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The Convergence of the Weiszfeld Algorithm

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Abstract—We investigate the convergence properties of the Weiszfeld procedure when it is applied to the approximated ℓ_p-norm single-facility location problem where $p > 2$. We show that convergence for $p > 2$ can be obtained by introducing a step size factor to the iterative procedure. Some numerical test results are also given.

Keywords—Weiszfeld algorithm, Convergence properties.

1. INTRODUCTION

A single facility minisum location problem (SFMLP) in the Euclidean plane ($\Re^2$) is stated as follows:

$$\min W(\mathbf{x}) = \sum_{j=1}^{n} w_j\, d(\mathbf{x}, \mathbf{a}_j), \tag{1}$$

where n is the number of fixed facilities; $\mathbf{a}_j = (a_{j1}, a_{j2})$, $j = 1, \dots, n$ are the fixed facility locations; $\mathbf{x} = (x_1, x_2)$ is the sought after location of the new facility; $w_j > 0$, $j = 1, \dots, n$ is the weight (demand) associated with the fixed facility j; and $d(\mathbf{u}, \mathbf{v})$ is some distance function used to calculate the distance between any two points $\mathbf{u}, \mathbf{v} \in \Re^2$.

As is readily seen in formulation (1), a distance predicting function is an important part of the objective function of a continuous location model. Since the model should represent the real situation as closely as possible, the accuracy of the distance predicting function employed plays a crucial role in terms of the validity and the applicability of locational decisions. A member of the family of ℓ_p-norm, $\ell_p(\mathbf{u}, \mathbf{v}) = [|u_1 - v_1|^p + |u_2 - v_2|^p]^{1/p}$, $p \geq 1$, is generally used as the distance function in continuous facility location models. The ℓ_2-norm (Euclidean distance) and the ℓ_1-norm (rectangular distance) are two well-studied special members of the ℓ_p-norm family. Using the notation in (1) an SFMLP with the ℓ_p-norm becomes

$$\min S(\mathbf{x}) = \sum_{j=1}^{n} w_j\, \ell_p(\mathbf{x}, \mathbf{a}_j). \tag{2}$$

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A weighted sum of order p, denoted by $\ell_{bp}(\mathbf{x})$, can be utilized to estimate distances in a transportation network. The ℓ_{bp} distance between any two points $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ in Euclidean plane is given by

$$\ell_{bp}(\mathbf{u}, \mathbf{v}) = \left[b_1 |u_1 - v_1|^p + b_2 |u_2 - v_2|^p\right]^{1/p}, \qquad b_1, b_2 > 0, \quad p \geq 1. \tag{3}$$

The ℓ_{bp}-norm is a generalization of the well-known weighted ℓ_p-norm. If for a fixed p, the equality ${b_1}^{1/p} = {b_2}^{1/p} = k$ holds, then one obtains the weighted ℓ_p-norm where k represents the weight or the stretch factor. Furthermore, if $b_1 = b_2 = 1$, the rectangular and Euclidean distances can be obtained from the ℓ_{bp}-norm by setting $p = 1$ and $p = 2$, respectively.

With the ℓ_{bp}-norm one introduces unequal weights or nonsymmetric distance irregularities along the axis directions. An empirical work on 17 geographic regions showed that the ℓ_{bp}-norm is better than the weighted ℓ_p-norm in terms of the accuracy of distance estimations [1]. Particularly in geographical regions with a predominant direction of nonlinearity (e.g., a mountain range), the gain in the accuracy of distance estimations with the ℓ_{bp}-norm is more pronounced.

In order to model distances in a geographical region a goodness-of-fit criterion is minimized. One such criterion known as "Sum of Squared Deviations" (SD) is given as follows:

$$\mathrm{SD} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{\left(d(\mathbf{a}_i, \mathbf{a}_j) - A(\mathbf{a}_i, \mathbf{a}_j)\right)^2}{A(\mathbf{a}_i, \mathbf{a}_j)} \tag{4}$$

where $d(\mathbf{a}_i, \mathbf{a}_j)$ and $A(\mathbf{a}_i, \mathbf{a}_j)$ are the predicted and actual distances between points $\mathbf{a}_i$ and $\mathbf{a}_j$, and n is the number of points in the data set. Although the ℓ_{bp}-norm is a three-parameter (b_1, b_2, and p) distance function as opposed to a two-parameter weighted ℓ_p-norm (k and p), the convexity of the goodness-of-fit criterion function SD in parameters b_1 and b_2 enables a fitting algorithm to determine the parameters of the three-parameter function almost as quickly as those of the two-parameter distance function [1].

General Solution Procedure with the ℓ_p-norm

In order to solve the single-facility location problem (2) a one-point iterative procedure is used [2]. This iterative procedure is a generalization of the Weiszfeld procedure which was originally devised for the Euclidean distance facility location problem [3]. The generalized Weiszfeld iterative procedure depends upon the convexity of the ℓ_p-norm, and thus, utilizes the first-order necessary and sufficient conditions. Since it is impossible to express the unknown variables (new facility locations) in the form of equations, the first-order derivatives cannot be solved directly. Instead, an iteration function is obtained by using these derivatives. Note that the first-order derivatives of $S(\mathbf{x})$ are not differentiable at the existing facility locations. Therefore, in order to avoid the problem caused by these discontinuities in the derivatives, an iterative procedure is devised by using a hyperbolic approximation of the ℓ_p-norm for actual computations.

A bound or a stopping rule is required to terminate the iterative procedure. There are several bounding methods examined in the literature. Among those, the rectangular bound, originally devised for single-facility Euclidean distance problems by Drezner [4] and extended to unapproximated ℓ_p-norm single-facility problems by Love and Dowling [5], is shown to be superior. The rectangular bound involves the solution of a rectangular distance location problem at each iteration of the Weiszfeld procedure.

In some cases, the optimal facility locations coincide with the existing facility locations. Thus, the existing facility locations are examined for optimality before applying the Weiszfeld procedure, and if an existing facility location is optimal then the rest of the solution procedure is not needed. This check is performed by using the fixed-point optimality conditions [6].

Solving the ℓ_{bp}-norm Location Models

There are two approaches to solve an ℓ_{bp} distance SFMLP.

APPROACH 1. USING THE PROCEDURES FOR THE ℓ_p-NORM. We first state the following equivalence property.

PROPERTY 1. *An equivalent ℓ_p-norm can be obtained from the $\ell_{bp}(\mathbf{x}, \mathbf{y})$ norm by scaling the horizontal and vertical components* $\mathbf{x}$ *and* $\mathbf{y}$ *by* $b_1^{1/p}$ *and* $b_2^{1/p}$, *respectively.*

PROOF. We rewrite the ℓ_{bp}-norm (3) as follows:

$$\ell_{bp}(\mathbf{z}) = \left[\left(b_1^{1/p}\left|z_1\right|\right)^p + \left(b_2^{1/p}\left|z_2\right|\right)^p\right]^{1/p},$$
$$\text{where } \mathbf{z} = (z_1, z_2) \text{ and } z_t = |x_t - y_t|, \qquad t = 1, 2.$$

Taking $|z_t'| = {b_t}^{1/p}\,|z_t|$, $t = 1, 2$, we obtain

$$\ell_{bp}(\mathbf{z}) = \left(|z_1'|^p + |z_2'|^p\right)^{1/p}.$$

Notice that $\ell_{bp}(\mathbf{z})$ is in the form of the ℓ_p-norm, i.e., we have

$$\ell_{bp}(\mathbf{z}) = \ell_p\left(\mathbf{z}'\right),$$

and the result follows.

Property 1 suggests that, after a scaling based modification in a location model's setting, the Weiszfeld procedure developed for the ℓ_p-norm single-facility location problem is readily applicable to the ℓ_{bp}-norm location problem. The existing facility locations $\mathbf{a}_j$, $j = 1, \dots, n$, in an ℓ_{bp}-norm location problem are first scaled in the corresponding directions by using the scale factors $b_1^{1/p}$ and $b_2^{1/p}$. The location problem with the new setting is solved by using the procedure developed for the ℓ_p-norm location model and the solution is then rescaled using the scale factors $b_1^{-1/p}$ and $b_2^{-1/p}$ for corresponding coordinates.

APPROACH 2. USING MODIFIED PROCEDURES FOR THE ℓ_{bp}-NORM. Making a direct use of Property 1, we can obtain modifications of the Weiszfeld procedure, the rectangular bounding method, and the fixed-point optimality condition for the ℓ_{bp}-norm location problem from the results developed for the ℓ_p-norm problem. This is done by including the scaling factors $b_1^{1/p}$ and $b_2^{1/p}$ in the expressions developed for the ℓ_p-norm location model.

The first approach is rather straightforward and utilizes the existing solution procedure for the SFMLP with the ℓ_p-norm. It additionally involves scaling and rescaling computations. The second approach uses specialized procedures for the ℓ_{bp}-norm location model.

This paper is organized as follows. In Section 2, we review the Weiszfeld procedure for the solution of the ℓ_p-norm single-facility location problem, and provide a generalization to the ℓ_{bp}-norm location model. In Section 3, we analyze the convergence properties of the Weiszfeld procedure when it is applied to single-facility minisum location problems.

2. MODIFIED WEISZFELD PROCEDURE

In this section, we review the iterative procedure for the ℓ_p-norm SFMLP and provide its generalization to the ℓ_{bp}-norm location model.

2.1. Procedure for SFMLP with the ℓ_p-norm

Since $\ell_p(\mathbf{x})$ is a norm, and thus, a convex function, it readily follows that problem (2) is a convex optimization problem. Furthermore, if the fixed facility locations are noncollinear, then $\ell_{bp}(\mathbf{x})$ is strictly convex. Therefore, assuming the optimal solution $\mathbf{x}^*$ is a differentiable point of $S(\mathbf{x})$, the first-order necessary and sufficient conditions require that

$$\frac{\partial S(\mathbf{x}^*)}{\partial x_t} = 0, \qquad t = 1, 2. \tag{5}$$

Evaluating the partial derivatives in (5) we have

$$\sum_{j=1}^{n} w_j \operatorname{sign}(x_t^* - a_{jt}) \frac{|x_t^* - a_{jt}|^{p-1}}{[\ell_p(\mathbf{x}^*, \mathbf{a}_j)]^{p-1}} = 0, \qquad t = 1, 2. \tag{6}$$

Noting that $(x_t - a_{jt}) = \operatorname{sign}(x_t - a_{jt})\, |x_t - a_{jt}|$, (6) can be rewritten as

$$\sum_{j=1}^{n} w_j \, (x_t^* - a_{jt}) \frac{|x_t^* - a_{jt}|^{p-2}}{[\ell_p(\mathbf{x}^*, \mathbf{a}_j)]^{p-1}} = 0, \qquad t = 1, 2. \tag{7}$$

Simplifying (7) and solving for $x_t{}^*$ we have

$$x_t^* = \frac{\sum\limits_{j=1}^{n} w_j \, |x_t^* - a_{jt}|^{p-2} \, [\ell_p(\mathbf{x}^*, \mathbf{a}_j)]^{1-p} \, a_{jt}}{\sum\limits_{j=1}^{n} w_j \, |x_t^* - a_{jt}|^{p-2} \, [\ell_p(\mathbf{x}^*, \mathbf{a}_j)]^{1-p}}, \qquad t = 1, 2. \tag{8}$$

Using (8) the one-point iteration scheme is devised as follows:

$$x_t^{k+1} = \frac{\sum\limits_{j=1}^{n} w_j \left|x_t^k - a_{jt}\right|^{p-2} \left[\ell_p(\mathbf{x}^k, \mathbf{a}_j)\right]^{1-p} a_{jt}}{\sum\limits_{j=1}^{n} w_j \left|x_t^k - a_{jt}\right|^{p-2} [\ell_p(\mathbf{x}^k, \mathbf{a}_j)]^{1-p}}, \qquad t = 1, 2; \tag{9}$$

where k represents the iteration number.

The iteration function (9) poses two main difficulties depending on the value of the parameter p in the application.

(i) If $p < 2$, then x_t^{k+1} is undefined along the hyperplanes $|x_t^k - a_{jt}| = 0$, where $j = 1, \dots, n$, and $t = 1, 2$.

(ii) If $p \geq 2$, then x_t^{k+1} is undefined at the existing facility locations a_j, $j = 1, \dots, n$.

In order to eliminate the obvious difficulty caused by the discontinuities in the derivatives, an approximation of the ℓ_p-norm is used in the objective function $S(\mathbf{x})$. The use of an approximation is discussed for rectangular distances by Wesolowsky and Love [7] and for Euclidean and rectangular distances by Eyster *et al.* [8]. Similar approximations are given for the ℓ_p-norm by Love and Morris [9], and Morris and Verdini [2]. Verdini [10] shows that the approximation given by Eyster *et al.* (for the Euclidean distance case) and Love and Morris is not appropriate when the Weiszfeld procedure is used for the ℓ_p distances problem with $p \geq 1$. Therefore, the approximation given here follows the one given by Morris and Verdini. We next present this approximation of $\ell_p(\mathbf{x})$, denoted by $\tilde{\ell}_p(\mathbf{x})$, and review its properties and the resulting iterative procedure.

Approximating Function for the ℓ_p-norm

Using the hyperbolic approximation of the ℓ_p-norm, the approximated distance $\tilde{\ell}_p$ between any two points $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ is given by

$$\tilde{\ell}_p(\mathbf{x}, \mathbf{y}) = \left[\left((x_1 - y_1)^2 + \epsilon\right)^{p/2} + \left((x_2 - y_2)^2 + \epsilon\right)^{p/2}\right]^{1/p}, \qquad p \geq 1, \quad \epsilon > 0. \tag{10}$$

Notice that the approximation to the ℓ_p distance is not a norm; it lacks the stationarity property, i.e., $\tilde{\ell}_p(\mathbf{0}) \neq 0$. However, it is still a convex function of $\mathbf{x}$ as shown by Morris and Verdini [2].

Iterative Procedure with Hyperbolic Approximation

Rewriting problem (2) with the hyperbolic approximating distance function $\tilde{\ell}_p(x)$, we have

$$\min \widetilde{S}(\mathbf{x}) = \sum_{j=1}^{n} w_j \left[\left((x_1 - a_{j1})^2 + \epsilon\right)^{p/2} + \left((x_2 - a_{j2})^2 + \epsilon\right)^{p/2}\right]^{1/p}, \quad \text{where } p \geq 1,\ \epsilon > 0 \text{ and } w_j \geq 0, \quad j = 1, \dots, n. \tag{11}$$

Clearly the function $\widetilde{S}(\mathbf{x})$, being a sum of n strictly convex functions, is a strictly convex function in $\mathbf{x} = (x_1, x_2)$. Therefore, again using the first-order necessary and sufficient conditions, and following the same steps given in (6)–(9) the modified iteration function is found as

$$x_t^{k+1} = \frac{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt}\right)^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{1-p} a_{jt}}{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt}\right)^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{1-p}}, \qquad t = 1, 2. \tag{12}$$

2.2. Generalization to SFMLP with the ℓ_{bp}-norm

We employ the following hyperbolic approximation of the ℓ_{bp}-norm. Using the notation given in (3)

$$\tilde{\ell}_{bp}(\mathbf{x}, \mathbf{y}) = \left[b_1 \left((x_1 - y_1)^2 + \epsilon\right)^{p/2} + b_2 \left((x_2 - y_2)^2 + \epsilon\right)^{p/2}\right]^{1/p}, \quad \text{where } b_1, b_2 > 0,\ p \geq 1,\ \epsilon > 0. \tag{13}$$

Similar to Property 1 given for the unapproximated case, we can write

$$\tilde{\ell}_{bp}(\mathbf{z}) = \tilde{\ell}_p\left(\mathbf{z}'\right), \tag{14}$$

where $\mathbf{z} = \mathbf{x} - \mathbf{y}$, $z_1' = b_1^{1/p} z_1$, $z_2' = b_2^{1/p} z_2$. We denote the small quantity associated with the ℓ_p-norm by ϵ', where $\epsilon' = \min\{b_1^{2/p}\,\epsilon,\ b_2^{2/p}\,\epsilon\}$. As suggested by relation (14), replacing the fixed facility locations a_{jt} with $b_t^{1/p} a_{jt}$ for $t = 1, 2$, $j = 1, \dots, n$, the unknown facility locations x_t with $b_t^{1/p} x_t$ for $t = 1, 2$, and ϵ with $\min\{b_1^{2/p}\,\epsilon,\ b_2^{2/p}\,\epsilon\}$ in (12), and simplifying, we obtain the modified iteration function for an ℓ_{bp}-norm SFMLP as

$$x_t^{k+1} = \frac{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt}\right)^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_{bp}\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{1-p} a_{jt}}{\sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt}\right)^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_{bp}\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{1-p}}, \qquad t = 1, 2. \tag{15}$$

3. CONVERGENCE OF THE WEISZFELD PROCEDURE

In this section, we examine the convergence properties of the modified Weiszfeld procedure for SFMLP. For the SFMLP with the approximated ℓ_p-norm, global convergence is shown by Morris [11]. For the unapproximated ℓ_p-norm single-facility problem, Brimberg and Love [12] prove local convergence of the Weiszfeld procedure. The authors also prove global convergence for the same problem [13]. Note that all of these convergence results apply for values of p in the interval [1,2]. This is not a restrictive condition since for a given transportation network the optimal value of p for the ℓ_p-norm can always be found in this interval [15]. The convergence of the modified Weiszfeld procedure with the ℓ_{bp}-norm can be shown similar to the ℓ_p-norm case for $p \in [1, 2]$. However, as shown by Üster and Love [16], when the ℓ_{bp}-norm is fitted to a region, it is possible to obtain an optimal p value greater than 2, i.e., the parameter p is not necessarily confined to the interval [1,2]. Therefore, for minisum location problems with the ℓ_{bp}-norm, convergence properties for values of p greater than 2 are of interest.

We have already shown that the Weiszfeld procedure used for the ℓ_p-norm SFMLP is applicable to the ℓ_{bp}-norm SFMLP after some modification. Therefore, our specific interest in this section is to analyze the convergence properties of the Weiszfeld procedure when it is used to solve the ℓ_p-norm SFMLP where $p > 2$.

We rewrite the iteration function (12) as follows:

$$x_t^{k+1} = x_t^k - \frac{1}{\Theta_t^k} \frac{\partial \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_t}, \qquad t = 1, 2, \tag{16}$$

where

$$\Theta_t^k = \sum_{j=1}^{n} w_j \left(\left(x_t^k - a_{jt}\right)^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_p\left(\mathbf{x}^k, \mathbf{a}_j\right)\right)^{1-p}, \qquad t = 1, 2. \tag{17}$$

Thus, the Weiszfeld procedure is indeed a steepest-descent procedure with a varying step size $1/\Theta_t^k$ at each iteration. It is well known that when the procedure is applied to the ℓ_p-norm SFMLP with $p > 2$, an iterate may overshoot [17]. In other words, the descent property of the objective function may not be guaranteed. The descent property is stated as $\widetilde{S}(\mathbf{x}^{k+1}) < \widetilde{S}(\mathbf{x}^k)$. In order to remedy this problem, the iteration function (16) can be modified in several ways: introduction of a factor that would change the step size; changing the direction of descent; changing both step size and the direction of descent. A recent book by Bertsekas [14, Chapter 1] includes a comprehensive review of these approaches.

Our primary concern is the convergence of the iterative procedure rather than the speed of its convergence. We already know that the Weiszfeld procedure performs the iterations by moving in the steepest-descent direction. Therefore, we choose to explore the first alternative. For that purpose, we aim to find a *good approximation* of the step size factor Ω to be introduced in (16) as follows:

$$x_t^{k+1} = x_t^k - \Omega \frac{1}{\Theta_t^k} \frac{\partial \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_t}, \qquad t = 1, 2. \tag{18}$$

Since the iterations are performed in the steepest descent direction, the existence of a step size that ensures the descent property is readily known [18, p. 243]. In order to find an approximate step size factor, we compare the Weiszfeld procedure with a modified Newton method. This approach was first suggested by Harris [19] in order to speed up the Weiszfeld iterative procedure when it is used for the Euclidean distance single-facility minisum location problem. It is later used in the context of solving minisum location problems where the distances are given by the powers of Euclidean distances [20], and functions of Euclidean distances [21]. For the former case, where the distances are given by $\ell_2^n(\mathbf{x})$, Chen was able to obtain fast convergence by using the Weiszfeld procedure with a step size factor $2/n$. If the step size factor is not considered, then convergence is ensured only for values of $n \in [1, 3]$ [22]. However, with the inclusion of the step size factor $2/n$, convergence is obtained for values of n up to 100 [20].

If the Newton method is used to solve the SFMLP, then the iteration function is given by

$$x_t^{k+1} = x_t^k - \mathrm{H}_{\mathbf{x}^k}^{-1} \frac{\partial \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_t}, \qquad t = 1, 2, \tag{19}$$

where the Hessian $H_{\mathbf{x}}$ is

$$H_{\mathbf{x}} = \begin{bmatrix} \dfrac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} & \dfrac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_2^2} \end{bmatrix}.$$

Following a similar analysis given by Chen [20], we devise a step size factor Ω by using modified Newton iterations (19). In order to prevent the possibility of oscillation in the iterative process, we consider only the the diagonal elements of the Hessian as suggested by Harris [19] and Chen [20]. This also greatly simplifies the Newton method, and ensures the semidefiniteness of the Hessian. Defining $\mathbf{s}_j = \mathbf{x} - \mathbf{a}_j$, the iteration function (19) becomes

$$x_t^{k+1} = x_t^k - \left(\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_t^2}\right)^{-1} \frac{\partial \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_t}, \qquad t = 1, 2, \tag{20}$$

where

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} = \sum_{j=1}^{n} w_j \left(s_{j1}^2 + \epsilon\right)^{p/2-2} \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^{1-2p} \left[(p-1)\, s_{j1}^2 \left(s_{j2}^2 + \epsilon\right)^{p/2} + \epsilon \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^p\right] \tag{21}$$

and

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_2^2} = \sum_{j=1}^{n} w_j \left(s_{j2}^2 + \epsilon\right)^{p/2-2} \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^{1-2p} \left[(p-1)\, s_{j2}^2 \left(s_{j1}^2 + \epsilon\right)^{p/2} + \epsilon \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^p\right]. \tag{22}$$

Consider first the diagonal entry given in (21). We approximate (21) by replacing the term (x_1^2) with $(x_1^2 + \epsilon)$ and by deleting the term $(\epsilon\, \tilde{\ell}_p(\mathbf{x}, \mathbf{a}_j))$ for small $\epsilon > 0$. Thus, we have

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} \approx \sum_{j=1}^{n} w_j \left(s_{j1}^2 + \epsilon\right)^{p/2-2} \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^{1-2p} (p-1) \left(s_{j1}^2 + \epsilon\right) \left(s_{j2}^2 + \epsilon\right)^{p/2}, \tag{23}$$

and with some further arrangement we obtain

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} \approx \sum_{j=1}^{n} w_j \left(s_{j1}^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^{1-p} (p-1)\, \mathcal{H}_j, \tag{24}$$

where

$$\mathcal{H}_j = \frac{\left(s_{j2}^2 + \epsilon\right)^{p/2}}{\left(s_{j1}^2 + \epsilon\right)^{p/2} + \left(s_{j2}^2 + \epsilon\right)^{p/2}}.$$

$\mathcal{H}_j$ can be approximated by letting $\epsilon \to 0$. Denoting the approximate value by $\widetilde{\mathcal{H}}_j$ and with some further rearrangement, we have

$$\widetilde{\mathcal{H}}_j = \frac{|s_{j2}|^p}{|s_{j1}|^p + |s_{j2}|^p} = \frac{1}{1 + |\cot(\phi_j)|^p},$$

where ϕ_j specifies the approximate value of the angle between the horizontal axis and the line connecting $\mathbf{x}$ and $\mathbf{a}_j$. If the existing facility locations are uniformly distributed over the region of interest, then ϕ_j can be taken as uniformly distributed for $0 \le \phi < 2\pi$. It can easily be verified that the average value of $\widetilde{\mathcal{H}}_j$ is $1/2$. Replacing $\mathcal{H}_j$ with $1/2$ in (24) we have

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} \approx \sum_{j=1}^{n} w_j \left(s_{j1}^2 + \epsilon\right)^{p/2-1} \left(\tilde{\ell}_p(\mathbf{s}_j)\right)^{1-p} \frac{(p-1)}{2}, \tag{25}$$

and equivalently, using (17)

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_1^2} \approx \frac{(p-1)}{2}\, \Theta_1^k. \tag{26}$$

Carrying similar steps to (23)–(25), it can be verified that

$$\frac{\partial^2 \widetilde{S}(\mathbf{x})}{\partial x_2^2} \approx \frac{(p-1)}{2}\, \Theta_2^k. \tag{27}$$

However, for this case, instead of $\mathcal{H}_j$, we will have $\mathcal{V}_j$ where

$$\widetilde{\mathcal{V}}_j = \frac{|s_{j1}|^p}{|s_{j1}|^p + |s_{j2}|^p} = \frac{1}{1 + |\tan(\phi_j)|^p}.$$

The average value of $\widetilde{\mathcal{V}}_j$ is also 1/2. (26) and (27) together with (20) suggest that the step size factor Ω can be taken as $2/(p-1)$, i.e., we have

$$x_t^{k+1} = x_t^k - \frac{2}{p-1}\frac{1}{\Theta_t^k}\frac{\partial \widetilde{S}\left(\mathbf{x}^k\right)}{\partial x_t}, \qquad t = 1, 2. \tag{28}$$

Note that, for $2 < p \leq 3$, we have $1 < 2/(p-1) \leq 2$, i.e., the new step size used in the Weiszfeld procedure is greater than the original step size. This may cause an iterate to fall outside the convex hull of the existing facility locations. In order to avoid this difficulty, we suggest that for $p \in [2, 3]$ a smaller step size factor $2/p$ should be used. For $p > 3$, the step size can be taken as $2/(p-1)$. Note that the step size is a decreasing function of p, suggesting that, as the p value increases we need to use a smaller step size factor in the Weiszfeld procedure.

Table 1. Modified Weiszfeld iterations: example in [13].

Ω	p																		
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2.00	11	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1.80	12	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1.60	14	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1.40	16	22	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1.20	19	12	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
1.00	24	10	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.95	25	9	19	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.90	27	6	15	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.85	29	9	12	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.80	31	9	10	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.75	33	9	8	19	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.70	36	9	7	15	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.65	39	10	7	10	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.60	42	12	8	8	13	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.55	46	15	9	7	12	X	X	X	X	X	X	X	X	X	X	X	X	X	X
0.50	51	18	9	7	8	11	X	X	X	X	X	X	X	X	X	X	X	X	X
0.45	57	21	12	5	7	11	X	X	X	X	X	X	X	X	X	X	X	X	X
0.40	64	25	15	11	8	9	13	X	X	X	X	X	X	X	X	X	X	X	X
0.35	74	30	19	14	11	8	9	12	14	X	X	X	X	X	X	X	X	X	X
0.30	87	37	24	18	14	11	9	10	13	17	X	X	X	X	X	X	X	X	X
0.25	105	47	31	23	19	15	13	11	10	11	13	17	X	X	X	X	X	X	X
0.20	132	61	42	32	26	22	19	16	15	13	12	12	13	16	19	X	X	X	X
0.15	176	85	59	47	40	34	30	27	24	21	20	18	17	16	15	14	16	17	18
0.10	266	132	94	76	66	59	54	50	48	45	43	42	40	39	38	36	31	28	27

To test the performance of the iteration function (28), we first used a pathological example given by Brimberg and Love [13]. The example is concerned with locating a single facility with respect to four existing facilities located at points $\mathbf{a}_1 = (0,0)$, $\mathbf{a}_2 = (0,10)$, $\mathbf{a}_3 = (10,10)$, and $\mathbf{a}_4 = (10,0)$, with weights $w_1 = 2$, $w_2 = 2$, $w_3 = 1$, and $w_4 = 1$. The authors start the iterations at $\mathbf{x} = (0,9)$ and observe that the iterates oscillate for $p > 2$. We ran the iterative procedure (18) with incremental values of Ω from 0.10 to 2.0. We employed the rectangular bounding method [5], and as the stopping criterion we used 0.01% difference between the bound value and the objective function value, or 300 as the maximum number of iterations, whichever is reached first. Our test results are given in Table 1. It can easily be verified that the use of a step size factor $2/(p-1)$ provides convergence. In order to see the effect of introducing a step size factor Ω into the Weiszfeld iterative procedure, we conducted some further numerical tests. For that purpose, we first generated six SFMLP's with random existing facility locations, three with unit weights and three with random weights. The number of existing facility locations in each group was 10, 25, and 50. As the initial point of iterations we used the center of gravity location. The pattern of the step sizes that provide convergence was very similar to the one given in Table 1. As the value of p increases the step size factor decreases, a pattern which resembles the function $2/(p-1)$. We also observed that for each value of p used in the tests a step size factor of $2/(p-1)$ provides convergence.

REFERENCES

1. H. Üster and R.F. Love, Application of weighted sum of order p to distance estimation, Technical Report 427, McMaster University, Faculty of Business, Hamilton, Ontario, Canada, (1998).
2. J.G. Morris and W.A. Verdini, Minisum ℓ_p distance location problems solved via a perturbed problem and Weiszfeld's algorithm, *Operations Research* **27**, 1180–1188, (1979).
3. E. Weiszfeld, Sur le point lequel la somme des distances de n points donnés est minimum, *Tôhoku Mathematical Journal* **43**, 355–386, (1937).
4. Z. Drezner, The planar two-center and two-median problems, *Transportation Science* **18**, 351–361, (1984).
5. R.F. Love and P.D. Dowling, A new bounding method for single facility location models, *Annals of Operations Research* **18**, 103–112, (1989).
6. H. Juel and R.F. Love, Fixed point optimality criteria for the location problem with arbitrary norms, *Journal of the Operational Research Society* **32**, 891–897, (1981).
7. G.O. Wesolowsky and R.F. Love, A nonlinear approximation method for solving a generalized rectangular distances Weber problem, *Management Science* **18**, 656–663, (1972).
8. J.W. Eyster, J.A. White and W.W. Wierwille, On solving multi-facility location problems using a hyperboloid approximation procedure, *AIIE Transactions* **5**, 1–6, (1973).
9. R.F. Love and J.G. Morris, Solving constrained multi-facility location problems involving ℓ_p distances using convex programming, *Operations Research* **23** (3), 581–587, (1975).
10. W.A. Verdini, On solving a class of location and location-allocation problems with extensions to clustering, Ph.D. Thesis, Kent State University, Graduate School of Business Administration, (1976).
11. J.G. Morris, Convergence of the Weiszfeld algorithm for Weber problems using a generalized "distance" function, *Operations Research* **29**, 37–48, (1981).
12. J. Brimberg and R.F. Love, Local convergence in a generalized Fermat-Weber problem, *Annals of Operations Research* **40**, 33–66, (1992).
13. J. Brimberg and R.F. Love, Global convergence of a generalized iterative procedure for the minisum location problem with ℓ_p distances, *Operations Research* **41** (6), 1153–1163, (1993).
14. D.M. Bertsekas, *Nonlinear Programming*, Athena Scientific, Belmont, MA, (1995).
15. J. Brimberg and R.F. Love, General considerations on the use of the weighted ℓ_p-norm as an empirical distance measure, *Transportation Science* **27**, 341–349, (1993).
16. H. Üster and R.F. Love, On the directional bias of the ℓ_{b_p}-norm, *European Journal of Operational Research* (to appear).
17. H.W. Kuhn, A note on Fermat's problem, *Mathematical Programming* **4**, 98–107, (1973).
18. J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).
19. B. Harris, Speeding up iterative algorithms—The generalized Weber problem, *Journal of Regional Science* **16**, 411–413, (1978).
20. R. Chen, Location problems with costs being sums of powers of euclidean distances, *Computers and Operations Research* **11**, 285–294, (1984).
21. R. Chen, Solution of location problems with radial cost functions, *Computers Math. Applic.* **10**, 87–94, (1984).
22. L. Cooper, An extension to the generalized weber problem, *Journal of Regional Science* **8**, 181–197, (1968).