimes H^0(r\omega_\ell), L(r_1(\omega_1 + \omega_\ell))) \\ &= \dim \text{Hom}_G(L(r_1(\omega_1 + \omega_\ell), H^0(r\omega_1) \otimes H^0(r\omega_\ell))) \\ &= \dim \text{Hom}_G(V(r_1(\omega_1 + \omega_\ell), H^0(r\omega_1) \otimes H^0(r\omega_\ell))) \\ &\leq 1, \end{aligned}$$

by (26) and the multiplicity property of good filtrations. \square

The proof of Theorem 1.3 can now be completed. Parts (a) and (b) have been proved in Section 8.2 and Section 8.3 and part (c) is obtained by substitution of Weyl module dimensions, given in Section 3.1.

9. Computations for type $A_\ell, \lambda = \omega_2 + \omega_{\ell-1}$

Let $\lambda = \omega_2 + \omega_{\ell-1}, \ell \geq 4$.

9.1. Evaluation of the Sum Formula

Here, we determine the contribution of each positive root to the right hand side of (7). We have $\lambda = (1, 1, 0, \dots, 0, -1, -1), \rho = (\frac{\ell}{2}, \frac{\ell}{2} - 1, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2} + 1, -\frac{\ell}{2})$, so $\lambda + \rho = (\frac{\ell}{2} + 1, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, \frac{\ell}{2} - i + 1, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2}, -\frac{\ell}{2} - 1)$. We search for m 's satisfying

$$0 < mp < \langle \lambda + \rho, \alpha \rangle. \tag{27}$$

Since $p \geq 2$ and $m \neq 0$ the inequality (27) is equivalent to

$$2 \leq mp < \langle \lambda + \rho, \alpha \rangle. \tag{28}$$

We find m 's and α 's satisfying (28). Then we find out whether $\lambda + \rho - mp\alpha$ has any two coordinates of the same value.

9.1.1. $\alpha = e_1 - e_2$ or $\alpha = e_\ell - e_{\ell+1}$

This case is trivial since $\langle \lambda + \rho, \alpha \rangle = 1$. Therefore for any p and ℓ , the contribution to the sum from the positive roots of this type is zero.

9.1.2. $\alpha = e_1 - e_i$ or $e_i - e_{\ell+1}, 2 < i < \ell$

For this case $\langle \lambda + \rho, \alpha \rangle = i$, hence

$$2 \leq mp < i. \tag{29}$$

We consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + 1 - mp, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, \underbrace{\frac{\ell}{2} - i + 1 + mp, \dots}_i \right).$$

By (29) we have

$$\frac{\ell}{2} + 1 - i < \frac{\ell}{2} + 1 - mp \leq \frac{\ell}{2} - 1.$$

Thus unless $\frac{\ell}{2} + 1 - mp = \frac{\ell}{2} - 1$, there is an s with $3 \leq s < i$ such that $v_1 = v_s$. If $\frac{\ell}{2} + 1 - mp = \frac{\ell}{2} - 1$, then $mp = 2$. Hence the equality holds if and only if $p = 2$ and $m = 1$. In this case,

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} - 1, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, \underbrace{\frac{\ell}{2} - i + 3, \dots}_i, -\frac{\ell}{2} + 2, -\frac{\ell}{2}, -\frac{\ell}{2} - 1 \right).$$

Since $i > 2$ we get

$$\frac{\ell}{2} + 2 - i < \frac{\ell}{2} + 3 - i < \frac{\ell}{2} + 1.$$

Hence there exists s with $1 \leq s \leq i - 1$ such that $v_i = v_s$ in this case also. Therefore for any p and ℓ , the contribution to the sum from the positive roots of this type is zero. By symmetry we see that contribution to the sum from the roots of type $\alpha = e_i - e_{\ell+1}$, $2 < i < \ell$ is also zero.

9.1.3. $\alpha = e_1 - e_\ell$ or $e_2 - e_{\ell+1}$

For this case $\langle \lambda + \rho, \alpha \rangle = \ell + 1$. Hence

$$2 \leq mp < \ell + 1. \tag{30}$$

We consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + 1 - mp, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2} + mp, -\frac{\ell}{2} - 1 \right).$$

From (30) we get

$$-\frac{\ell}{2} + 2 \leq -\frac{\ell}{2} + mp < \frac{\ell}{2} + 1.$$

The only way that v_ℓ is different from the other coordinates of v is that $v_\ell = -\frac{\ell}{2} + mp = \frac{\ell}{2} - 1$ which gives us $mp = \ell - 1$. In this case,

$$v_1 = \frac{\ell}{2} + 1 - mp = -\frac{\ell}{2} + 2 = v_{\ell-1}.$$

Hence there is at least two equal coordinates of v . Therefore for any p and ℓ , the contribution to the sum from the positive roots of this type is zero. By symmetry, the contribution to the sum from the root $\alpha = e_2 - e_{\ell+1}$ is also zero.

9.1.4. $\alpha = e_2 - e_i$ or $e_i - e_\ell$, $2 < i < \ell$

For this case $\langle \lambda + \rho, \alpha \rangle = i - 1$. Hence

$$2 \leq mp < i - 1. \tag{31}$$

We consider

$$(v_s)_{s=1}^\ell := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + 1, \frac{\ell}{2} - mp, \frac{\ell}{2} - 2, \dots, \frac{\ell}{2} - i + 1 + mp, \dots \right).$$

From (31) we get

$$\frac{\ell}{2} - i + 1 < \frac{\ell}{2} - mp \leq \frac{\ell}{2} - 2.$$

Thus, there is an s with $3 \leq s < i$ such that $v_2 = v_s$. Therefore for any p and ℓ , the contribution to the sum from the positive roots of this type is zero. By symmetry, the contribution to the sum from the roots of type $\alpha = e_i - e_\ell$, $2 < i < \ell$ is also zero.

9.1.5. $\alpha = e_1 - e_{\ell+1}$

For this case $\langle \lambda + \rho, \alpha \rangle = \ell + 2$. Hence,

$$2 \leq mp < \ell + 2. \tag{32}$$

We consider

$$(v_s)_{s=1}^{\ell} := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + 1 - mp, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2}, -\frac{\ell}{2} - 1 + mp \right).$$

With (32) we get

$$-\frac{\ell}{2} - 1 < \frac{\ell}{2} + 1 - mp \leq \frac{\ell}{2} - 1.$$

Thus the v_i are distinct if and only if one of the following conditions hold.

- (i) $v_1 = \frac{\ell}{2} - 1$.
- (ii) $v_1 = -\frac{\ell}{2} + 1$.

The condition (i) holds if and only if $mp = 2$. Thus unless $p = 2$ and $m = 1$, v_1 is equal to one of the other v_i . If $p = 2$ and $m = 1$, then

$$(v_s)_{s=1}^{\ell} := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} - 1, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2}, -\frac{\ell}{2} + 1 \right).$$

When we conjugate this element into X_+ with the permutation $(1, 2)(\ell, \ell + 1)$ and subtract ρ we obtain $(0, 0, \dots, 0)$. The permutation has positive sign.

The condition (ii) holds if and only if $mp = \ell$. In this case,

$$(v_s)_{s=1}^{\ell} := \lambda + \rho - mp\alpha = \left(-\frac{\ell}{2} + 1, \frac{\ell}{2}, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2}, \frac{\ell}{2} - 1 \right).$$

When we conjugate this element into X_+ with the permutation $(1, \ell - 1, \ell, 2)$ and subtract ρ we obtain $(0, 0, \dots, 0)$. The permutation has negative sign.

9.1.6. $\alpha = e_2 - e_{\ell}$

For this case $\langle \lambda + \rho, \alpha \rangle = \ell$. Hence

$$2 \leq mp < \ell. \tag{33}$$

We consider

$$(v_s)_{s=1}^{\ell} := \lambda + \rho - mp\alpha = \left(\frac{\ell}{2} + 1, \frac{\ell}{2} - mp, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2} + mp, -\frac{\ell}{2} - 1 \right).$$

Hence $\lambda - mp\alpha$ has distinct coordinates if and only if $v_2 = -\frac{\ell}{2} + 1$. That is $mp + 1 = \ell$. In this case,

$$\lambda - mp\alpha = \left(\frac{\ell}{2} + 1, -\frac{\ell}{2} + 1, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, \frac{\ell}{2} - 1, -\frac{\ell}{2} - 1 \right).$$

So its Weyl conjugate in X_+ is

$$\left(\frac{\ell}{2} + 1, \frac{\ell}{2} - 1, \frac{\ell}{2} - 2, \dots, -\frac{\ell}{2} + 2, -\frac{\ell}{2} + 1, -\frac{\ell}{2} - 1\right)$$

and on subtraction of ρ we obtain $\omega_1 + \omega_\ell$. The Weyl group element has negative sign.

9.2. Contributions combined

If $p = 2$ and ℓ is even, we have

$$\sum_{i>0} \text{Ch } V(\lambda)^i = (v_2(\ell) - 1)\chi(0). \tag{34}$$

Thus, $\text{Ch}(\text{rad } V(\lambda)) = e_0\chi(0)$, where $e_0 \geq 0$. For $p = 2$ and odd ℓ ,

$$\sum_{i>0} \text{Ch } V(\lambda)^i = -\chi(0) + v_2(\ell - 1)\chi(\omega_1 + \omega_\ell). \tag{35}$$

Thus, $\text{Ch}(\text{rad } V(\lambda)) = -\chi(0) + e_1\chi(\omega_1 + \omega_\ell)$, where $e_1 \geq 1$.

If p is odd $V(\lambda)$ is simple unless $\ell \equiv 0 \pmod{p}$ or $\ell \equiv 1 \pmod{p}$. If $\ell \equiv 0 \pmod{p}$ then

$$\sum_{i>0} \text{Ch } V(\lambda)^i = v_p(\ell)\chi(0). \tag{36}$$

Thus, $\text{Ch}(\text{rad } V(\lambda)) = e_2\chi(0)$, where $e_2 \geq 1$.

If $\ell \equiv 1 \pmod{p}$ then

$$\sum_{i>0} \text{Ch } V(\lambda)^i = v_p(\ell - 1)\chi(\omega_1 + \omega_\ell). \tag{37}$$

Thus, $\text{Ch}(\text{rad } V(\lambda)) = e_3\chi(\omega_1 + \omega_\ell)$, where $e_3 \geq 1$.

It is a well-known special case of Theorem 1.3 that

$$\chi(\omega_1 + \omega_\ell) = \begin{cases} \text{Ch } L(\omega_1 + \omega_\ell) + \chi(0) & \text{if } \ell \equiv -1 \pmod{p}, \\ \text{Ch } L(\omega_1 + \omega_\ell) & \text{otherwise.} \end{cases} \tag{38}$$

Thus, when $\ell \equiv 1 \pmod{p}$, we see that $V(\omega_1 + \omega_\ell)$ is simple if p is odd and has radical isomorphic to k if $p = 2$.

9.3. Structure of $V(\lambda)$

In this subsection we shall prove Theorem 1.4. We begin with some general facts which do not depend on p . We observe that $V(\omega_2)$ is simple and isomorphic to $\wedge^2(V)$, where V is the standard $\ell + 1$ -dimensional module, and that $V(\omega_{\ell-1}) \cong \wedge^{\ell-1}(V)$ is isomorphic to the dual of $V(\omega_2)$. Thus,

$$\text{Hom}_G(k, L(\omega_2) \otimes L(\omega_{\ell-1})) \cong \text{Hom}_G(L(\omega_2), L(\omega_2)) \cong k. \tag{39}$$

A fundamental property of Weyl modules and induced modules [8, II.4.19] is that the tensor product of two Weyl modules or two induced modules has a filtration by modules of the same kind, with

the factors being the same as for the corresponding tensor product in characteristic zero. Thus, the classical identity [6, p. 225]:

$$\chi(\omega_2)\chi(\omega_{\ell-1}) = \chi(\omega_2 + \omega_{\ell-1}) + \chi(\omega_1 + \omega_\ell) + \chi(0) \tag{40}$$

implies that $M := V(\omega_2) \otimes V(\omega_{\ell-1})$ has a filtration $E_1 \subset E_2 \subset M$ with factors $E_1 \cong V(\omega_2 + \omega_{\ell-1})$, $E_2/E_1 \cong V(\omega_1 + \omega_\ell)$ and $M/E_2 \cong k$. It follows that

$$\begin{aligned} \dim \text{Hom}_G(k, V(\omega_2 + \omega_{\ell-1})) &\leq \dim \text{Hom}_G(k, M) \\ &= \dim \text{Hom}_G(L(\omega_2), L(\omega_2)) \\ &= 1. \end{aligned} \tag{41}$$

Dually, we know that M has a quotient isomorphic to $H^0(\omega_2 + \omega_{\ell-1})$. Since the highest weight $\omega_2 + \omega_{\ell-1}$ occurs in $M/\text{rad } E_1$, the latter also has $H^0(\omega_2 + \omega_{\ell-1})$ as a quotient. Therefore, if we write $[A : L(\mu)]$ for the multiplicity of the simple G -module $L(\mu)$ as a composition factor of the G -module A , we have

$$\begin{aligned} [V(\omega_2 + \omega_{\ell-1}) : L(\omega_1 + \omega_\ell)] &= [H^0(\omega_2 + \omega_{\ell-1}) : L(\omega_1 + \omega_\ell)] \\ &\leq [M/\text{rad } E_1 : L(\omega_1 + \omega_\ell)] \\ &= [M/E_1 : L(\omega_1 + \omega_\ell)] \\ &= [E_2/E_1 : L(\omega_1 + \omega_\ell)] + [M/E_2 : L(\omega_1 + \omega_\ell)] \\ &= 1. \end{aligned} \tag{42}$$

Now we are ready to prove each part of Theorem 1.4. Parts (c) and (f) are immediate, since by Section 9.2 the Sum Formula evaluates to zero. In (a) and (d), we see from (34) and (36) that $\text{rad } V(\lambda)$ has only trivial composition factors, with positive multiplicity. Since $\text{Ext}_G^1(k, k) = 0$, it follows from (41) that the multiplicity is one. In (b) we know from (37) and (38) that the only composition factor of $\text{rad } V(\lambda)$, with positive multiplicity, is $L(\omega_1 + \omega_\ell) \cong V(\omega_1 + \omega_\ell)$. By (42) the multiplicity is one. In (g) we see from (35) and (38) that the only composition factor of $\text{rad } V(\lambda)$, with positive multiplicity, is $L(\omega_1 + \omega_\ell)$. By (42) the multiplicity is one. The proof of (e) is more involved. Since $\dim L(\omega_2) = \binom{\ell+1}{2}$ is odd, $M \cong \text{End}(L(\omega_2))$ decomposes as the direct sum of the scalar multiples of the identity map and the space of trace zero endomorphisms. The indecomposable submodule E_1 must therefore be isomorphic to a submodule of the nontrivial summand. Then, by (39), it follows that

$$\text{Hom}_G(k, V(\lambda)) = 0. \tag{43}$$

The Sum Formula tells us that the composition factors (with positive multiplicities) of $\text{rad } V(\lambda)$ are k and $L(\omega_1 + \omega_\ell)$. By (42), we know that $[V(\lambda) : L(\omega_1 + \omega_\ell)] = 1$. Then by (43), it follows that $\text{soc}(\text{rad } V(\lambda)) \cong L(\omega_1 + \omega_\ell)$. Since $\text{Ext}_G^1(k, k) = 0$, the module $(\text{rad } V(\lambda))/\text{soc}(\text{rad } V(\lambda))$ must be a trivial module of dimension $\leq \dim \text{Ext}_G^1(k, L(\omega_1 + \omega_\ell))$. By (38), the self-duality of $L(\omega_1 + \omega_\ell)$ and a standard property of Weyl modules [8, II.2.14], we know that

$$\text{Ext}_G^1(k, L(\omega_1 + \omega_\ell)) \cong \text{Ext}_G^1(L(\omega_1 + \omega_\ell), k) \cong k. \tag{44}$$

Hence $\text{rad } V(\lambda)$ is a nonsplit extension of k by $L(\omega_1 + \omega_\ell)$. Such a module is unique up to isomorphism, by (44), and isomorphic to $H^0(\omega_1 + \omega_\ell)$, by (38). This completes the proof of (e).

Finally, (h) is obtained by substitution of Weyl module dimensions, given in Section 3.1 and Theorem 1.3. This completes the proof of Theorem 1.4.

10. Further computations for Type A_4

The aim of this section is to prove Theorem 1.5. Let $\lambda = (p - 1)(\omega_2 + \omega_3)$ and $\lambda_1 = (p - 2)(\omega_2 + \omega_3)$. First we observe that for $p = 2$, we have $\lambda = \omega_2 + \omega_3$, so the structure of $V(\lambda)$ has been given in Theorem 1.4. Its radical is isomorphic to k . So (a) is proved, as also is (e). We assume that p is odd from now on. We begin with (c). Let $\lambda_2 = (p - 2)(\omega_1 + \omega_4)$. Then $V(\lambda_2)$ is simple, by Theorem 1.3. Next, we compute the Sum Formula (7) for $\lambda_1 = (p - 2, p - 2, 0, 2 - p, 2 - p)$. We have $\rho = (2, 1, 0, -1, -2)$. Using Lemma 5.1 we see that the only nonzero contribution is from the positive root $(0, 1, 0, -1, 0)$ and for $m = 1$. This yields

$$\sum_{i>0} \text{Ch } V(\lambda_1)^i = \chi(\lambda_2).$$

It follows that $\text{rad } V(\lambda_1) \cong V(\lambda_2)$, proving (c). To prove (b) we compute (7) for $\lambda = (p - 1, p - 1, 0, 1 - p, 1 - p)$. Direct calculations show that for all α , except $\alpha = (1, 0, 0, 0, -1)$, either there is no m satisfying $0 < mp < \langle \lambda + \rho, \alpha \rangle$, or $\lambda + \rho - mp\alpha$ has repeated entries. When $\alpha = (1, 0, 0, 0, -1)$, m can be 1 or 2. Consider the case $\alpha = (1, 0, 0, 0, -1)$, and $m = 1$. Then $\lambda + \rho - mp\alpha = (1, p, 0, -p, -1)$, so the permutation sending $\lambda + \rho - mp\alpha$ to its conjugate $(p, 1, 0, -1, -p) \in X_+$ has positive sign. Subtraction of ρ from this conjugate yields $(p - 2)(\omega_1 + \omega_4)$. Hence $\chi(\lambda - mp\alpha) = \chi((p - 2)(\omega_1 + \omega_4))$. In the case $m = 2$, we get $\lambda + \rho - mp\alpha = (-p + 1, p, 0, -p, p - 1)$ which is sent to its conjugate $(p, p - 1, 0, 1 - p, -p) \in X_+$ by a permutation with negative sign. Subtraction of ρ from this conjugate yields $(p - 2)(\omega_2 + \omega_3)$. Hence $\chi(\lambda - mp\alpha) = -\chi((p - 2)(\omega_2 + \omega_3))$. For both cases $v_p(mp) = 1$, since $p > 2$. Thus,

$$\sum_{i>0} \text{Ch } V(\lambda)^i = \chi(\lambda_1) - \chi(\lambda_2) = \text{Ch } L(\lambda_1), \tag{45}$$

where the last equality is by (c). It follows that $\text{rad } V(\lambda) \cong L(\lambda_1)$ and (b) is proved.

By (b) and (c) we have

$$\dim L(\lambda) = \dim V(\lambda) - \dim V(\lambda_1) + \dim V(\lambda_2).$$

Then the proof of (d) is completed by substituting the dimensions of Weyl modules from Section 3.1.

11. p -Ranks of incidence matrices

In this section, we explain how the theorems on p -ranks stated in Section 1.2 may be deduced from our results on the dimensions of simple modules. We shall use some of the notation from that earlier discussion.

11.1. The permutation module on singular points

We recall that P is either the set of isotropic 1-dimensional subspaces in the n -dimensional space $V(q)$ with respect to a nondegenerate quadratic form f , or else the set of singular 1-dimensional subspaces in $V(q^2)$ with respect to a nonsingular Hermitian form h . In both cases, the set of polar hyperplanes is denoted by P^* . The map sending each $p \in P$ to its polar hyperplane $p^\perp \in P^*$ is clearly a permutation isomorphism with respect to the action of the projective orthogonal group $\text{PO}(V(q), f)$ in the quadratic case or the projective unitary group $\text{PGU}(V(q^2), h)$ in the Hermitian case. For technical reasons, rather than working with these groups, it will be more convenient for us to work with certain subgroups which are Chevalley groups, under which term we shall include the twisted as well as the untwisted types. Let $\Omega(V(q), f)$ denote the commutator subgroup of the orthogonal group $\text{O}(V(q), f)$ and let $\text{P}\Omega(V(q), f)$ be its image in $\text{PO}(V(q), f)$. By [3, Theorems 11.3.2 and 14.5.2], $\text{P}\Omega(V(q), f)$ is, with one exception, a Chevalley group of type $B_\ell(q)$, $D_\ell(q)$ or ${}^2D_\ell(q)$,

with $\dim V(q) = 2\ell + 1$ in the first case and 2ℓ in the other two cases. The exceptional case is when $q = 2$ and $\dim V(q) = 5$, in which case the Chevalley group of type $B_2(2)$ is a subgroup of $\text{PO}(V(q), f)$ containing $\text{P}\Omega(V(q), f)$ with index 2. By [3, Theorems 14.5.1], $\text{PSU}(V(q^2), h)$ is a (twisted) Chevalley group of type ${}^2A_\ell(q)$, where $\dim V(q^2) = \ell + 1$. In each case, we shall denote the Chevalley group by $\overline{G}(q)$.

The permutation module over k with basis P can be identified with the space $k[P]$ of all functions from P to k by identifying a point with its characteristic function. Being a permutation module, it is a self-dual $k\overline{G}(q)$ -module, with P as an orthonormal basis of a nonsingular $\overline{G}(q)$ -invariant symmetric bilinear form. Since $|P| \equiv 1 \pmod{p}$, we have an orthogonal direct sum decomposition of $k\overline{G}(q)$ -modules

$$k[P] = k\mathbf{1} \oplus Y_P, \tag{46}$$

where $\mathbf{1}$ is the constant function with value 1 and Y_P is the subspace of functions f such that $\sum_{p \in P} f(p) = 0$. Let $\pi_{\mathbf{1}}$ and π_{Y_P} be the projections with respect to this decomposition. Thus, $\pi_{\mathbf{1}}$ maps each $p \in P$ to $\mathbf{1}$. The head of Y_P , $\text{head}(Y_P)$, is defined as the largest semisimple quotient of Y_P and the socle, $\text{soc}(Y_P)$, is the maximal semisimple submodule. Let $L = \text{soc}(Y_P)$. The self-duality of $k[P]$ and (46) imply that Y_P is self-dual and hence that $\text{head}(Y_P) \cong L^*$. We claim that $L^* \cong L$ as $k\overline{G}(q)$ -modules. If $\overline{G}(q)$ is of type $B_\ell(q)$ then every simple module is self-dual. This is also true for type $D_\ell(q)$ or ${}^2D_\ell(q)$ when ℓ is even. Suppose $\overline{G}(q) \cong \text{P}\Omega(V(q), f)$ is of type $D_\ell(q)$ or ${}^2D_\ell(q)$ with ℓ odd. Then there is an automorphism δ of $\Omega(V(q), f)$, such that twisting a simple module by δ yields the dual simple module. If M is the module of a representation of $\Omega(V(q), f)$, we denote by M^δ the twisted module obtained by composing the representation with δ . Then $V(q) \cong V(q)^\delta$ and since $k[P]$ is constructed from $V(q)$, it follows that $k[P] \cong k[P]^\delta$. Hence $Y_P \cong (Y_P)^\delta$ and $L \cong L^\delta \cong L^*$. Similarly, in the Hermitian case, the group $\text{SU}(V(q^2), h)$ has an automorphism δ which dualizes simple modules. Therefore, the form h gives an isomorphism of $V(q^2)^\delta$ with the Galois twist $V(q^2)^{(q)}$. Since $k[P]$ is constructed from $V(q^2)$, it follows that $k[P]^{(q)} \cong k[P]^\delta$. Then, because $k[P]$ is isomorphic to all of its Galois conjugates, we have $k[P] \cong k[P]^\delta$, from which we get $L \cong L^\delta \cong L^*$. This completes the proof of our claim.

Lemma 11.1.

- (a) The action of $\overline{G}(q)$ on P has permutation rank 3.
- (b) $\dim \text{End}_{k\overline{G}(q)}(k[P]) = 3$.
- (c) $\dim \text{End}_{k\overline{G}(q)}(Y_P) = 2$.

Proof. Part (a) is well known [5]. (We should point out that our hypotheses on ℓ ($\ell \geq 3$ for type A_ℓ , $\ell \geq 2$ for type B_ℓ , and $\ell \geq 3$ for type D_ℓ) are needed here.) Then (b) follows from the general fact that the endomorphism ring of a transitive permutation module has a natural basis in bijection with the set of double cosets of a stabilizer. Therefore, the cardinality of the basis equals the rank of the permutation representation. Part (c) now follows from the decomposition (46). \square

By means of the isomorphism of $\overline{G}(q)$ -sets between P^* and P the incidence matrix between P^* and P can be interpreted as the map

$$\phi : k[P] \rightarrow k[P], \quad p \mapsto \sum_{p' \in p^\perp} p',$$

which clearly belongs to $\text{End}_{k\overline{G}(q)}(k[P])$.

Let $\overline{G}(q)_p$ be the stabilizer of an element of P . Thus, the possible choices for $\overline{G}(q)_p$ form a conjugacy class in $\overline{G}(q)$. Later we shall make a definite choice.

Lemma 11.2.

- (a) $k[P]$ is not semisimple.
- (b) L is simple.
- (c) L is, up to isomorphism, the unique nontrivial simple $k\bar{G}(q)$ -module on which $\bar{G}(q)_p$ has nonzero fixed points.
- (d) $\text{Im } \phi = k\mathbf{1} \oplus L$.

Proof. A 3-dimensional semisimple k -algebra is a direct product of 3 copies of k so has no nilpotent elements. We see that $\phi^2(p) = \sum_{p'' \in P} a_{p,p''} p''$, with

$$a_{p,p''} = |\{p' \mid p' \in p^\perp, p'' \in p'^\perp\}| = |p^\perp \cap p''^\perp| \pmod{p} = \mathbf{1}.$$

Similarly, we see that $\phi \circ \pi_1 = \pi_1 = \pi_1 \circ \phi$. Thus $(\phi - \pi_1)^2 = 0$, which proves (a). To prove (b), we fix an isomorphism from $\text{head}(Y_p)$ with $L = \text{soc}(Y_p)$ and let $\psi \in \text{End}_{k\bar{G}(q)}(Y_p)$ be the composite map $Y_p \rightarrow \text{head}(Y_p) \cong L \subset Y_p$. Given a decomposition of L into a direct sum of m simple modules, the m projections onto these simple summands are linearly independent elements of $\text{End}_{k\bar{G}(q)}(L)$. Their compositions with ψ , together with id_{Y_p} , give $m + 1$ linearly independent elements of $\text{End}_{k\bar{G}(q)}(Y_p)$. Hence $m = 1$, by Lemma 11.1(c). Part (c) follows from (b) and Frobenius reciprocity, since $k[P]$ can be viewed as the induced module $\text{ind}_{\bar{G}(q)_p}^{\bar{G}(q)}(k)$. To prove (d), observe that π_1, π_{Y_p} and $\psi \circ \pi_{Y_p}$ form a basis of $\text{End}_{k\bar{G}(q)}(k[P])$. It follows the image of a nonzero $k\bar{G}(q)$ -endomorphisms of $k[P]$ must be one of $\mathbf{1}, Y_p, L, k[P]$ and $k\mathbf{1} \oplus L$. Since $\phi \neq \pi_1$ and $\phi^2 = \pi_1$, it follows $\text{Im } \phi = k\mathbf{1} \oplus L$. \square

Corollary 11.3. $\text{rank}_p A_{11} = 1 + \dim L$, where L is the unique nontrivial simple $k\bar{G}(q)$ -module on which $\bar{G}(q)_p$ has nontrivial fixed points.

11.1.1. *The dual Hermitian quadrangle*

Let $(V(q^2), h)$ be a 5-dimensional Hermitian space and now define P to be is the set of 2-dimensional totally singular subspaces, instead of the one-dimensional ones considered above. This new P is the set of points of the dual Hermitian quadrangle. The above discussion carries through with $\bar{G}(q) = \text{SU}(V(q^2), h)$, which acts with rank 3 on P , and we obtain Corollary 11.3 in that case too.

Remark 11.4. In all of the cases above, the simplicity of the socle and head of Y_p can also be derived from the very general theory in [4]. Indeed that theory shows that for the permutation module on the cosets of any maximal parabolic subgroup in a finite group with a split (B, N) -pair of characteristic p , there is a decomposition (46) such that the socle and head of the nontrivial summand are simple.

11.2. *Representations of finite Chevalley groups*

The aim of this subsection is to obtain further information about the simple module L . To each of the Chevalley groups $\bar{G}(q)$ in the previous subsection, there corresponds a central extension $G(q)$ called the universal Chevalley group. Any representation of $\bar{G}(q)$ gives a representation of $G(q)$ by composing with the natural map $G(q) \rightarrow \bar{G}(q)$. If we let $G(q)_p$ be the full preimage in $G(q)$ of $\bar{G}(q)_p$, then $G(q)_p$ is the point stabilizer in $G(q)$ and L is the unique simple $G(q)$ -module on which $G(q)_p$ has nonzero fixed points. For technical reasons, it is preferable to work with $G(q)$. Since our goal is to determine $\dim L$, there is no loss in replacing $\bar{G}(q)$ by $G(q)$.

A basic fact [18, 13.1] about simple $kG(q)$ -modules is that they are the restrictions of certain simple rational modules for $G(k)$, the ambient Chevalley group over k . By this we mean that $G(k)$ has an endomorphism σ , such that $G(q)$ is the subgroup of fixed points [18, 12.4]. Further, we may assume that the subgroups T and B are σ -stable and denote by $T(q)$ and $B(q)$ the subgroups of T

and B fixed by σ . $G(k)$ is a simply connected algebraic group, since $G(q)$ is universal. In the notation Section 2, let $L(\lambda)$ denote the simple $G(k)$ -module with highest weight $\lambda \in X_+$. The simple module $L(\lambda)$ is characterized by the property that it has a unique B -stable line, and T acts on this line by the character λ . By Steinberg’s theorem [18, 13.3], the simple $G(k)$ modules with highest weights in the set

$$X_q = \left\{ \lambda = \sum_{i=1}^{\ell} a_i \omega_i \in X_+ \mid 0 \leq a_i \leq q - 1 \ (\forall i) \right\}$$

remain irreducible upon restriction to $G(q)$ and this gives a complete set of mutually nonisomorphic simple $kG(q)$ -modules.

Our immediate goal is to identify the highest weight $\lambda \in X_q$ such that the restriction of the $G(k)$ -module $L(\lambda)$ to $G(q)$ is isomorphic to L .

Lemma 11.5.

- (a) For $G(q)$ in Theorem 1.6 we have $L = L((q - 1)\omega_1)$.
- (b) For $G(q)$ in Theorem 1.8 we have $L = L((q - 1)(\omega_1 + \omega_\ell))$.
- (c) For $G(q)$ in Theorem 1.9 we have $L = L((q - 1)(\omega_2 + \omega_3))$.

Proof. Consider the action of the $G(k)$ on its standard module V . We may assume that V is obtained from the standard module of $G(q)$ by extending scalars from \mathbf{F}_q or \mathbf{F}_{q^2} to k . Then the highest weight space is isotropic. Let $G(k)_P$ be its stabilizer. Because T and B are σ -stable, the highest weight space contains a vector v in the standard module of $G(q)$. The subgroup $G(q)_P$ was previously defined only up to conjugacy in $G(q)$. Now we fix our choice by setting $G(q)_P = G(k)_P \cap G(q)$. The weight of v is ω_1 , so the element $v^{q-1} \in S^{q-1}(V)$ is a weight vector of weight $(q - 1)\omega_1$, this weight being the highest in $S^{q-1}(V)$ and having multiplicity one. It follows that $S^{q-1}(V)$ has a unique composition factor isomorphic to $L((q - 1)\omega_1)$ and that $G(k)_P$ stabilizes the highest weight space in this simple module. Since the restriction of the weight $(q - 1)\omega_1$ to $T(q)$ is trivial, it follows that $G(q)_P$ acts trivially on this highest weight space. Therefore $L = L((q - 1)\omega_1)$.

In the case of the unitary group the standard module V of $G(k)$ has highest weight ω_1 and the dual module V^* has highest weight ω_ℓ . Let v and δ be, respectively, weight vectors with these weights. Then the vector $v^{q-1} \otimes \delta^{q-1}$ spans the highest weight space in $S^{q-1}(V) \otimes S^{q-1}(V^*)$. Now there is a $kG(q)$ -isomorphism $V^{(q)} \cong V^*$, mapping the highest weight vector $v \in V^{(q)}$ to the highest weight vector $\delta \in V^*$. Hence $G(q)_P$ stabilizes the span of $v^{q-1} \otimes \delta^{q-1}$, which is the weight space of weight $(q - 1)(\omega_1 + \omega_\ell)$, the highest in $S^{q-1}(V) \otimes S^{q-1}(V^*)$. It follows that $G(q)_P$ stabilizes the highest weight space in the simple module $L((q - 1)(\omega_1 + \omega_\ell))$ and since the restriction of $(q - 1)(\omega_1 + \omega_\ell)$ to $T(q)$ is equal to the restriction of $(q - 1)(\omega_1 + q\omega_1) = (q^2 - 1)\omega_1$, which is trivial, we have shown that $G(q)_P$ fixes a nonzero vector in $L((q - 1)(\omega_1 + \omega_\ell))$, which implies that $L \cong L((q - 1)(\omega_1 + \omega_\ell))$.

Finally, we give an analogous argument of which the case $\ell = 4$ applies in the case (A'') . In this case we can find a totally singular 2-dimensional subspace of the standard module $V(q^2)$ of $G(q)$ such that $\wedge^2 Z$ spans the highest weight space in $\wedge^2(V) = \wedge^2(V(q^2) \otimes_{\mathbf{F}_{q^2}} k)$. We define $G(q)_P$ to be the stabilizer in $G(q)$ of Z and $G(k)_P$ to be the stabilizer of the highest weight space (of weight ω_2) in $\wedge^2(V)$. Then $G(k)_P$ also stabilizes the highest weight space in $S^{q-1}(\wedge^2(V))$. Since $V^* \cong V^{(q)}$ as $kG(q)$ -modules, it follows that $G(q)_P$ stabilizes the one-dimensional highest weight (of weight $\omega_{\ell-1}$) in $\wedge^2(V^*)$ and thus also the space of weight $(q - 1)\omega_3$ in $S^{q-1}(\wedge^2(V^*))$ and finally the highest weight space of weight $(q - 1)(\omega_2 + \omega_{\ell-1})$ in $S^{q-1}(\wedge^2(V)) \otimes S^{q-1}(\wedge^2(V^*))$. This implies that $G(q)_P$ stabilizes the highest weight space in $L((q - 1)(\omega_2 + \omega_{\ell-1}))$, and we see by inspection that this space affords the trivial character of $T(q)$. We conclude that $L \cong L((q - 1)(\omega_2 + \omega_{\ell-1}))$ in this case. \square

Next, we obtain a further refinement using Steinberg's Tensor Product Theorem [17, Theorem 1.1]. In the orthogonal cases the p -adic expression of the highest weight of L is

$$(q-1)\omega_1 = (p-1)\omega_1 + p(p-1)\omega_1 + \cdots + p^{t-1}(p-1)\omega_1.$$

So by Steinberg's Tensor Product Theorem, we have

$$L = L((q-1)\omega_1) = L((p-1)\omega_1) \otimes L((p-1)\omega_1)^{(p)} \otimes \cdots \otimes L((p-1)\omega_1)^{(p^{t-1})}.$$

This is a twisted tensor product, with t tensor factors, in which the factor $L((p-1)\omega_1)^{(p^i)}$ is the module obtained from the simple $G(k)$ -module of highest weight $(p-1)\omega_1$ by composing the representation with the i -th power of the Frobenius endomorphism of $G(k)$.

In the unitary case we have the analogous result with $L((p-1)\omega_1)$ replaced by $L((p-1)(\omega_1 + \omega_\ell))$ and in the case of the dual Hermitian quadrangle we replace $L((p-1)\omega_1)$ with $L((p-1)(\omega_2 + \omega_3))$.

We can summarize these findings in the following statement.

Theorem 11.6.

- (a) $\text{rank}_p A_{11} = 1 + (\dim_k L((p-1)\omega_1))^t$ in the orthogonal cases (Theorem 1.6).
- (b) $\text{rank}_p A_{11} = 1 + (\dim_k L((p-1)(\omega_1 + \omega_\ell)))^t$ in the Hermitian case (Theorem 1.8).
- (c) $\text{rank}_p A_{11} = 1 + (\dim_k L((p-1)(\omega_2 + \omega_3)))^t$ in case of the dual Hermitian generalized quadrangle (Theorem 1.9).

Now the theorems of Section 1.2 follow from Theorems 1.1, 1.2, 1.3, and 1.5.

Acknowledgments

We wish to thank Eric Moorhouse for helpful discussions and for showing us his computer calculations. We are also indebted to the developers of the computer algebra system SAGE [16], which was used to work out many examples.

References

- [1] H.H. Andersen, J.C. Jantzen, Cohomology of induced representations for algebraic groups, *Math. Ann.* 269 (1984) 487–525.
- [2] A. Blokhuis, G.E. Moorhouse, Some p -ranks related to orthogonal spaces, *J. Algebraic Combin.* 4 (1995) 295–316.
- [3] R.W. Carter, *Simple Groups of Lie Type*, John Wiley, London, 1972.
- [4] C.W. Curtis, Modular representations of finite groups with split BN-pairs, in: *Seminar on Algebraic Groups and Related Finite Groups*, in: *Lecture Notes in Math.*, vol. 131, Springer-Verlag, Berlin, 1969, pp. 57–95.
- [5] J. Dieudonné, *La Géométrie des Groupes Classiques*, *Ergeb. Math. Grenzgeb.*, vol. 5, Springer-Verlag, Berlin, 1955.
- [6] W. Fulton, J. Harris, *Representation Theory, a First Course*, *Grad. Texts in Math.*, vol. 129, Springer-Verlag, New York, 1991.
- [7] J.M. Goethals, P. Delsarte, On a class of majority-logic decodable cyclic codes, *IEEE Trans. Inform. Theory* 14 (1968) 182–188.
- [8] J.C. Jantzen, *Representations of Algebraic Groups*, Academic Press, London, 1987.
- [9] J.C. Jantzen, Darstellungen halbeinfacher Gruppen und kontravariante Formen, *J. Reine Angew. Math.* 290 (1977) 117–141.
- [10] A. Kleshchev, J. Sheth, On extensions of simple modules over symmetric and algebraic groups, *J. Algebra* 221 (1999) 705–722.
- [11] F.J. MacWilliams, H.B. Mann, On the p -rank of the design matrix of a difference set, *Inform. and Control* 12 (1968) 474–489.
- [12] G.E. Moorhouse, Some p -ranks related to Hermitian varieties, *J. Statist. Plann. Inference* 56 (1996) 229–241.
- [13] G. Navarro, P.H. Tiep, p -Brauer characters of p' -degree and self-normalizing Sylow p -subgroups, *J. Group Theory*, in press.
- [14] S. Payne, J. Thas, *Finite Generalized Quadrangles*, Pitman, London, 1984.
- [15] K.J.C. Smith, On the p -rank of the incidence matrix of points and hyperplanes in a finite projective geometry, *J. Combin. Theory* 1 (1969) 122–129.
- [16] W. Stein, et al. Sage Mathematics Software (Version 3.4.1), The Sage Development Team, 2009, <http://www.sagemath.org/>.
- [17] R. Steinberg, Representations of algebraic groups, *Nagoya Math. J.* 22 (1963) 33–56.
- [18] R. Steinberg, Endomorphisms of linear algebraic groups, *Mem. Amer. Math. Soc.* 80 (1968).