On asymptotic properties of matrix semigroups with an invariant cone

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Abstract

Recently, several research efforts showed that the analysis of joint spectral characteristics of sets of matrices can be simplified when these matrices share an invariant cone. We prove new results in this direction.

We prove that the joint spectral subradius is continuous in the neighborhood of sets of matrices that leave an embedded pair of cones invariant.

We show that both the averaged maximal spectral radius, as well as the maximal trace, where the maximum is taken on all the products of the same length \( t \), converge towards the joint spectral radius when \( t \) increases, provided that the matrices share an invariant cone, and additionally one of them is primitive.

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1. Introduction

The Perron–Frobenius theorem is one of the most well known theorems in linear algebra. It states strong properties that matrices with nonnegative entries enjoy. Starting in the fifties, it has become clear that not only this theorem, but many other properties of nonnegative matrices can be generalized, the fundamental feature being of importance for a matrix \( A \) is that there exists an invariant proper cone \( K \), that is, a proper cone such that \( AK \subset K \). (A cone \( K \) is said proper if it is closed, convex, with nonempty interior, and contains no straight line. Unless explicitly mentioned, all cones are supposed to be proper...
in the following.) If \( K \) is an invariant cone for \( A \), we say that \( A \) is \( K \)-nonnegative. If \( A(K \setminus \{0\}) \subset \text{int} K \), we say that \( A \) is \( K \)-positive. Finally if there exists a natural number \( t \) such that \( A^t \) is \( K \)-positive, we say that \( A \) is primitive. (All these definitions are obvious generalizations of the case of nonnegative matrices with \( K = \mathbb{R}^n_+ \).) As an example of the good properties that such matrices enjoy, let us mention the generalized Perron–Frobenius theorem (see for instance [1,2]).

**Theorem 1** (Generalized Perron–Frobenius theorem). Let \( K \) be a proper cone. If a matrix \( P \) is \( K \)-primitive, then it has a single eigenvalue of largest modulus, which moreover is a real positive number.

The interested reader can find a survey of properties of \( K \)-nonnegative matrices in [3] (see also [4,5] for the study of situations where several matrices share a common invariant cone).

Our goal is to exploit the assumption of \( K \)-nonnegativity for the study of finitely generated matrix semigroups, and more precisely of joint spectral characteristics. The joint spectral characteristics of a set of matrices are quantities that allow to describe the asymptotic behaviour of the semigroup generated by this set, when the length of the products increases. In this paper we will restrict our attention to two of these quantities:

**Definition 1.** For a bounded set of matrices \( \Sigma \subset \mathbb{R}^{n \times n} \), the joint spectral radius \( \rho(\Sigma) \) and joint spectral subradius \( \tilde{\rho}(\Sigma) \) are respectively defined as:

\[
\rho(\Sigma) \triangleq \lim_{t \to \infty} \sup \{ \|A_1 \cdots A_t\|^{1/t} : A_i \in \Sigma \},
\]

\[
\tilde{\rho}(\Sigma) \triangleq \lim_{t \to \infty} \inf \{ \|A_1 \cdots A_t\|^{1/t} : A_i \in \Sigma \}.
\]

Both these limits exist and do not depend on the norm chosen. They are natural generalizations of the notion of spectral radius of a matrix (i.e., the maximal modulus of the eigenvalues) to a set of matrices. The joint spectral radius appeared in [6], and the joint spectral subradius in [7]. Not only these quantities account for stability issues of switching linear systems, but they have also found many applications in various different fields of Engineering, Mathematics, and Computer Science. See [8] for a recent survey on these quantities.

The issue of computing the joint spectral radius has been largely studied, and several negative results are available in the literature. For instance, its exact computation, as well as the computation of the subradius, are both known to be Turing-impossible. The hardness of this task has led to a rich literature, where techniques from different fields (from Control Theory to Ergodic Theory, Automata Theory, etc.) have been applied to the topic (see [9–14] as a few examples).

The joint spectral subradius has been less studied until recently. Though some related quantities like the so-called Bohl exponent have been studied in the literature, it seems that the only available methods that allow to effectively compute the joint spectral subradius have been proposed only very recently [15,16]. These methods are proved to be efficient only in some favorable cases, namely, when the set of matrices share a common invariant embedded pair of cones (see below for definitions). It appears that this assumption allows for more appealing properties, and in this note we prove (Section 2) that under the same hypothesis, the joint spectral subradius is a continuous function. This explains at least in part why the numerical computation of the subradius seems easier in this case. Then, in Section 3, we study the joint spectral radius, and we generalize a recent result on nonnegative matrices to any set leaving a cone invariant. This result is about the convergence of the maximum trace, and the maximum spectral radius of the products towards the joint spectral radius, when the length of the products increases.

The joint spectral characteristics have attracted much attention in recent years, especially in the case of nonnegative matrices because many applications involve such matrices [17–19,9]. These applications make the theoretical study of \((K-)\)nonnegative matrices of particular importance.
2. Continuity of the subradius

The continuity of the joint spectral radius (w.r.t. the Hausdorff distance\(^2\)) is well-known:

**Proposition 1** [20]. For any bounded set of matrices \(\Sigma \in \mathbb{R}^n\), and for any \(\epsilon > 0\), there is a \(\delta > 0\) such that
\[
D(\Sigma, \Sigma') < \delta \Rightarrow |\rho(\Sigma) - \rho(\Sigma')| < \epsilon.
\]

Less is known on the subradius. This quantity is not continuous, as shown on the next example, drawn from [8]:

**Example 1.** The sequence of sets
\[
\Sigma_k = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -\frac{1}{k} & 1 \end{pmatrix} \right\}, \ k \in \mathbb{N},
\]
converges towards
\[
\Sigma = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}
\]
when \(k \to \infty\). For any \(k \in \mathbb{N}\), we have \(\check{\rho}(\Sigma_k) = 0\) because the product \((A_1A_k^0)^2\) is the zero matrix. However \(\check{\rho}(\Sigma) = 1\) (In order to see this, observe that the lower right entry of any product is equal to one). Hence, the joint spectral subradius is not continuous in the neighborhood of \(\Sigma\).

Thus, the nonnegativity of matrices is not sufficient for ensuring the continuity of the subradius in the neighborhood of the set. However, we show in this section that if the matrices share a second invariant cone, then we have the continuity. More precisely, let us consider a cone \(K \subset \mathbb{R}^n\). We say that a convex closed cone \(K'\) is *embedded in* \(K\) if \((K' \setminus \{0\}) \subset \text{int}K\). In this case, following [15], we call \(\{K, K'\}\) an embedded pair. Note that the embedded cone \(K'\) may be degenerate, i.e., may have an empty interior. An embedded pair \(\{K, K'\}\) is called an *invariant pair* for a matrix \(A\) (or a set of matrices \(\Sigma\)) if the cones \(K\) and \(K'\) are both invariant for \(A\) (for the matrices in \(\Sigma\)). Note that the zero matrix trivially leaves invariant any embedded pair of cones.

The following definition aims at characterizing the “embeddedness” of the pair \(\{K, K'\}\); see Fig. 1 for an illustration.

**Definition 2** [15]. For a given embedded pair \(\{K, K'\}\) the value \(\beta(K, K')\) is the smallest number such that for any line intersecting \(K\) and \(K'\) by segments \([x, y]\) and \([x', y']\) respectively (with \([x, x'] \subset [x, y']\)) one has
\[
1 \leq \frac{|x - y'|}{|x - x'|} \leq \beta.
\]

It is quite easy to see that for any embedded pair \(\{K, K'\}\), the constant \(\beta(K, K')\) is finite. (It follows from the compactness of the unit ball in \(\mathbb{R}^n\).)

\(^2\) The *Hausdorff distance* measures the distance between sets of points in a metric space:
\[
D(\Sigma, \Sigma') \triangleq \max \left\{ \sup_{A \in \Sigma'} \inf_{A' \in \Sigma} ||A - A'||, \sup_{A \in \Sigma} \inf_{A' \in \Sigma'} ||A - A'|| \right\}.
\]
In what follows we denote by $\Sigma^t$ the set of products of length $t$ of matrices in $\Sigma$. Also, we note $\geq_K$ the partial order defined by a cone $K$:

$$x \geq_K y \iff x - y \in K.$$ 

In the developments below we use the two following results:

**Lemma 1** [21]. Let $\Sigma$ be a compact set of matrices that share an invariant cone $K$. If there exists a real number $r > 0$, and a nonzero vector $x \in K$ such that

$$\forall A \in \Sigma, \quad Ax \geq_K r x,$$

then $\tilde{\rho}(\Sigma) \geq r$.

**Theorem 2** [15, Theorem 2.12]. For any compact set $\Sigma$ with an invariant pair $\{K, K'\}$, there exists a nonzero vector $x \in K'$ such that

$$\forall A \in \Sigma, \quad Ax \geq_K \left( \tilde{\rho}(\Sigma)/\beta \right) x,$$

where $\beta = \beta(K, K')$.

We are now in position to prove the main theorem of this section:

**Theorem 3.** Let $\Sigma$ be a compact set of matrices in $\mathbb{R}^{n \times n}$, and let $\Sigma_k$ be a sequence of sets in $\mathbb{R}^{n \times n}$ that converges to $\Sigma$ in the Hausdorff metric. If $\Sigma$ leaves an embedded pair of cones invariant, then,

$$\tilde{\rho}(\Sigma_k) \to \tilde{\rho}(\Sigma) \quad \text{as } k \to \infty.$$

**Proof.** Let us consider a set $\Sigma$ that leaves an embedded pair of cones $(K, K')$ invariant, such that $\beta(K, K') = \beta$. If $\tilde{\rho}(\Sigma) = 0$ then the joint spectral subradius is continuous at $\Sigma$, being upper semicontinuous and nonnegative in general [8]. We can then suppose that $\tilde{\rho}(\Sigma) > 0$ and, by scaling the set of matrices, we suppose that $\tilde{\rho}(\Sigma) = 2$ (the joint spectral subradius is a homogeneous function of the entries of the matrices).

Fix $\epsilon > 0$. By upper semicontinuity, we know that there exists a $\delta > 0$ such that

$$D(\Sigma', \Sigma) < \delta \quad \Rightarrow \quad \tilde{\rho}(\Sigma') < \tilde{\rho}(\Sigma) + \epsilon.$$
We still have to show that for some $\delta' > 0$, if $D(\Sigma', \Sigma) < \delta'$, then
\[
\tilde{\rho}(\Sigma') > \tilde{\rho}(\Sigma) - \epsilon.
\]
It follows directly from the definition (1) that
\[
\tilde{\rho}(\Sigma') = \tilde{\rho}(\Sigma).
\]
Thus, replacing $\Sigma$ with $\Sigma^t$ in Theorem 2, one obtains that
\[
\forall x \in K, \quad \exists A \in \Sigma^t, \quad Ax \geq_K (2^t/\beta)x.
\]
Hence, there exists an integer $t$ and a vector $x \in K^t$, $|x| = 1$, such that
\[
Ax \geq_K (2 - \epsilon/2)^t x.
\]
Take also $t$ large enough such that $(2 - \epsilon/2)^t > (2 - \epsilon)^t + \epsilon/2$. Since $x \in K^t \setminus \{0\} \subseteq \text{int}(K)$, we can define $\eta > 0$ such that for any vector $y$:
\[
|y| < \eta, \quad \Rightarrow \quad (\epsilon/2)x + y \in K.
\]
Then, $\tilde{\rho}(\Sigma') = 2 - \epsilon$. \hfill $\square$

**Corollary 1.** The joint spectral subradius is continuous in the neighborhood of any compact set of matrices with positive entries.

**Proof.** It is known (see [15, Corollary 2.14]) that if a matrix has positive entries and in each column of any matrix the ratio between the greatest and the smallest elements does not exceed $c$, then $\Sigma$ has an invariant pair of embedded cones, which are $\mathbb{R}^n_+$ and
\[
\mathbb{R}^n_{+,-c} = \{ x \in \mathbb{R}^n_+ | x_{\max} \leq cx_{\min} \}.
\]
Moreover, $\beta(\mathbb{R}^n_+, \mathbb{R}^n_{+,-c}) = c^2$.

Now, by compactness of the set of matrices, there is a finite $c^*$ such that $\mathbb{R}^n_{+,-c^*}$ is an invariant pair for all matrices in $\Sigma$, which concludes the proof. \hfill $\square$

### 3. Asymptotic regularity

It is well known that for any bounded set of matrices, the joint spectral radius can be alternatively defined in terms of the maximal spectral radius (instead of the maximal norm):

**Theorem 4** (Joint spectral radius theorem, [22]). For any bounded set of matrices $\Sigma$,
\[
\limsup_{t \to \infty} \sup \{ \rho^{1/t}(A_1 \ldots A_t) : A_t \in \Sigma^t \} = \rho(\Sigma).
\]
Some attention has been recently given to a peculiarity in the alternative definition provided by Theorem 4: the maximal averaged spectral radius asymptotically converges towards the value of the joint spectral radius in limit superior, but not at every time steps. Simple examples illustrate this fact, as for instance

$$\Sigma = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$  \hfill (3)

For this set of matrices $\rho(\Sigma) = 1$ but all products of odd length have a spectral radius equal to zero. It is thus natural to try to understand when the maximal averaged spectral radius actually converges or to give sufficient conditions for it. Moreover, the next proposition shows that the joint spectral radius can also be defined as the limit superior of the rate of growth of the maximal trace in the semigroup:

**Proposition 2** [23]. For any finite set of matrices, the joint spectral radius satisfies

$$\rho(\Sigma) = \limsup_{t \to \infty} \max_{A \in \Sigma^t} \left\{ \frac{\text{tr}(A)}{t} \right\}. \hfill (4)$$

Here again, the joint spectral radius is defined as a limit superior, and one could wonder when the sequence actually converges. A sufficient condition relying on the nonnegativity of the matrices has been proposed recently:

**Theorem 5** [24]. Let $\Sigma$ be a finite set of nonnegative matrices. If one of them is primitive, then

$$\max_{A \in \Sigma^t} \left\{ \frac{\text{tr}(A)}{t} \right\} \to \rho(\Sigma) \hfill (5)$$

as $t \to \infty$.

Note that already for one matrix, and for the simpler common spectral radius, the assumption of $A$ being primitive is critical. For instance, if $A$ is a cyclic permutation matrix, the theorem above does not hold.

In this section we generalise Theorem 5 to arbitrary invariant cones:

**Theorem 6.** Consider a bounded set of matrices $\Sigma$ leaving a cone $K$ invariant. If there is a matrix $A \in \Sigma$ which is $K$-primitive, then, both the quantity

$$\max_{A \in \Sigma^t} \left\{ \frac{\text{tr}(A)}{t} \right\} \hfill (6)$$

and

$$\max_{A \in \Sigma^t} \left\{ \frac{\rho}{t} \right\} \hfill (7)$$

converge towards $\rho$ when $t$ tends to $\infty$.

The proof is inspired from developments in [24] and results from [25] which we now recall.

**Definition 3** (Kronecker product). Let $A \in \mathbb{R}^{n_1 \times n_2}$, $B \in \mathbb{R}^{m_1 \times m_2}$. The Kronecker product of $A$ and $B$ is a matrix in $\mathbb{R}^{n_1m_1 \times n_2m_2}$ defined as

$$(A \otimes B) \triangleq \begin{pmatrix} A_{1,1}B & \ldots & A_{1,n_2}B \\ \vdots & \ddots & \vdots \\ A_{n_1,1}B & \ldots & A_{n_1,n_2}B \end{pmatrix}.$$
The \textit{kth Kronecker power of} $A$, denoted $A^{\otimes k}$, is defined inductively as 

$$A^{\otimes k} = A \otimes A^{\otimes (k-1)} = A^{\otimes 1}.$$ 

The following proposition is well known.

\textbf{Proposition 3.} \hspace{1em} (1) For any matrices $A$, $B$, $C$, $D$, 

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

(2) For any matrix $A$, we have 

$$\text{tr}(A^{\otimes k}) = \text{tr}^k(A).$$

The main reason for introducing Kronecker products in our context is the following theorem:

\textbf{Theorem 7 [25].} Let $\Sigma = \{A_1, \ldots, A_m\}$ be a set of matrices that leaves a cone invariant. Then, we have the following property:

$$\rho(\Sigma) = \lim_{k \to \infty} \rho^{1/k}(A_1^{\otimes k} + \cdots + A_m^{\otimes k}). \tag{8}$$

We prove a last technical result that will be necessary in the proof of Theorem 6. (The Kronecker product of vectors is obtained by simply considering a vector as a $1 \times n$ matrix and by applying Definition 3.)

\textbf{Lemma 2.} Let $K$ be a proper cone. If a matrix $A$ is $K$-primitive, then for any $s$, $A^{\otimes s}$ is $K^{\otimes s}$-primitive, where 

$$K^{\otimes s} \triangleq \text{Conv}\{x_1 \otimes \cdots \otimes x_s : x_i \in K, \ i = 1, \ldots, s\}$$

is a proper cone.

\textbf{Proof.} The fact that $K^{\otimes s}$ is a proper cone is proved in [25, Lemma 4].

It is clear that if $A$ is $K$-nonnegative, then $A^{\otimes s}$ is $K^{\otimes s}$-nonnegative, since 

$$A^{\otimes s}(x_1 \otimes \cdots \otimes x_s) = Ax_1 \otimes \cdots \otimes Ax_s.$$ 

In order to prove that $A^{\otimes s}$ is primitive, we will show that if $A$ is $K$-primitive, all extremal points of $K^{\otimes s}$ (except the origin) are mapped in the interior of $K^{\otimes s}$ by a suitable power of $A^{\otimes s}$.

So, let us take an extremal point $\tilde{x} = x_1 \otimes \cdots \otimes x_s$ of $K^{\otimes s}$, and show that for some $t > 0$, 

$$(A^{\otimes s})^t \tilde{x} \in \text{int} K^{\otimes s}.$$ 

and $A$ is primitive, it suffices to show that 

$$x_i \in \text{int} K \implies x_1 \otimes \cdots \otimes x_s \in \text{int} K^{\otimes s}.\tag{9}$$

We will in fact show the slightly stronger property that for any set of proper cones $K_i$ in finite dimensional vector spaces,

$$x_i \in \text{int} K_i \implies x_1 \otimes \cdots \otimes x_s \in \text{int}(K_1 \otimes \cdots \otimes K_s),$$

where 

$$K_1 \otimes \cdots \otimes K_s \triangleq \text{Conv}\{x_1 \otimes \cdots \otimes x_s : x_i \in K_i\}.$$
By induction, it is sufficient to prove it for $s = 2$. Let us be given two proper cones $K_1 \in \mathbb{R}^{n_1}$, $K_2 \in \mathbb{R}^{n_2}$, $x_1 \in \text{int}(K_1)$. Suppose by contradiction that $x_1 \otimes x_2 \in K_1 \otimes K_2 \setminus \text{int}(K_1 \otimes K_2)$. Then, there exists a vector $z \in \mathbb{R}^{n_1 \times n_2}$, $z \neq 0$, such that $(z, x_1 \otimes x_2) = 0$, and for all $w \in K_1 \otimes K_2$, $(z, w) \geq 0$. Now, since $x_1 \in \text{int}K_1$, for any $i, 1 \leq i \leq n_1$, there exists a $\delta > 0$ such that for both $\lambda = \delta$, $\lambda = -\delta$, $(x_1 + \lambda e_i) \otimes x_2 \in K_1 \otimes K_2$ (i.e. the $i$th standard basis vector). This implies that

$$(z, (x_1 + \lambda e_i) \otimes x_2) = (z, x_1 \otimes x_2) + (z, \lambda e_i \otimes x_2) = (z, \lambda e_i \otimes x_2) \geq 0,$$

and thus $(z, e_i \otimes x_2) = 0$ (because $\lambda$ can be positive and negative). For the same reason, for any $j$, $(z, x_1 \otimes e_j) = 0$. Now,

$$(z, (x_1 + \lambda e_i) \otimes (x_2 + \gamma e_j)) = (z, x_1 \otimes x_2) + (z, \lambda e_i \otimes x_2) + (z, x_1 \otimes \gamma e_j) + (z, \lambda \gamma e_i \otimes e_j) = (z, \lambda \gamma e_i \otimes e_j) \geq 0.$$

Thus, $(z, e_i \otimes e_j) = 0$ for all $i, j$, which implies that $z = 0$, a contradiction. □

We can now prove the main result of this section.

**Proof.** (Proof of Theorem 6). We first restrict our attention to finite sets of matrices, and we fix $\Sigma = \{A_1, \ldots, A_n\}$.

We claim that for any natural number $k$, the $k$th Kronecker powers of the matrices in $\Sigma$ satisfy the property

$$\lim_{t \to \infty} \text{tr}^{1/t}(A_1^\otimes k + \cdots + A_m^\otimes k)^t = \rho(A_1^\otimes k + \cdots + A_m^\otimes k).$$

By Lemma 2 above, one of the matrices $A_i^\otimes k$ is $K^\otimes k$-primitive. Moreover, it is well known that for any cone $K$, a sum of $K$-nonnegative matrices, one of which is $K$-primitive, is also $K$-primitive, then, so is $A_1^\otimes k + \cdots + A_m^\otimes k$.

Thus, by Theorem 1, the claim is proved.

Now, note that for any $k, t \in \mathbb{N}$,

$$\text{tr}\left(\left(\sum_{A \in \Sigma} A^\otimes k\right)^t\right) = \sum_{A \in \Sigma^t} \text{tr}(A^\otimes k) \leq m^t \left(\max_{A \in \Sigma^t} \text{tr}^k(A)\right). \quad (9)$$

For deriving the above relations, we successively used items 1 and 2 of Proposition 3. Dividing this equation by $m^t$ and taking the power $1/(kt)$, we obtain

$$\text{tr}^{1/(kt)}\left(\left(\sum_{A \in \Sigma} A^\otimes k/m\right)^t\right) \leq \max_{A \in \Sigma^t} \{\text{tr}^{1/t}(A)\}. \quad (10)$$

Now, by the claim, for any $k \in \mathbb{N}$, the left hand side in (10) tends towards $\rho^{1/k}(\sum_{A \in \Sigma} A^\otimes k/m)$ as $t \to \infty$. 

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This proves part one of the theorem, as the latter quantity is arbitrarily close to \( \rho(\Sigma) \) for large \( k \) (Theorem 7).

For the second part of the theorem, since \( \rho(A) \geq \text{tr}A/n \) (n is the dimension of the matrices), we have

\[
\max_{A \in \Sigma^t} \{(\text{tr}(A)/n)^{1/t}\} \leq \max_{A \in \Sigma^t} \{\rho^{1/t}(A)\},
\]

which immediately implies that \( \max_{A \in \Sigma^t} \{\rho^{1/t}(A)\} \to \rho \) when \( t \to \infty \).

We now relax the assumption that \( \Sigma \) is finite. We present the proof for (7). The proof for the maximum trace is the same word by word.

We fix an arbitrary \( \epsilon > 0 \). Consider a sequence of finite sets \( \Sigma_s \) which converges to \( \Sigma \) in Hausdorff metric. We choose such a sequence such that \( \Sigma_s \subset \Sigma_{s+1} \). Also, we suppose \( A \in \Sigma_1 \), where \( A \) is the primitive matrix in \( \Sigma \). Thus, for all \( s \in \mathbb{N} \), we have

\[
\max_{A \in \Sigma_s^t} \{\rho^{1/t}(A)\} \leq \max_{A \in \Sigma_{s+1}^t} \{\rho^{1/t}(A)\} \leq \max_{A \in \Sigma^t} \{\rho^{1/t}(A)\}.
\]

By continuity of the JSR, one can take \( s \) such that \( \rho(\Sigma_s) \geq \rho(\Sigma) - \epsilon/2 \). We also take \( t \) such that

\[
\forall t' \geq t, \quad \max_{A \in \Sigma_{s}^{t'}} \{\rho^{1/t'}(A)\} \geq \rho(\Sigma_s) - \epsilon/2.
\]

We obtain, putting the two equations above together, that

\[
\forall t' \geq t, \quad \max_{A \in \Sigma_{s}^{t'}} \{\rho^{1/t'}(A)\} \geq \rho(\Sigma) - \epsilon. \quad \square
\]

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References