The Convergence of the Weiszfeld Algorithm

H. ÜSTER AND R. F. LOVE
Management Science/Information Systems Area
Michael G. DeGroote School of Business
McMaster University
Hamilton, L8S 4M4
Ontario, Canada
(Received December 1999; accepted January 2000)

Abstract—We investigate the convergence properties of the Weiszfeld procedure when it is applied to the approximated \( \ell_p \)-norm single-facility location problem where \( p > 2 \). We show that convergence for \( p > 2 \) can be obtained by introducing a step size factor to the iterative procedure. Some numerical test results are also given. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Weiszfeld algorithm, Convergence properties.

1. INTRODUCTION

A single facility minimization problem (SFMLP) in the Euclidean plane (\( \mathbb{R}^2 \)) is stated as follows:

\[
\min W(x) = \sum_{j=1}^{n} w_j d(x, a_j),
\]

where \( n \) is the number of fixed facilities; \( a_j = (a_{j1}, a_{j2}), j = 1, \ldots, n \) are the fixed facility locations; \( x = (x_1, x_2) \) is the sought after location of the new facility; \( w_j > 0, j = 1, \ldots, n \) is the weight (demand) associated with the fixed facility \( j \); and \( d(u, v) \) is some distance function used to calculate the distance between any two points \( u, v \in \mathbb{R}^2 \).

As is readily seen in formulation (1), a distance predicting function is an important part of the objective function of a continuous location model. Since the model should represent the real situation as closely as possible, the accuracy of the distance predicting function employed plays a crucial role in terms of the validity and the applicability of locational decisions. A member of the family of \( \ell_p \)-norm, \( \ell_p(u, v) = |u_1 - v_1|^p + |u_2 - v_2|^p \), \( p \geq 1 \), is generally used as the distance function in continuous facility location models. The \( \ell_2 \)-norm (Euclidean distance) and the \( \ell_1 \)-norm (rectangular distance) are two well-studied special members of the \( \ell_p \)-norm family. Using the notation in (1) an SFMLP with the \( \ell_p \)-norm becomes

\[
\min S(x) = \sum_{j=1}^{n} w_j \ell_p(x, a_j).
\]
A weighted sum of order \( p \), denoted by \( \ell_{bp}(x) \), can be utilized to estimate distances in a transportation network. The \( \ell_{bp} \) distance between any two points \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in Euclidean plane is given by

\[
\ell_{bp}(u, v) = |b_1| u_1 - v_1 |^p + |b_2| u_2 - v_2 |^p |^{1/p}, \quad b_1, b_2 > 0, \quad p \geq 1.
\]

(3)

The \( \ell_{bp} \)-norm is a generalization of the well-known weighted \( \ell_p \)-norm. If for a fixed \( p \), the equality \( b_1^{1/p} = b_2^{1/p} = k \) holds, then one obtains the weighted \( \ell_p \)-norm where \( k \) represents the weight or the stretch factor. Furthermore, if \( b_1 = b_2 = 1 \), the rectangular and Euclidean distances can be obtained from the \( \ell_{bp} \)-norm by setting \( p = 1 \) and \( p = 2 \), respectively.

With the \( \ell_{bp} \)-norm one introduces unequal weights or nonsymmetric distance irregularities along the axis directions. An empirical work on 17 geographic regions showed that the \( \ell_{bp} \)-norm is better than the weighted \( \ell_0 \)-norm in terms of the accuracy of distance estimations [1]. Particularly in geographical regions with a predominant direction of nonlinearity (e.g., a mountain range), the gain in the accuracy of distance estimations with the \( \ell_{bp} \)-norm is more pronounced.

In order to model distances in a geographical region a goodness-of-fit criterion is minimized. One such criterion known as "Sum of Squared Deviations" (SD) is given as follows:

\[
SD = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{(d(a_i, a_j) - A(a_i, a_j))^2}{A(a_i, a_j)}
\]

(4)

where \( d(a_i, a_j) \) and \( A(a_i, a_j) \) are the predicted and actual distances between points \( a_i \) and \( a_j \), and \( n \) is the number of points in the data set. Although the \( \ell_{bp} \)-norm is a three-parameter \( (b_1, b_2, \text{and} \, p) \) distance function as opposed to a two-parameter weighted \( \ell_p \)-norm \( (k \text{and} \, p) \), the convexity of the goodness-of-fit criterion function \( SD \) in parameters \( b_1 \) and \( b_2 \) enables a fitting algorithm to determine the parameters of the three-parameter function almost as quickly as those of the two-parameter distance function [1].

**General Solution Procedure with the \( \ell_p \)-norm**

In order to solve the single-facility location problem (2) a one-point iterative procedure is used [2]. This iterative procedure is a generalization of the Weiszfeld procedure which was originally devised for the Euclidean distance facility location problem [3]. The generalized Weiszfeld iterative procedure depends upon the convexity of the \( \ell_p \)-norm, and thus, utilizes the first-order necessary and sufficient conditions. Since it is impossible to express the unknown variables (new facility locations) in the form of equations, the first-order derivatives cannot be solved directly. Instead, an iteration function is obtained by using these derivatives. Note that the first-order derivatives of \( S(x) \) are not differentiable at the existing facility locations. Therefore, in order to avoid the problem caused by these discontinuities in the derivatives, an iterative procedure is devised by using a hyperbolic approximation of the \( \ell_p \)-norm for actual computations.

A bound or a stopping rule is required to terminate the iterative procedure. There are several bounding methods examined in the literature. Among those, the rectangular bound, originally devised for single-facility Euclidean distance problems by Drezner [4] and extended to unapproximated \( \ell_p \)-norm single-facility problems by Love and Dowling [5], is shown to be superior. The rectangular bound involves the solution of a rectangular distance location problem at each iteration of the Weiszfeld procedure.

In some cases, the optimal facility locations coincide with the existing facility locations. Thus, the existing facility locations are examined for optimality before applying the Weiszfeld procedure, and if an existing facility location is optimal then the rest of the solution procedure is not needed. This check is performed by using the fixed-point optimality conditions [6].
Solving the $\ell_{bp}$-norm Location Models

There are two approaches to solve an $\ell_{bp}$ distance SFMLP.

**Approach 1. Using the procedures for the $\ell_p$-norm.** We first state the following equivalence property.

**Property 1.** An equivalent $\ell_p$-norm can be obtained from the $\ell_{bp}(x, y)$ norm by scaling the horizontal and vertical components $x$ and $y$ by $b_1^{1/p}$ and $b_2^{1/p}$, respectively.

**Proof.** We rewrite the $\ell_{bp}$-norm (3) as follows:

$$\ell_{bp}(z) = \left( \left( b_1^{1/p} |z_1| \right)^p + \left( b_2^{1/p} |z_2| \right)^p \right)^{1/p},$$

where $z = (z_1, z_2)$ and $z_t = |x_t - y_t|, \quad t = 1, 2.$

Taking $|z_t^*| = b_t^{1/p} |z_t|, \quad t = 1, 2$, we obtain

$$\ell_{bp}(z) = \left( |z_1^*|^p + |z_2^*|^p \right)^{1/p}.$$  

Notice that $\ell_{bp}(z)$ is in the form of the $\ell_p$-norm, i.e., we have

$$\ell_{bp}(z) = \ell_p(z'),$$

and the result follows.

Property 1 suggests that, after a scaling based modification in a location model’s setting, the Weiszfeld procedure developed for the $\ell_p$-norm single-facility location problem is readily applicable to the $\ell_{bp}$-norm location problem. The existing facility locations $a_j, j = 1, \ldots, n,$ in an $\ell_{bp}$-norm location problem are first scaled in the corresponding directions by using the scale factors $b_1^{1/p}$ and $b_2^{1/p}$. The location problem with the new setting is solved by using the procedure developed for the $\ell_p$-norm location model and the solution is then rescaled using the scale factors $b_1^{1/p}$ and $b_2^{1/p}$ for corresponding coordinates.

**Approach 2. Using modified procedures for the $\ell_{bp}$-norm.** Making a direct use of Property 1, we can obtain modifications of the Weiszfeld procedure, the rectangular bounding method, and the fixed-point optimality condition for the $\ell_{bp}$-norm location problem from the results developed for the $\ell_p$-norm problem. This is done by including the scaling factors $b_1^{1/p}$ and $b_2^{1/p}$ in the expressions developed for the $\ell_p$-norm location model.

The first approach is rather straightforward and utilizes the existing solution procedure for the SFMLP with the $\ell_p$-norm. It additionally involves scaling and rescaling computations. The second approach uses specialized procedures for the $\ell_{bp}$-norm location model.

This paper is organized as follows. In Section 2, we review the Weiszfeld procedure for the solution of the $\ell_p$-norm single-facility location problem, and provide a generalization to the $\ell_{bp}$-norm location model. In Section 3, we analyze the convergence properties of the Weiszfeld procedure when it is applied to single-facility minisum location problems.

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**2. Modified Weiszfeld Procedure**

In this section, we review the iterative procedure for the $\ell_p$-norm SFMLP and provide its generalization to the $\ell_{bp}$-norm location model.

**2.1. Procedure for SFMLP with the $\ell_p$-norm**

Since $\ell_p(x)$ is a norm, and thus, a convex function, it readily follows that problem (2) is a convex optimization problem. Furthermore, if the fixed facility locations are noncollinear, then $\ell_{bp}(x)$ is
strictly convex. Therefore, assuming the optimal solution $x^*$ is a differentiable point of $S(x)$, the first-order necessary and sufficient conditions require that

$$\frac{\partial S(x^*)}{\partial x_t} = 0, \quad t = 1, 2. \quad (5)$$

Evaluating the partial derivatives in (5) we have

$$\sum_{j=1}^{n} w_j \text{sign}(x^*_t - a_{jt}) \left| \frac{x^*_t - a_{jt}}{\ell_p(x^*, a_j)} \right|^{p-1} = 0, \quad t = 1, 2. \quad (6)$$

Noting that $(x_t - a_{jt}) = \text{sign}(x_t - a_{jt}) |x_t - a_{jt}|$, (6) can be rewritten as

$$\sum_{j=1}^{n} w_j (x^*_t - a_{jt}) \left| \frac{x^*_t - a_{jt}}{\ell_p(x^*, a_j)} \right|^{p-2} = 0, \quad t = 1, 2. \quad (7)$$

Simplifying (7) and solving for $x^*_t$ we have

$$x^*_t = \frac{\sum_{j=1}^{n} w_j |x^*_t - a_{jt}|^{p-2} \left[ \ell_p(x^*, a_j) \right]^{1-p}}{\sum_{j=1}^{n} w_j |x^*_t - a_{jt}|^{p-2} \left[ \ell_p(x^*, a_j) \right]^{1-p}}, \quad t = 1, 2. \quad (8)$$

Using (8) the one-point iteration scheme is devised as follows:

$$x^{k+1}_t = \frac{\sum_{j=1}^{n} w_j |x^k_t - a_{jt}|^{p-2} \left[ \ell_p(x^k, a_j) \right]^{1-p} a_{jt}}{\sum_{j=1}^{n} w_j |x^k_t - a_{jt}|^{p-2} \left[ \ell_p(x^k, a_j) \right]^{1-p}}, \quad t = 1, 2. \quad (9)$$

where $k$ represents the iteration number.

The iteration function (9) poses two main difficulties depending on the value of the parameter $p$ in the application.

(i) If $p < 2$, then $x^{k+1}_t$ is undefined along the hyperplanes $|x^k_t - a_{jt}| = 0$, where $j = 1, \ldots, n$, and $t = 1, 2$.

(ii) If $p \geq 2$, then $x^{k+1}_t$ is undefined at the existing facility locations $a_j$, $j = 1, \ldots, n$.

In order to eliminate the obvious difficulty caused by the discontinuities in the derivatives, an approximation of the $\ell_p$-norm is used in the objective function $S(x)$. The use of an approximation is discussed for rectangular distances by Wesolowsky and Love [7] and for Euclidean and rectangular distances by Eyster et al. [8]. Similar approximations are given for the $\ell_p$-norm by Love and Morris [9], and Morris and Verdini [2]. Verdini [10] shows that the approximation given by Eyster et al. (for the Euclidean distance case) and Love and Morris is not appropriate when the Weiszfeld procedure is used for the $\ell_p$ distances problem with $p \geq 1$. Therefore, the approximation given here follows the one given by Morris and Verdini. We next present this approximation of $\ell_p(x)$, denoted by $\tilde{\ell}_p(x)$, and review its properties and the resulting iterative procedure.

Approximating Function for the $\ell_p$-norm

Using the hyperbolic approximation of the $\ell_p$-norm, the approximated distance $\tilde{\ell}_p$ between any two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is given by

$$\tilde{\ell}_p(x, y) = \left[ ((x_1 - y_1)^2 + \epsilon)^{p/2} + ((x_2 - y_2)^2 + \epsilon)^{p/2} \right]^{1/p}, \quad p \geq 1, \quad \epsilon > 0. \quad (10)$$

Notice that the approximation to the $\ell_p$ distance is not a norm; it lacks the stationarity property, i.e., $\ell_p(0) \neq 0$. However, it is still a convex function of $x$ as shown by Morris and Verdini [2].
Iterative Procedure with Hyperbolic Approximation

Rewriting problem (2) with the hyperbolic approximating distance function \( \tilde{d}_p(x) \), we have

\[
\min S(x) = \sum_{j=1}^{n} w_j \left( \left((x_1 - a_{jt})^2 + \epsilon\right)^{p/2} + \left((x_2 - a_{jt})^2 + \epsilon\right)^{p/2} \right)^{1/p},
\]

where \( p \geq 1, \epsilon > 0 \) and \( w_j \geq 0, \quad j = 1, \ldots, n \). (11)

Clearly the function \( S(x) \), being a sum of \( n \) strictly convex functions, is a strictly convex function in \( x = (x_1, x_2) \). Therefore, again using the first-order necessary and sufficient conditions, and following the same steps given in (6)–(9) the modified iteration function is found as

\[
x_{t+1}^k = \frac{\sum_{j=1}^{n} w_j \left( \left((x_t^k - a_{jt})^2 + \epsilon\right)^{p/2-1} \left( \tilde{d}_p(x_t^k, a_j) \right)^{1-p} a_{jt} \right)}{\sum_{j=1}^{n} w_j \left( \left((x_t^k - a_{jt})^2 + \epsilon\right)^{p/2-1} \left( \tilde{d}_p(x_t^k, a_j) \right)^{1-p} \right)}, \quad t = 1, 2. \quad (12)
\]

2.2. Generalization to SFMLP with the \( \ell_{bp} \)-norm

We employ the following hyperbolic approximation of the \( \ell_{bp} \)-norm. Using the notation given in (3)

\[
\tilde{d}_{bp}(x, y) = \left[ b_1 \left((x_1 - y_1)^2 + \epsilon\right)^{p/2} + b_2 \left((x_2 - y_2)^2 + \epsilon\right)^{p/2} \right]^{1/p},
\]

where \( b_1, b_2 > 0, \ p \geq 1, \epsilon > 0 \). (13)

Similar to Property 1 given for the unapproximated case, we can write

\[
\tilde{d}_{bp}(x) = \tilde{d}_p(x'),
\]

where \( x = x - y, \ x'_1 = b_1^{1/p} z_1, \ x'_2 = b_2^{1/p} z_2 \). We denote the small quantity associated with the \( \ell_p \)-norm by \( \epsilon' \), where \( \epsilon' = \min\{b_1^{2/p} \epsilon, b_2^{2/p} \epsilon\} \). As suggested by relation (14), replacing the fixed facility locations \( a_{jt} \) with \( b_1^{1/p} a_{jt} \) for \( t = 1, 2, j = 1, \ldots, n \), the unknown facility locations \( x_t^k \) with \( b_1^{1/p} x_t^k \) for \( t = 1, 2 \), and \( \epsilon \) with \( \min\{b_1^{2/p} \epsilon, b_2^{2/p} \epsilon\} \) in (12), and simplifying, we obtain the modified iteration function for an \( \ell_{bp} \)-norm SFMLP as

\[
x_{t+1}^k = \frac{\sum_{j=1}^{n} w_j \left( \left((x_t^k - a_{jt})^2 + \epsilon\right)^{p/2-1} \left( \tilde{d}_{bp}(x_t^k, a_j) \right)^{1-p} a_{jt} \right)}{\sum_{j=1}^{n} w_j \left( \left((x_t^k - a_{jt})^2 + \epsilon\right)^{p/2-1} \left( \tilde{d}_{bp}(x_t^k, a_j) \right)^{1-p} \right)}, \quad t = 1, 2. \quad (15)
\]

3. CONVERGENCE OF THE WEISZFELD PROCEDURE

In this section, we examine the convergence properties of the modified Weiszfeld procedure for SFMLP. For the SFMLP with the approximated \( \ell_p \)-norm, global convergence is shown by Morris [11]. For the unapproximated \( \ell_p \)-norm single-facility problem, Brimberg and Love [12] prove local convergence of the Weiszfeld procedure. The authors also prove global convergence for the same problem [13]. Note that all of these convergence results apply for values of \( p \) in the interval \([1,2]\). This is not a restrictive condition since for a given transportation network the optimal value of \( p \) for the \( \ell_p \)-norm can always be found in this interval [15]. The convergence of the modified Weiszfeld procedure with the \( \ell_{bp} \)-norm can be shown similar to the \( \ell_p \)-norm case for \( p \in [1,2] \). However, as shown by Üster and Love [16], when the \( \ell_{bp} \)-norm is fitted to a region, it is possible to obtain an optimal \( p \) value greater than 2, i.e., the parameter \( p \) is not necessarily confined to the interval \([1,2]\). Therefore, for minisum location problems with the \( \ell_{bp} \)-norm, convergence properties for values of \( p \) greater than 2 are of interest.
We have already shown that the Weiszfeld procedure used for the $\ell_p$-norm SFMLP is applicable to the $\ell_{gp}$-norm SFMLP after some modification. Therefore, our specific interest in this section is to analyze the convergence properties of the Weiszfeld procedure when it is used to solve the $\ell_p$-norm SFMLP where $p > 2$.

We rewrite the iteration function (12) as follows:

$$x_{t+1}^k = x_t^k - \frac{1}{\Theta_t^k} \frac{\partial S(x_t^k)}{\partial x_t}, \quad t = 1, 2,$$

where

$$\Theta_t^k = \sum_{j=1}^n w_j \left( (x_t^k - a_j)^2 + \epsilon \right)^{p/2-1} \left( \ell_p(x_t^k, a_j) \right)^{1-p}, \quad t = 1, 2.$$  

Thus, the Weiszfeld procedure is indeed a steepest-descent procedure with a varying step size $1/\Theta_t^k$ at each iteration. It is well known that when the procedure is applied to the $\ell_p$-norm SFMLP with $p > 2$, an iterate may overshoot [17]. In other words, the descent property of the objective function may not be guaranteed. The descent property is stated as $S(x_{t+1}^k) < S(x_t^k)$. In order to remedy this problem, the iteration function (16) can be modified in several ways: introduction of a factor that would change the step size; changing the direction of descent; changing both step size and the direction of descent. A recent book by Bertsekas [14, Chapter 1] includes a comprehensive review of these approaches.

Our primary concern is the convergence of the iterative procedure rather than the speed of its convergence. We already know that the Weiszfeld procedure performs the iterations by moving in the steepest-descent direction. Therefore, we choose to explore the first alternative. For that purpose, we aim to find a good approximation of the step size factor $\Omega$ to be introduced in (16) as follows:

$$x_{t+1}^k = x_t^k - \frac{1}{\Theta_t^k} \frac{\partial S(x_t^k)}{\partial x_t}, \quad t = 1, 2.$$  

Since the iterations are performed in the steepest descent direction, the existence of a step size that ensures the descent property is readily known [18, p. 243]. In order to find an approximate step size factor, we compare the Weiszfeld procedure with a modified Newton method. This approach was first suggested by Harris [19] in order to speed up the Weiszfeld iterative procedure when it is used for the Euclidean distance single-facility minisum location problem. It is later used in the context of solving minisum location problems where the distances are given by the powers of Euclidean distances [20], and functions of Euclidean distances [21]. For the former case, where the distances are given by $\ell_2^p(x)$, Chen was able to obtain fast convergence by using the Weiszfeld procedure with a step size factor $2/n$. If the step size factor is not considered, then convergence is ensured only for values of $n \in [1, 3]$ [22]. However, with the inclusion of the step size factor $2/n$, convergence is obtained for values of $n$ up to 100 [20].

If the Newton method is used to solve the SFMLP, then the iteration function is given by

$$x_{t+1}^k = x_t^k - H_{x_t}^{-1} \left( \frac{\partial S(x_t^k)}{\partial x_t} \right), \quad t = 1, 2,$$

where the Hessian $H_x$ is

$$H_x = \begin{bmatrix}
\frac{\partial^2 S(x)}{\partial x_1^2} & \frac{\partial^2 S(x)}{\partial x_1 \partial x_2} \\
\frac{\partial^2 S(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 S(x)}{\partial x_2^2}
\end{bmatrix}.$$  

Following a similar analysis given by Chen [20], we devise a step size factor $\Omega$ by using modified Newton iterations (19). In order to prevent the possibility of oscillation in the iterative process, we
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consider only the the diagonal elements of the Hessian as suggested by Harris [19] and Chen [20]. This also greatly simplifies the Newton method, and ensures the semidefiniteness of the Hessian. Defining \( s_j = x - a_j \), the iteration function (19) becomes

\[
x_k^{t+1} = x_k^t - \left( \frac{\partial^2 \tilde{S}(x)}{\partial x_i^2} \right)^{-1} \frac{\partial \tilde{S}(x_k^t)}{\partial x_i}, \quad t = 1, 2,
\]

where

\[
\frac{\partial^2 \tilde{S}(x)}{\partial x_i^2} = \sum_{j=1}^{n} w_j \left( s_{j1}^2 + e \right)^{p/2-2} \left( \tilde{\ell}_p(s_j) \right)^{1-2p} \left( (p-1) s_{j1}^2 (s_{j2}^2 + e) + e \left( \tilde{\ell}_p(s_j) \right)^p \right),
\]

and

\[
\frac{\partial^2 \tilde{S}(x)}{\partial x_i^2} = \sum_{j=1}^{n} w_j \left( s_{j2}^2 + e \right)^{p/2-2} \left( \tilde{\ell}_p(s_j) \right)^{1-2p} \left( (p-1) s_{j2}^2 (s_{j1}^2 + e) + e \left( \tilde{\ell}_p(s_j) \right)^p \right).
\]

Consider first the diagonal entry given in (21). We approximate (21) by replacing the term \( (x_i^2 + e) \) with \( (2x_i^2 + e) \) and by deleting the term \( (\ell_p(x, a_j)) \) for small \( e > 0 \). Thus, we have

\[
\frac{\partial^2 S(x)}{\partial x_i^2} \approx \sum_{j=1}^{n} w_j \left( s_{j1}^2 + e \right)^{p/2-2} \left( \tilde{\ell}_p(s_j) \right)^{1-2p} \left( (p-1) s_{j1}^2 (s_{j2}^2 + e) + e \left( \tilde{\ell}_p(s_j) \right)^p \right),
\]

and with some further arrangement we obtain

\[
\frac{\partial^2 S(x)}{\partial x_i^2} \approx \sum_{j=1}^{n} w_j \left( s_{j1}^2 + e \right)^{p/2-1} \left( \tilde{\ell}_p(s_j) \right)^{1-p} \left( (p-1) \mathcal{H}_j \right),
\]

where

\[
\mathcal{H}_j = \frac{(s_{j2}^2 + e)^{p/2}}{(s_{j1}^2 + e)^{p/2} + (s_{j2}^2 + e)^{p/2}}.
\]

\( \mathcal{H}_j \) can be approximated by letting \( e \to 0 \). Denoting the approximate value by \( \tilde{\mathcal{H}}_j \) and with some further rearrangement, we have

\[
\tilde{\mathcal{H}}_j = \frac{|s_{j2}|^p}{|s_{j1}|^p + |s_{j2}|^p} = \frac{1}{1 + |\cot(\phi_j)|^p},
\]

where \( \phi_j \) specifies the approximate value of the angle between the horizontal axis and the line connecting \( x \) and \( a_j \). If the existing facility locations are uniformly distributed over the region of interest, then \( \phi_j \) can be taken as uniformly distributed for \( 0 \leq \phi < 2\pi \). It can easily be verified that the average value of \( \tilde{\mathcal{H}}_j \) is 1/2. Replacing \( \mathcal{H}_j \) with 1/2 in (24) we have

\[
\frac{\partial^2 S(x)}{\partial x_i^2} \approx \sum_{j=1}^{n} w_j \left( s_{j1}^2 + e \right)^{p/2-1} \left( \tilde{\ell}_p(s_j) \right)^{1-p} \left( \frac{(p-1)}{2} \right),
\]

and equivalently, using (17)

\[
\frac{\partial^2 S(x)}{\partial x_i^2} \approx \left( \frac{(p-1)}{2} \right) \Theta_i^k.
\]

Carrying similar steps to (23)–(25), it can be verified that

\[
\frac{\partial^2 S(x)}{\partial x_i^2} \approx \left( \frac{(p-1)}{2} \right) \Theta_i^k.
\]
However, for this case, instead of $H_j$, we will have $V_j$ where

$$
\tilde{V}_j = \frac{|s_{j1}|^p}{|s_{j1}|^p + |s_{j2}|^p} = \frac{1}{1 + |\tan(\phi_j)|^p}.
$$

The average value of $\tilde{V}_j$ is also 1/2. (26) and (27) together with (20) suggest that the step size factor $\Omega$ can be taken as $2/(p - 1)$, i.e., we have

$$
x_t^{k+1} = x_t^k - \frac{2}{p - 1} \frac{1}{\Omega_t^k} \frac{\partial \tilde{S}(x^k)}{\partial x_t}, \quad t = 1, 2.
$$

Note that, for $2 < p \leq 3$, we have $1 < 2/(p - 1) \leq 2$, i.e., the new step size used in the Weiszfeld procedure is greater than the original step size. This may cause an iterate to fall outside the convex hull of the existing facility locations. In order to avoid this difficulty, we suggest that for $p \in [2, 3]$ a smaller step size factor $2/p$ should be used. For $p > 3$, the step size can be taken as $2/(p - 1)$. Note that the step size is a decreasing function of $p$, suggesting that, as the $p$ value increases we need to use a smaller step size factor in the Weiszfeld procedure.

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To test the performance of the iteration function (28), we first used a pathological example given by Brimberg and Love [13]. The example is concerned with locating a single facility with respect to four existing facilities located at points $a_1 = (0,0)$, $a_2 = (0,10)$, $a_3 = (10,10)$, and $a_4 = (10,0)$, with weights $w_1 = 2$, $w_2 = 2$, $w_3 = 1$, and $w_4 = 1$. The authors start the iterations at $x = (0,9)$ and observe that the iterates oscillate for $p > 2$. We ran the iterative
procedure (18) with incremental values of $\Omega$ from 0.10 to 2.0. We employed the rectangular bounding method [5], and as the stopping criterion we used 0.01% difference between the bound value and the objective function value, or 300 as the maximum number of iterations, whichever is reached first. Our test results are given in Table 1. It can easily be verified that the use of a step size factor $2/(p - 1)$ provides convergence. In order to see the effect of introducing a step size factor $\Omega$ into the Weiszfeld iterative procedure, we conducted some further numerical tests. For that purpose, we first generated six SFMLP's with random existing facility locations, three with unit weights and three with random weights. The number of existing facility locations in each group was 10, 25, and 50. As the initial point of iterations we used the center of gravity location. The pattern of the step sizes that provide convergence was very similar to the one given in Table 1. As the value of $p$ increases the step size factor decreases, a pattern which resembles the function $2/(p - 1)$. We also observed that for each value of $p$ used in the tests a step size factor of $2/(p - 1)$ provides convergence.

REFERENCES