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On large sets of Kirkman triple systems and 3-wise balanced design

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Abstract

In this paper, the existence of large sets of Kirkman triple system is transformed to the existence of finite OLKFs and LGKSs. Our main result is: If there exist both an OLKF(6^k) and an LGKS($\{3\},\{6+3\},6^{k-1},6(k-1)+3,3$) for all $k\in\{6,7,\ldots,40\}\setminus\{14,17,21,22,25,26\}$, then there exists an LKTS(v) for any $v\equiv 3 \pmod 6$, $v\neq 21$. As well, we present a construction of 3-wise balanced design.

Keywords: Large set of Kirkman triple systems; 3-Wise balanced design; Generalized frame; Large set of generalized Kirkman systems

1. Introduction

In 1850, T.P. Kirkman posed the following problem (see [1]):

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two will walk twice abreast.

In the same year, a solution of the problem was given by A. Cayley and T.P. Kirkman, respectively. As well, J.J. Sylvester posed the further problem:

Every three girls walk together exactly once in a 13 week period. (*It is also required that every two girls walk together once in a week.*)

The problem is just the known so-called *Sylvester's problem of the 15 schoolgirls*, which was not solved until 1974. Denniston gave the following solution in [4]:

$$\mathrm{LKTS}(15)=\{(Z_{13}\cup\{a,b\},\mathscr{B}_i)\colon\ i\in Z_{13}\},$$

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where $\mathscr{B}_i = \mathscr{B}_0 + i\,(\mathrm{mod}\,13)$, $i \in Z_{13}$, and $\mathscr{B}_0$ is as follows:

0 1 9		0 2 7		0 3 11		0 4 6
2 4 12		3 4 8		1 7 12		1 8 11
5 10 11		5 6 12		6 8 10		2 9 10
7 8 a		9 11 a		2 5 a		3 12 a
3 6 b		1 10 b		4 9 b		5 7 b

0 5 8	0 10 12	1 4 5
1 2 3	3 5 9	2 6 11
6 7 9	4 7 11	3 7 10
4 10 a	1 6 a	8 9 12
11 12 b	2 8 b	0 a b

"Sylvester's problem of the 15 schoolgirls" is the first large set problem in mathematical history. The generalized case of the problem, unlimited number of points is 15, is called "large set of disjoint Kirkman triple systems (LKTS)". Following it, many problems of a similar nature have been raised and solved. For example, a large set of disjoint Steiner triple systems, a large set of disjoint transitive triple systems, a large set of disjoint Mendelsohn triple systems, a large set of disjoint group-divisible designs, etc. (see [12–14,16,21,22,25]). But until now, the research on LKTS has not advanced very far. By the end of 1979, Denniston had given the direct constructions for several small values and a tripling recursive construction of LKTS, where an LKTS was called a "double resolvable complete 3-designs" (see [4–7]). Meanwhile, S. Schreiber had also given the existence of an LKTS(33) (see [9]). Furthermore, some infinite classes can be gotten by using the Denniston's recursive construction. In near year, some new constructions on LKTS have been published (see [2,8,17–20,26,27]). In this paper, using those known constructions and results, the problem of existence on large sets of Kirkman triple system will be transformed to the problem of existence on finite OLKFs and LGKSs.

2. Definitions and known results

A *group-divisible design* GDD$(t,K,v;r_1\{k_1\},\ldots,r_s\{k_s\})$ is a triple $(X,\mathscr{G},\mathscr{B})$, where $\sum_{i=1}^{s} r_ik_i = v$, $K \subseteq N$, and for any $k \in K, k \geqslant t$, such that

(1) X is a set of v points,
(2) $\mathscr{G}$ is a partition of X into r_i sets of k_i points (called groups), $i = 1,2,\ldots,s$,
(3) $\mathscr{B}$ is a set of subsets of X (called blocks), such that $|B| \in K$, $|B \cap G| \leqslant 1$ for all $B \in \mathscr{B}$ and $G \in \mathscr{G}$, and such that any t-subset T of X with $|T \cap G| \leqslant 1$ for all $G \in \mathscr{G}$, is contained in exactly one block.

Usually, we write GDD$(t,k,v;r_1\{k_1\},\ldots,r_s\{k_s\})$ instead of GDD$(t,\{k\},v;r_1\{k_1\},\ldots,r_s\{k_s\})$ and write GDD$(k_1^{r_1}\cdots k_s^{r_s})$ instead of GDD$(2,3,v;r_1\{k_1\},\ldots,r_s\{k_s\})$. A group-divisible design, $(X,\mathcal{G},\mathcal{B})$, is *resolvable* if there exists a partition $\Gamma=\{P_1,P_2,\ldots,P_r\}$ of $\mathcal{B}$ such that each part P_i is itself a partition of X. The parts P_i are called *parallel classes*, and the partition Γ is called a *resolution*.

A GDD$(t,K,v;v\{1\})(X,\mathcal{G},\mathcal{B})$ is often called *t-wise balanced design* and denoted by $S(t,K,v)$. As well, $\mathcal{G}$ in the triple $(X,\mathcal{G},\mathcal{B})$ can be omitted and we write $(X,\mathcal{B})$ instead of $(X,\mathcal{G},\mathcal{B})$. An $S(t,\{k\},v)$ is denoted by $S(t,k,v)$, and is called *Steiner system*.

Lemma 2.1 (Colbourn and Dinitz [3]). *Let q be a prime power.*

(1) *There exists an $S(3,q+1,q^n+1)$ for any integer $n\geqslant 2$.*
(2) *If there exists an $S(3,q+1,v+1)$, then there exists an $S(3,q+1,qv+1)$.*
(3) *If an $S(3,q+1,v+1)$ and an $S(3,q+1,w+1)$ both exist, then there exists an $S(3,q+1,vw+1)$.*
(4) *There exists an $S(3,6,22)$, an $S(3,5,26)$ and an $S(3,6,26)$.*
(5) *There exists an $S(3,K,v)$ for $v\geqslant 5$, where $K=\{5,6,\ldots,40\}\setminus\{17,21,22,25,26\}$.*
(6) *There exists an $S(3,4,v)$ if and only if $v\equiv 2,4 \pmod 6$.*

An $S(2,3,v)$ is called a *Steiner triple system* and denoted by STS(v) briefly. A resolvable STS(v) is called a *Kirkman triple system* and denoted by KTS(v). It is well known that there are exactly $v(v-1)/6$ blocks in a KTS(v), and there are exactly $(v-1)/2$ elements in a resolution (i.e., there are $(v-1)/2$ parallel classes).

Lemma 2.2 (Ray-Chandhuri and Wilson [23]). *There exists a* KTS(v) *if and only if $v\equiv 3 \pmod 6$.*

Two KTS(v)s on the same set X are said to be *disjoint* if there is no common block. It is easy to see that the maximum number of pairwise disjoint KTS(v)s on the same point set is $v-2$. And a set containing $v-2$ pairwise disjoint KTS(v)s is called a *large set of disjoint Kirkman triple systems*, briefly an LKTS(v).

The known results are summarized as follows (see [2,4–9,17–20,26,27]).

Lemma 2.3. *There exists an* LKTS$(3^n m(2\times 13^{n_1}+1)(2\times 13^{n_2}+1)\cdots(2\times 13^{n_t}+1))$ *for $n\geqslant 1$, $m\in M=\{1,5,11,17,25,35,43,67,91,123\}\cup\{2^{2r+1}25^s+1 : r\geqslant 0, s\geqslant 0\}$, $t\geqslant 0$ and $n_i\geqslant 1 (i=1,2,\ldots,t)$.*

Recently, the author [17] introduced two concepts which are called a large set of generalized Kirkman systems (LGKS) and an overlarge set of Kirkman frames (OLKF). And we given a construction to obtain an LKTS from an $S(3,K,v)$, an LGKS and an OLKF. In next section, we will apply the construction and known results of $S(3,K,v)$ to obtain our theorem.

Theorem A. *If there exist both an* OLKF(6^k) and an LGKS$(\{3\},\{6+3\},6^{k-1},6(k-1)+3,3)$ *for all $k\in\{6,7,\ldots,40\}\setminus\{14,17,21,22,25,26\}$, then there exists an* LKTS(v) *for any $v\equiv 3 \pmod 6, v\neq 21$.*

3. Construct LKTS using $S(3, K, v)$

We first summarize the content of [8].

An $S(2,K,v)(X,\mathscr{B})$ is called a *modified resolvable* (briefly an MRS(K,v)), if there exists a set Γ which consists of subsets of $\mathscr{B}$, let $\Gamma=\{P_1,P_2,\ldots,P_h\}$, such that

(i) each P_i is itself a partition of X;
(ii) $\bigcup_{i=1}^{h} P_i=\mathscr{B}$;
(iii) for each $B\in\mathscr{B}$, there are exactly $(|B|-1)/2$ elements P_i of Γ such that $B\in P_i$.

The set Γ is called a *resolution* of $\mathscr{B}$. Each element P_i of Γ is called a *parallel class*. It is obvious that if $(X,\mathscr{B})$ is an MRS(K,v), then for any $k\in K$, k must be odd and $k\geqslant 3$. And an MRS$(\{3\},v)$ is just a Kirkman triple system KTS(v).

Lemma 3.1 (Lei Jianguo [17, Lemma 2]). *The resolution of an* MRS(K,v) *must contain* $(v-1)/2$ *parallel classes. Thus* $v\equiv 1\,(\mathrm{mod}\,2)$ *is the necessary condition for the existence of an* MRS(K,v).

A *generalized frame* $F(t,k,v\{m\})$ is a collection $\{(X,\mathscr{G},\mathscr{B}_r)\colon r\in R\}$, where X is a vm-set, $\mathscr{G}$ is a partition of X into v sets of m points (called groups), such that

(1) each $(X\setminus G,\mathscr{G}\setminus G,\mathscr{B}_r)$, $G\in\mathscr{G}$, is a GDD$(t-1,k,(v-1)m;(v-1)\{m\})$;
(2) $(X,\mathscr{G},\bigcup_{r\in R}\mathscr{B}_r)$ is a GDD$(t,k,vm;v\{m\})$;
(3) all $\mathscr{B}_r$, $r\in R$, are pairwise disjoint.

It is known that an $F(t,k,v\{m\})$ contains $vm/(k-t+1)$ GDD$(t-1,k,(v-1)m;(v-1)\{m\})$s, i.e. $|R|=vm/(k-t+1)$. Let $R_G=\{r\in R\colon \mathscr{B}_r$ have the same group set $\mathscr{G}\setminus G\}$, it is known that $|R_G|=m/(k-t+1)$.

An $F(2,3,v\{m\})$ is called a *Kirkman frame*, briefly a KF(m^v), and each element (i.e., a GDD$(1,3,(v-1)m;(v-1)\{m\})$) of the $F(2,3,v\{m\})$ is called a *partial parallel class* (or *holey parallel class*). In fact, each partial parallel class is a partition of $X\setminus G$ for some group G. It is well known that there exists a KF(m^v) if and only if $m\in 0\,(\mathrm{mod}\,2)$, $v\geqslant 4$ and $m(v-1)\equiv 0\,(\mathrm{mod}\,3)$ (see [24]).

If each element (a GDD$(2,3,(v-1)m;(v-1)\{m\})$) of an $F(3,3,v\{m\})$ is also a KF(m^{v-1}), then this $F(3,3,v\{m\})$ is called an *overlarge set of Kirkman frames* and denoted by OLKF(m^v) briefly.

Lemma 3.2 (Lei Jianguo [17,18]). *There exists an* OLKF(6^5) and an OLKF(2^{14}).

Now, we recalled the definition of large set of generalized Kirkman systems.

Let Y be an mu-set and Y_1 be a w-set, where $mu>w\geqslant 2$, $Y\cap Y_1=\phi$, $X=Y\cup Y_1$. A *large set of generalized Kirkman systems* LGKS$(K,\{m+w\},m^u,mu+w,w)$, is a collection $\{(X,\mathscr{B}_i):1\leqslant i\leqslant mu\}\cup\{(Y,\mathscr{G},\mathscr{A}_j):1\leqslant j\leqslant w-2\}$, satisfying the following conditions:

(I) Each $(Y,\mathscr{G},\mathscr{A}_j)$ is a KF(m^u). And if $w=2$, let $m=1$.

(II) Each $(X, \mathscr{B}_i)$ is an MRS$(K \cup \{m+w\}, mu+w)$ and $\mathscr{B}_i$ contains a block $G \cup Y_1$ for some $G \in \mathscr{G}$.
(III) For any 3-subset (or triple) T of X with $|T \cap Y_1| \leqslant 2$, T is contained in exactly one block of $(\bigcup_{i=1}^{mu} \mathscr{B}_i) \cup (\bigcup_{j=1}^{w-2} \mathscr{A}_j)$.
(IV) For any block $B \in (\bigcup_{i=1}^{mu} \mathscr{B}_i)$, such that if $Y_1 \not\subset B$, then

$$|\{\mathscr{B}_i : B \in \mathscr{B}_i\}| = |B| - 2$$

and $Y_1 \subset B$ if and only if $B \setminus Y_1 \in \mathscr{G}$ and $|\{\mathscr{B}_i : B \in \mathscr{B}_i\}| = m$.

Note: If $w = 2$, it is obvious that the elements of an LGKS$(K, \{3\}, 1^u, u+2, 2)$ are only some MRS$(K \cup \{3\}, u+2)$s; thus, we simply write it as LGKS$(K \cup \{3\}, u+2)$.
It is easy to see that an LGKS$(\{3\}, u+2)$ is just a large set of disjoint Kirkman triple systems LKTS$(u+2)$.

Lemma 3.3 (Lei Jianguo [17, Construction 3]). *If there exists an* LGKS$(K, \{m+w\}, m^u, mu+w, w)$, *and there exist both an* LGKS$(K' \cup \{w\}, m+w)$ *and an* LGKS(K', k) *for any* $k \in K$, *then there exists an* LGKS$(K' \cup \{w, 3\}, mu+w)$.

Corollary 3.4 (Lei Jianguo [17, Corollary 4]). *If there exist an* LGKS(K, u) *and an* LKTS(k) *for any* $k \in K$, *then there exists an* LKTS(u).

Corollary 3.5. *If there exist an* LGKS$(K, \{m+3\}, m^u, mu+3, 3)$ *and both an* LKTS$(m+3)$ *and an* LKTS(k) *for any* $k \in K$, *then there exists an* LKTS$(mu+3)$.

Proof. Taking $w = 3$ and $K' = \{3\}$ in Lemma 3.3, it is our result. □

Lemma 3.6 (Lei Jianguo [17, Theorem 3]). *There exists an* LGKS$(\{3\}, \{9\}, 6^{(v-3)/6}, v, 3\}$ *for* $v \in \{3^m(2 \times 13^k + 1)^t; t = 0, 1, k \geqslant 0, m \geqslant 1\}$.

Let X be a u-set and $\infty \notin X$. Then an $S(3, K, u+1)$ $(X \cup \{\infty\}, \mathscr{B})$ may be denoted as $S(3, K_0, K_1, u+1)$, where $K_0 = \{|B|; B \in \mathscr{B}, \infty \notin B\}$, $K_1 = \{|B|; B \in \mathscr{B}, \infty \in B\}$.

Theorem 3.7 (Lei Jianguo [17, Theorem 4]). *If there exists an* $S(3, K_0, K_1, u+1)$, *and there exist both an* OLKF(m^k) *for all* $k \in K_0$ *and an* LGKS$(K', \{m+w\}, m^{k-1}, m(k-1) + w, w)$ *for all* $k \in K_1$, *then there exists an* LGKS$(K' \cup \{3\}, \{m+w\}, m^u, mu+w, w)$.

Proof of Theorem A. Let $v = 6u - 3$. For $u = 1, 2$ and 3, LKTS(3), LKTS(9) and LKTS(15) have been given in Lemma 2.3. Below, consider the case $u \geqslant 5$. By Lemma 2.1(5), there exists an $S(3, K, u)$ for $u \geqslant 5$, where $K = \{5, 6, \ldots, 40\} \setminus \{17, 21, 22, 25, 26\}$. So if there exist both an OLKF(6^k) and an LGKS$(\{3\}, \{6+3\}, 6^{k-1}, 6(k-1)+3, 3)$ for all $k \in \{5, 6, 7, \ldots, 40\} \setminus \{17, 21, 22, 25, 26\}$, then by Theorem 3.7, there exists an LGKS$(\{3\}, \{6+3\}, 6^{u-1}, 6(u-1)+3, 3)$. Furthermore, since there exists an LKTS(9), by Corollary 3.5, there exists an LKTS$(6u-3)$ for any $u \geqslant 5$. An OLKF(6^5) and an OLKF(6^{14}) have existed by Lemma 3.2, and an LGKS$(\{3\}, \{6+3\}, 6^4, 6 \times 4 + 3, 3)$ and LGKS$(\{3\}, \{6+3\}, 6^{13}, 6 \times 13 + 3, 3)$ have been given in Lemma 3.6. □

4. The construction of OLKF

We can see that the overlarge set of Kirkman frames play an important role in constructing LKTS. In the section, we will give some constructions of OLKF(m^v).

Theorem 4.1. *If there exists an* OLKF(m^v), *then there exists an* OLKF($(mt)^v$), *where* $t \neq 2, 6$.

Proof. Let X be a v-set and $\{(X \times Z_m, \bar{\mathscr{G}}, \mathscr{B}_x^i)$: $x \in X, i \in Z_m\}$ be an OLKF(m^v), where $\bar{\mathscr{G}} = \{\bar{G}_z = \{z\} \times Z_m$: $z \in X\}$. Then $((X \setminus \{x\}) \times Z_m, \bar{\mathscr{G}} \setminus \bar{G}_x, \mathscr{B}_x^i)$ is a KF(m^{v-1}). Furthermore, each $\mathscr{B}_x^i$ can be partitioned into partial parallel classes $\{A_x^i(y,t)$: $y \in X \setminus \{x\}, 1 \leqslant t \leqslant m/2\}$, where $A_x^i(y,t)$ is a partition of $(X \setminus \{x,y\}) \times Z_m$, $1 \leqslant t \leqslant m/2$. We will construct an OLKF($(mt)^v$) over the set $X \times Z_m \times Z_t$ with group set $\mathscr{G} = \{G_z = \{z\} \times Z_m \times Z_t$: $z \in X\}$. The structure of its block sets $\mathscr{A}_x^i(k)$ are as follows, where $x \in X$, $i \in Z_m$, $k \in Z_t$.

It is well known that there exists an orthogonal array OA$(4,t)$ for $t \neq 2, 6$. Then

$$\begin{aligned}\mathscr{A}_x^i(k) = \{&\{(y_1,j_1,h_1),(y_2,j_2,h_2),(y_3,j_3,h_3)\}\\ &: \{(y_1,j_1),(y_2,j_2),(y_3,j_3)\} \in \mathscr{B}_x^i, y_1 < y_2 < y_3, (h_1,h_2,h_3+k,h_4)\\ &\in OA(4,t)\},\end{aligned}$$

and each $\mathscr{A}_x^i(k)$ can be partitioned into following partial parallel classes:

$$\begin{aligned}A_x^i(k;y,t,l) = \{&\{(y_1,j_1,h_1),(y_2,j_2,h_2),(y_3,j_3,h_3)\} : \{(y_1,j_1),(y_2,j_2),(y_3,j_3)\}\\ &\in A_x^i(y,t), y_1 < y_2 < y_3, (h_1,h_2,h_3+k,l) \in OA(4,t)\}.\end{aligned}$$

Thus $\{(X \times Z_m \times Z_t, \mathscr{G}, \mathscr{A}_x^i(k))$: $x \in X, i \in Z_m, k \in Z_t\}$ is an OLKF($(mt)^v$). □

Theorem 4.2. *If there exists an* $S(3,K,v)$, *and there exists an* OLKF(m^k) *for all* $k \in K$, *then there exists an* OLKF(m^v).

Proof. Let X be a v-set, and $(X, \mathscr{B})$ be an $S(3,K,v)$. We will construct an OLKF(m^v) over the set $X \times Z_m$.

For any $B \in \mathscr{B}$, we have $|B| \in K$, so there exists an OLKF($m^{|B|}$) $= \{(B \times Z_m, \mathscr{G}_B, \mathscr{A}_B(x,i))$: $x \in B, i \in Z_m\}$, where each $((B \setminus \{x\}) \times Z_m, \mathscr{G}_B \setminus G_x, \mathscr{A}_B(x,i))$ is a KF($m^{|B|-1}$) over set $(B \setminus \{x\}) \times Z_m$, and $\mathscr{G}_B = \{G_z = \{z\} \times Z_m$: $z \in B\}$. Let each $\mathscr{A}_B(x,i)(x \in B, i \in Z_m)$ can be partitioned into the partial parallel classes $\{A_B(x,i;y,t)$: $y \in B \setminus \{x\}, 1 \leqslant t \leqslant m/2\}$, where each $A_B(x,i;y,t)$ is a partition of $(B \setminus \{x,y\}) \times Z_m$. Define

$$\mathscr{A}(x,i) = \bigcup_{x \in B \in \mathscr{B}} \mathscr{A}_B(x,i), \quad x \in X, \quad i \in Z_m,$$

$$\mathscr{G} = \{G_z = \{z\} \times Z_m\text{: } z \in X\},$$

$$A(x,i;y,t) = \bigcup_{x \in B \in \mathscr{B}} A_B(x,i;y,t), \quad x \in X, \quad i \in Z_m, \quad y \in X \setminus \{x\}, \quad 1 \leqslant t \leqslant \frac{m}{2}.$$

It is easy to check that each $((X \setminus \{x\}) \times Z_m, \mathscr{G} \setminus G_x, \mathscr{A}(x,i))$ is a GDD(m^{v-1}). And each $\mathscr{A}(x,i)$ can be partitioned into partial parallel classes $A(x,i;y,t)$, $y \in X \setminus \{x\}, 1 \leqslant t \leqslant m/2$, i.e., each $((X \setminus \{x\}) \times Z_m, \mathscr{G} \setminus G_x, \mathscr{A}(x,i))$ is a KF(m^{v-1}). Thus $\{(X \times Z_m, \mathscr{G}, \mathscr{A}(x,i))\colon x \in X, i \in Z_m\}$ is an OLKF(m^v). □

Theorem 4.3. *There exists an* OLKF(6^v) *for* $v \in \{13^k+1\colon k \geqslant 1\} \cup \{4^n 25^k+1; n,k \geqslant 0\}$.

Proof. By Lemma 2.1(1) and (4), there exist an $S(3,5,26)$, an $S(3,5,4^n+1)$ and an $S(3,14,13^n+1)$ for $n \geqslant 2$. Furthermore, there exists an $S(3,5,v)$ for any $v \in \{4^n 25^k + 1{:}k,n \geqslant 0\}$ by Lemma 2.1(2). An OLKF(6^5) and an OLKF(2^{14}) have been given by Lemma 3.2. Thus we can get our results by using Theorems 4.1 and 4.2. □

5. A construction for 3-wise balanced design

Lemma 5.1 (Hanani [10]). *There exists a* GDD$(3, q+1, q^2+q; (q+1)\{q\})$ *for prime power* q.

Corollary 5.2. *There exists a* GDD$(3, K, q^2+q-s; r_1\{q\}, r_2\{q-1\}, \ldots, r_h\{q+1-h\})$, *for prime power* q, *where* $K \subseteq \{q+1, q, \ldots, q+1-m\}$, $m = r_1 + r_2 + \cdots + r_h < q, q^2 + q - s = r_1 q + r_2(q-1) + \cdots + r_h(q+1-h)$.

Proof. By Lemma 5.1, there exists a GDD$(3, q+1, q^2+q; (q+1)\{q\})(X, \mathscr{G}, \mathscr{B})$ for prime power q. Fixing r_1 groups and deleting $i-1$ points from r_i groups $(2 \leqslant i \leqslant h)$, the required GDD will be obtained. □

Let X be a $v-$set, $\infty_1, \infty_2 \notin X$, $(X \cup \{\infty_1, \infty_2\}, \mathscr{B})$ be an $S(3, K, v+2)$. Denote

$$K_0 = \{|B|\colon B \in \mathscr{B}, \infty_1, \infty_2 \notin B\}, \qquad K_{\infty_1} = \{|B|\colon B \in \mathscr{B}, \infty_1 \in B\},$$

$$K_{\infty_2} = \{|B|\colon B \in \mathscr{B}, \infty_2 \in B\}, \qquad K_2 = \{|B|\colon B \in \mathscr{B}, \{\infty_1, \infty_2\} \subset B\}.$$

Then, this $S(3, K, v+2)$ may be denoted as $S(3, K_0, K_{\infty_1}, K_{\infty_2}, K_2, v+2)$.

Let X be a v-set, $\infty_1 \notin X$ and $(X \cup \{\infty_1\}, \mathscr{B})$ be an $S(3, K, v+1)$. Denote $\mathscr{B}_{\infty_1} = \{B \setminus \{\infty_1\}\colon \infty_1 \in B \in \mathscr{B}\}$. If there exists a subfamily $\mathscr{C}$ of $\mathscr{B}_{\infty_1}$, which partition the set X, this $S(3, K, v+1)$ may be denoted as $S(3, K, v+1; K')$, where $K' = \{|C|\colon C \in \mathscr{C}\}$.

Lemma 5.3 (Hanani [10]). *There exists an* $S(3, q+1, q^k+1; \{q\})$ *for prime power* q.

Theorem 5.4. *Let* q *be prime power. Suppose an* $S(3, K_0, K_{\infty_1}, K_{\infty_2}, K_2, v+2)$ *exists. If there exist an* $S(3, K, q(k-2)+2)$ *for any* $k \in K_2$ *and an* $S(3, K, q(k-1)+1; \{q\})$ *for any* $k \in K_{\infty_2}$, *and* $q+1 \geqslant k$ *for any* $k \in K_0 \cup K_{\infty_1}$, *then there exists an* $S(3, K', qv+2)$, *where* $K' = K \cup K_0 \cup \{k, k-1\colon k \in K_{\infty_1}\}$.

Construction. Let X be a v-set, Y be a q-set, $\infty_1, \infty_2 \notin X$, and $(X \cup \{\infty_1, \infty_2\}, \mathscr{B})$ be an $S(3, K_0, K_{\infty_1}, K_{\infty_2}, K_2, v+2)$. We will construct an $S(3, K', qv+2)$ on set $(X \times Y) \cup \{\infty_1, \infty_2\}$ by the following four steps.

Step 1: For any $B\in\mathscr{B}$, $\{\infty_1,\infty_2\}\subset B$ (i.e. $|B|\in K_2$), by the hypothesis, there exists an $S(3,K,q(|B|-2)+2)$ on set $((B\setminus\{\infty_1,\infty_2\})\times Y)\cup\{\infty_1,\infty_2\}$. Let its block set be $\mathscr{B}_B$.

Step 2: For any $B\in\mathscr{B}$, $\infty_2\in B$, and $\infty_1\notin B$, (i.e. $|B|\in K_{\infty_2}$), by the hypothesis, there exists an $S(3,K,q(|B|-1)+1;\{q\})$ on set $((B\setminus\{\infty_2\})\times Y)\cup\{\infty_2\}$. Let its block set be $\bar{\mathscr{B}}_B$ and $\mathscr{C}_B=\{\{b\}\times Y : b\in B\}\subset\{A\setminus\{\infty_2\} : \infty_2\in A\in\bar{\mathscr{B}}_B\}$. Let $\mathscr{B}_B=\bar{\mathscr{B}}_B\setminus\{A\text{: }(A\setminus\{\infty_2\})\in\mathscr{C}_B\}$.

Step 3: For any $B\in\mathscr{B}$, $\infty_1\in B$, and $\infty_2\notin B$, (i.e. $|B|\in K_{\infty_1}$), by the hypothesis, $q+1\geqslant|B|$, so there exists a GDD$(3,K_B,q(|B|-1)+1;(|B|-1)\{q\},1\{1\})$ over set $X_B=\{(B\setminus\{\infty_1\})\times Y\}\cup\{\infty_1\}$ by Corollary 5.2, where $K_B=\{|B|,|B|-1\}$, its group set $\mathscr{G}_B=\{\{b\}\times Y : b\in B\setminus\{\infty_1\}\}\cup\{\{\infty_1\}\}$, and let its block set be $\mathscr{B}_B$.

Step 4: For any $B\in\mathscr{B}$, $\infty_1,\infty_2\notin B$, (i.e. $|B|\in K_0$), by the hypothesis, $q+1\geqslant|B|$, so there exists a GDD$(3,|B|,q|B|;|B|\{q\})$ over the set $X_B=B\times Y$ by Corollary 5.2, its group set $\mathscr{G}_B=\{\{b\}\times Y : b\in B\}$, and let its block set be $\mathscr{B}_B$.

Define

$$\mathscr{A}=\bigcup_{B\in\mathscr{B}}\mathscr{B}_B.$$

Then $((X\times Y)\cup\{\infty_1,\infty_2\},\mathscr{A})$ will be an $S(3,K',qv{+}2)$, where $K'{=}K\cup K_0\cup\{k,k{-}1 : k\in K_{\infty_1}\}$.

Proof. For any triple T of set $(X\times Y)\cup\{\infty_1,\infty_2\}$, we only need to show that there exists exactly one block of $\mathscr{A}$ containing triple T. All the possibilities of T are exhausted as follows:

(1) $T=\{(x,i),\infty_1,\infty_2\}$, where $x\in X$, $i\in Y$. Since there is exactly one block B in $\mathscr{B}$ containing triple $\{x,\infty_1,\infty_2\}$, of course $|B|\in K_2$, and $(((B\setminus\{\infty_1,\infty_2\})\times Y)\cup\{\infty_1,\infty_2\},\mathscr{B}_B)$ is an $S(3,K,q(|B|-2)+2)$, T is exactly contained in one block of the $\mathscr{B}_B$ in Step 1.

(2) $T=\{(x,i),(y,j),\infty_2\}$, where $x,y\in X$, $i,j\in Y$, and $(x,i)\neq(y,j)$. If $x=y$, we have $i\neq j$. There is exactly one block B in $\mathscr{B}$ containing triple $\{x,\infty_1,\infty_2\}$. Similar to (1), we can show that T is exactly contained in one block of the $\mathscr{B}_B$ in Step 1. If $x\neq y$, then there is exactly one block B in $\mathscr{B}$ containing triple $\{x,y,\infty_2\}$. When $\infty_1\in B$, i.e. $|B|\in K_2$, similar to (1). When $\infty_1\notin B$, i.e. $|B|\in K_{\infty_2}$, by Step 2, since $(((B\setminus\{\infty_2\})\times Y)\cup\{\infty_2\},\bar{\mathscr{B}}_B)$ is an $S((3,K,q(|B|-1)+1;\{q\})$, there exists exactly one block $A\in\bar{\mathscr{B}}_B$ containing T. It is easy to see that $A\notin\mathscr{C}_B$, i.e., $A\in\mathscr{B}_B$. Thus, T is exactly contained in one block of the $\mathscr{B}_B$ in Step 2.

(3) $T=\{(x,i),(y,j),\infty_1\}$, where $x,y\in X$, $i,j\in Y$, and $(x,i)\neq(y,j)$. If $x=y$, we have $i\neq j$. There is exactly one block B in $\mathscr{B}$ containing triple $\{x,\infty_1,\infty_2\}$. Similar to (1), we can show that T is exactly contained in one block of the $\mathscr{B}_B$ in Step 1. If $x\neq y$, then there is exactly one block B in $\mathscr{B}$ containing triple $\{x,y,\infty_1\}$. When $\infty_2\in B$, i.e. $|B|\in K_2$, similar to (1). When $\infty_2\notin B$, i.e. $|B|\in K_{\infty_1}$, by Step 3, $(X_B,\mathscr{G}_B,\mathscr{B}_B)$ is a GDD$(3,K_B,q(|B|-1)+1;(|B|-1)\{q\},1\{1\})$, where $X_B=\{(B\setminus\{\infty_1\})\times Y\}\cup\{\infty_1\}$, $\mathscr{G}_B=\{\{b\}\times Y : b\in B\setminus\{\infty_1\}\}\cup\{\{\infty_1\}\}$. Thus T is exactly contained in one block of the $\mathscr{B}_B$ in Step 3.

(4) $T=\{(x,i),(y,j),(z,h)\}$, where $x,y,z\in X$, $i,j,h\in Y$, and $(x,i)\neq(y,j)\neq(z,h)\neq(x,i)$. If $x=y=z$, we have $i\neq j\neq h\neq i$. There is exactly one block B in $\mathscr{B}$ containing triple $\{x,\infty_1,\infty_2\}$. Similar to (1), we can show that T is exactly contained in one block of the $\mathscr{B}_B$ in Step 1. If $x=y\neq z$, then there is exactly one block B in $\mathscr{B}$ containing triple $\{x,z,\infty_2\}$. Similar to (2), we can show that T is exactly contained in one block of the $\mathscr{B}_B$ in Step 2. If $x\neq y\neq z\neq x$, then there is exactly one block B in $\mathscr{B}$ containing triple $\{x,y,z\}$. When $|B|\in K_2$ or $|B|\in K_{\infty_2}$ or $|B|\in K_{\infty_1}$, similar to (1) or (2) or (3), respectively, we can show that T is exactly contained in one block of the $\mathscr{B}_B$ in Steps 1 or 2 or 3. When $|B|\in K_0$, since $(B\times Y,\mathscr{G}_B,\mathscr{B}_B)$ is a GDD$(3,|B|,q|B|;|B|\{q\})$, so T is exactly contained in one block of the $\mathscr{B}_B$ in Step 4. This completes the proof. □

Corollary 5.5. *If there exists an* $S(3,6,v+2)$, *then there exists an* $S(3,\{5,6\},5v+2)$.

Proof. There exists an $S(3,6,v+2)$, i.e. there exists an $S(3,\{6\},\{6\},\{6\},\{6\},v+2)$. By Lemma 2.1, there exists an $S(3,6,4\times 5+2)$, and there exists an $S(3,5+1,26;\{5\})$ by Lemma 5.3. Thus, there exists an $S(3,\{5,6\},5v+2)$ by Theorem 5.4. □

6. Uncited references

[11,15]

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