JOURNAL OF FUNCTIONAL ANALYSIS 87, 149–169 (1989)

Non-linear Elliptic and Parabolic Equations Involving Measure Data

LUCIO BOCCARDO

Dipartimento di Matematica, Università di Roma I, Piazza A. Moro 2, 00185 Roma, Italy

AND

THIERRY GALLOUËT

Département de Mathématiques, Université de Chambery, BP 1104, 73011 Chambery Cedex, France

Communicated by H. Brezis

Received November 1987; revised February 1988

In this paper we prove the existence of solutions for equations of the type $-\operatorname{div}(a(\cdot, Du)) = f$ in a bounded open set Ω, $u = 0$ on $\partial\Omega$, where a is a possibly non-linear function satisfying some coerciveness and monotonicity assumptions and f is a bounded measure. We also consider the equation $-\operatorname{div}(a(\cdot, Du)) + g(\cdot, u) = f$ in Ω, $u = 0$ on $\partial\Omega$ (with $f \in L^1(\Omega)$, or $f \in M(\Omega)$, $g(\cdot, u)\cdot u \geqq 0$) and the parabolic equivalent of the first (elliptic) equation.

I. INTRODUCTION

1. Throughout this paper Ω is a bounded open set of $\mathbb{R}^N$ ($N \geqq 2$). We begin with some remarks on the well-known problem

$$\begin{aligned} Au &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where A is a *linear*, uniformly elliptic operator, with bounded coefficients and $f \in M(\Omega)$. $M(\Omega)$ denotes the set of bounded measures on Ω (finite Radon measures).

Problem (1) is known to have a solution in a suitable sense, but this

0022-1236/89 $3.00

solution is not obtained as easily as in the case where f lies in $H^{-1}(\Omega)$. If for example A is taken to be $-\Delta$, it seems "natural" to introduce

$$E(u)=\frac{1}{2}\int_\Omega |Du|^2\,dx-\langle f,u\rangle, \qquad u\in H_0^1(\Omega)\cap C(\bar{\Omega}).$$

Unfortunately the solutions of (1) are not in general critical points of E. For instance, the minimum value of E can be $-\infty$.

A solution of (1) will be obtained by solving (1) with f in $H^{-1}(\Omega)$ and obtaining estimates on u, that will only depend on A, Ω, and $\|f\|_{L^1}$. A classical method (see [S] for instance) yields a solution of (1) through a *duality* and and a *$C^{0,\alpha}$-regularity argument.* Indeed, if $f\in W^{-1,q'}$ with $q'>N$, then u, solution of (1), lies in $C^{0,\alpha}$ and the mapping $f\to u$ is a linear continuous map from $W^{-1,q'}$ in $C^{0,\alpha}$. A duality argument then implies that the adjoint operator maps $M(\Omega)$ into $W_0^{1,q}(\Omega)$.

Such a method leads to estimates on u in $W_0^{1,q}(\Omega)$, for all $1\leqq q<N/(N-1)$, that only depend on A, Ω, and $\|f\|_{L^1}$.

This method is however restricted to a *linear* setting, at least when f lies in $M(\Omega)$. In the case of a non-linear operator A with f in $M(\Omega)$, it is usually assumed that the principal part of A is *linear* (see, e.g., [BS, BBC, GM1, GM2, G, BP]).

2. Our first goal in the present study (Section II) is to obtain a solution of (1) with f in $M(\Omega)$ and a non-linear operator A of the form

$$Au=-\operatorname{div}(\underline{a}(x,Du)),$$

with a function $\underline{a}\colon \Omega\times\mathbb{R}^N\to\mathbb{R}^N$ satisfying the following set of hypotheses:

$\underline{a}(x,\xi)$ is measurable in $x\in\Omega$, for all ξ in $\mathbb{R}^N$ and continuous in $\xi\in\mathbb{R}^N$, for a.e. x in Ω; (2)

there exist three constants p, M, α, with $p\in(2-1/N, N]$, $M\geqq 0$, $\alpha>0$, such that, for any ξ in $\mathbb{R}^N$ with $|\xi|>M$, $\underline{a}(x,\xi)\,\xi\geqq\alpha\,|\xi|^p$, for a.e. x in Ω, $\underline{a}(x,0)=0$; (3)

there exists a function b in $L^{p'}(\Omega)$, $p'=p/(p-1)$, and a constant $K\geqslant 0$ such that, for any ξ in $\mathbb{R}^N$, $|\underline{a}(x,\xi)|\leqslant K(b(x)+|\xi|^{p-1})$, for a.e. x in Ω; (4)

there exist three constants s, γ, K and a function d such that $s\geqq 2$, $d\in L^1(\Omega)$, $\gamma<(s-1)(N/(N-1))(p-1)$, $K\geqq 0$, and $(\underline{a}(x,\xi)-\underline{a}(x,\eta))(\xi-\eta)\geqslant(1/\beta(x,\xi,\eta))\,|\xi-\eta|^s$ for a.e. x in Ω and any ξ,η in $\mathbb{R}^N$, $0\leqslant\beta(x,\xi,\eta)\leqq K(d^{s-1}(x)+|\xi|^\gamma+|\eta|^\gamma)$ for a.e. x in Ω and ξ,η in $\mathbb{R}^N$. (5)

The hypotheses (2), (3), (4) are classical in the study of non-linear operators in divergence form (see [LL]). The additional assumption on p, i.e., $p \in (2 - 1/N, N]$, is motivated, as far as the lower bound is concerned, by Remark 1 in Section II. The upper bound $p \leqq N$ is not a limitation, because if $p > N$ problem (1) is known to have a unique (variational) weak solution in $W_0^{1,p}(\Omega)$ (see, e.g., [LL]), since $M(\Omega)$ is included in $W^{-1,p'}(\Omega)$.

Hypothesis (5) is more technical. It is more restrictive than strict monotonicity and less restrictive than strong monotonicity. It should be noted that an hypothesis such as (5) is almost never satisfied for "small $|\xi - \eta|$," if s is strictly less than 2.

The model example of function $\underset{\sim}{a}$ satisfying (2)–(5) is $\underset{\sim}{a}(x, \xi) = |\xi|^{p-2}\xi$ (p as in (3)), in which case (5) is satisfied with $s = p$, $\gamma = 0$ when $p \geqq 2$, and with $s = 2$, $\gamma = 2 - p$ when $p < 2$. Furthermore $\gamma < (N/(N-1))(p-1)$ since $p > 2 - 1/N$. The corresponding operator is $Au = -\Delta_p u = -\operatorname{div}(|Du|^{p-2}Du)$.

The proof of the existence of a solution of (1) when f lies in $M(\Omega)$ is divided in three steps. First (1) is shown to have a unique weak solution u in $W_0^{1,p}(\Omega)$ for f in $W^{-1,p'}(\Omega)$ (cf. [LL]). Then estimates on u in $W^{1,q}(\Omega)$, for all $1 \leqq q < (N/(N-1))(p-1)$, that only depend on Ω, $\underset{\sim}{a}$, and $\|f\|_{L^1}$ are obtained. In the last step, an arbitrary f of $M(\Omega)$ is considered and approximated by a sequence (f_n) in $W^{-1,p'}(\Omega)$ which converges to f. The limiting process hinges of the proof of the almost pointwise convergence of the sequence (Du_n), where u_n is the weak solution of (1) with $f = f_n$.

The second part of this paper (Section III) is devoted to a generalization of a few results of [BS, GM2, G] to the case of non-linear operators with non-linearity on the principal part of the operator with the help of the method introduced in Section II. Roughly speaking, we investigate equations of the kind

$$\begin{aligned} Au + g(x, u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{6}$$

A is as in Section II, $g(\cdot, u)\, u \geqq 0$, and f in $L^1(\Omega)$. It should be emphasized that, even when A is taken to be $-\Delta$, the existence of u cannot be expected for f in $M(\Omega)$ whenever g increases too rapidly at infinity. The reader is referred to [GM1, BP] for a characterization of the "admissible" measures f when g is an increasing function of u and A is $-\Delta$.

The third part of this paper (Section IV) examines the parabolic analogous of the equations studied in Section II.

II. Existence of Solutions for a Non-linear Elliptic Operator in Divergence Form and a Right Hand Side f in $M(\Omega)$

3. Let us consider, for an arbitrary f in $M(\Omega)$, the equation

$$\begin{aligned} Au &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{7}$$

where $Au = -\operatorname{div}(\underline{a}(\cdot\, Du))$, and $\underline{a}$ satisfies (2)–(5).

A function u will be called a weak solution of (7) if it satisfies

$$\begin{aligned} &u \in W_0^{1,1}(\Omega),\ \underline{a}(\cdot, Du) \in L^1_{\text{loc}}(\Omega) \text{ and} \\ &\int_\Omega \underline{a}(x, Du)\, Dv = \langle f, v \rangle, \text{ for any } v \text{ in } C_0^\infty(\Omega). \end{aligned} \tag{8}$$

Our first result is the following.

Theorem 1. *Let $\underline{a}$ satisfy* (2)–(5) *and f be an element of $M(\Omega)$. Then there exists a weak solution u of* (7) *with the regularity $u \in W_0^{1,q}(\Omega)$ for all $1 \leqq q < (N/(N-1))(p-1)$.*

Remark 1. As already mentioned in the Introduction, Theorem 1 is also true when $p > N$ in (3), in which case hypothesis (5) is not necessary. The existence of u is then an easy consequence of the results of Leray–Lions [LL], since $M(\Omega)$ is included in $W^{-1,p'}(\Omega)$. The limitation $p > 2 - 1/N$ stems from the requirement that u lie in $W_0^{1,1}(\Omega)$. Then the distribution Du is a function and the quantity $\underline{a}(\cdot, Du)$ is meaningful.

If f lies in $W^{-1,p'}(\Omega)$, (7) is known to have a unique weak solution u (see [LL]), such that

$$\begin{aligned} &u \in W_0^{1,p}(\Omega) \\ &\int_\Omega \underline{a}(x, Du)\, Dv = \langle f, v \rangle, \text{ for any } v \text{ in } W_0^{1,p}(\Omega). \end{aligned} \tag{9}$$

The first step in the proof of Theorem 1 consists in deriving a $W_0^{1,q}(\Omega)$ estimate on u for $1 \leqq q < (N/(N-1))(p-1)$ which only depends on $q, \underline{a}, \Omega, \|f\|_{L^1}$, whenever f lies in $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ and u is the solution of (9). This is the object of Subsection 4. In Subsection 5 we take f in $M(\Omega)$ a sequence (f_n) in $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ which converges to f and we pass to the limit in the equation $Au_n = f_n$.

4. In this subsection we prove the following estimate on u ($\underline{a}$ satisfies hypotheses (2)–(5)):

> for any $1 \leqq q < (N/(N-1))(p-1)$, for any $B > 0$, there exists $C > 0$, depending on q, $\underline{a}$, Ω, and B such that if f lies in $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ and u is the solution of (9), then $\|u\|_{W_0^{1,q}} \leqq C$ whenever $\|f\|_{L^1} \leqq B$. (10)

Remark 2. p is given in (3) and the assumption $p > 2 - 1/N$ implies that $1 < (N/(N-1))(p-1)$.

In order to prove (10), let f be an element of $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ and let u be the corresponding solution of (9), and assume that $\|f\|_{L^1} \leqq B$. From now onward we denote by $c_1, c_2, \ldots$ various constants which only depend on q, $\underline{a}$, Ω, and B.

Let n be a fixed integer and define ψ as

$$\begin{aligned} \psi(s) &= n && \text{if } s > n \\ \psi(s) &= s && \text{if } -n \leqq s \leqq n \\ \psi(s) &= -n && \text{if } s < -n. \end{aligned}$$

The choice of $\psi(u)$ as test function in (9) yields

$$\int_\Omega \psi'(u)\, \underline{a}(x, Du)\, Du = \int_\Omega f\psi(u). \tag{11}$$

By virtue of (3), (11) yields

$$\int_{D_n} |Du|^p \leqq \frac{n}{\alpha} \|f\|_1 = nc_1 \tag{12}$$

with

$$D_n = \{x \in \Omega, |u(x)| \leqq n, |Du(x)| \geqq M\}. \tag{13}$$

Now we define ψ as

$$\begin{aligned} \psi(s) &= 1 && \text{if } s > n+1 \\ \psi(s) &= s - n && \text{if } n \leqq s \leqq n+1 \\ \psi(s) &= 0 && \text{if } -n \leqq s \leqq n \\ \psi(s) &= s + n && \text{if } -n-1 \leqq s \leqq -n \\ \psi(s) &= -1 && \text{if } s < -n-1. \end{aligned}$$

Then, if $c_1 = (1/\alpha)\,\|f\|_1$,

$$\int_{B_n} |Du|^p\,dx \leqq c_1 \tag{14}$$

with

$$B_n = \{x \in \Omega,\ n \leqq |u(x)| \leqq n+1,\ |Du(x)| \geqq M\}. \tag{15}$$

The estimate (12) can be proved with the help of (14), (15) since

$$D_n = B_0 \cup B_1 \cup \cdots \cup B_{n-1}.$$

For any $q < p$, Hölder's inequality implies that

$$\int_{B_n} |Du|^q \leqslant \left(\int_{B_n} |Du|^p\right)^{q/p} (\mathrm{meas}(B_n))^{(p-q)/p}.$$

But, if $1/q^* = 1/q - 1/N$ $(q < p \leqslant N)$, $\mathrm{meas}(B_n) \leqq (1/n^{q^*}) \int_{B_n} |u|^{q^*}$. Setting $c_2 = c_1^{q/p}$, we obtain

$$\int_{B_n} |Du|^q \leqq c_2 \left(\int_{B_n} |u|^{q^*}\right)^{(p-q)/p} \frac{1}{n^{q^*((p-q)/p)}}. \tag{16}$$

Applying Hölder's inequality with the exponents $p/(p-q)$ and p/q we find that, for all positive integers n_0,

$$\sum_{n=n_0}^{\infty} \int_{B_n} |Du|^q \leqq c_2 \left(\sum_{n=n_0}^{\infty} \int_{B_n} |u|^{q^*}\right)^{(p-q)/p} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(p-q)/q}}\right)^{q/p}.$$

The above estimate, together with estimate (12), yields

$$\int_{\Omega} |Du|^q \leqq c_3 + n_0^{q/p} c_4 + c_2\,\|u\|_{L^{q^*}}^{q^*(p-q)/p} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(p-q)/q}}\right)^{q/p}, \tag{17}$$

where $c_3 = M^q\,\mathrm{meas}(\Omega)$, $c_4 = c_1^{q/p}(\mathrm{meas}(\Omega))^{(p-q)/p}$.

Sobolev imbedding Theorem implies that

$$\|u\|_{L^{q^*}}^q \leqq c_5 \left(n_0^{q/p} + \|u\|_{L^{q^*}}^{q^*(p-q)/p} \left(\sum_{n=n_0}^{\infty} \frac{1}{n^{q^*(p-q)/q}}\right)^{q/p}\right). \tag{18}$$

Recall that $2 - 1/N < p \leqq N$. Two cases have to be distinguished. If p is equal to N, then $q^*(p-q)/p = (qN/(N-q))/((N-q)/N) = q$ and $q^*((p-q)/q) = (qN/(N-q))/((N-q)/q) = N \geqq 2$.

A proper choice of n_0 in (18) gives the estimate

$$\|u\|_{L^{q^*}} \leqq c_6.$$

Then, by virtue of (17),

$$\|Du\|_{L^q} \leqq c_7,$$

which proves (10). Note that $q < p = (N/(N-1))(p-1)$, since $p = N$. If p is strictly less than N, then $q^*((p-q)/q) = (qN/(N-q))/((p-q)/p) < q$ and $q^*((p-q)/q) = N(p-q)/(N-q) > 1$, provided $q < (N/(N-1))(p-1)$.

We conclude with the help of (18), written for $n_0 = 1$, that

$$\|u\|_{L^{q^*}} \leqq c_8$$

and thus that

$$\|Du\|_{L^q} \leqq c_9,$$

which also proves (10). Note that in the latter case $(N/(N-1))(p-1) < p$.

The above arguments prove the following

LEMMA 1. *Let* $2 - 1/N < p \leqq N$ $(N \geqq 2)$, M and c_1 *positive constants and* Ω *be a bounded open set of* R^N.

If $1 \leqq q < (p-1)(N/(N-1))$, *there exists a constant* C *depending only on* p, M, c_1, Ω, q *such that, whenever* $u \in W_0^{1,q}(\Omega)$ *satisfies* (14) *for all* $n \in N$, *then*

$$\|u\|_{W_0^{1,q}} \leqq C.$$

Remark 3. In the case where $p = N$, q is restricted to be strictly less than $(N/(N-1))(p-1)$, so as to be in position to apply Sobolev imbedding theorem $(q < N)$. In the other case the limitation on q guarantees the convergence of the series in the right hand side of (18).

Remark 4. The proof of estimate (10) only uses the coerciveness of $\underline{a}$ (hypothesis (3)) and hypothesis (4), which makes (9) meaninful. Then it is easy to see that these estimates are still true for a general "Leray–Lions operator" (see [LL]). In particular the function $\underline{a}$ can depend on u.

Remark 5. We thank Idelfonso Diaz who informed us, after completion of this work, that R. Gariepy and M. Pierre have obtained the same estimate in the case $A = -\Delta_p$ with a different method.

5. In this subsection we prove Theorem 1. Let $f \in M(\Omega)$ and $\underline{a}$ let satisfy (2)–(5).

A sequence $(f_n) \subset W^{-1,p'}(\Omega) \cap L^1(\Omega)$ that converges to f in the distribution sense is considered. It is further assumed that $\|f_n\|_{L^1} \leqq B = \|f\|_{M(\Omega)}$.

Let u_n be the solution of (9) with $f = f_n$. Then for every n integer, $\underline{a}(\cdot, Du_n) \in L^1(\Omega)$,

$$-\operatorname{div}(\underline{a}(\cdot, Du_n)) = f_n \qquad \text{in the distribution sense.} \tag{19}$$

By virtue of the estimate (10), $\|u_n\|_{W_0^{1,q}} \leqq C$ where C only depends on q, $\underline{a}$, Ω, and B and $1 \leqq q < (N/(N-1))(p-1)$.

Then there exist u in $W_0^{1,q}(\Omega)$ and some subsequence (still denoted (u_n)) such that

$$\begin{aligned} &u_n \to u && \text{in } W_0^{1,q}(\Omega) - \text{weak} \\ &u_n \to u && \text{in } L^q(\Omega) \\ &u_n \to u && \text{a.e.} \end{aligned} \tag{20}$$

The above convergence does not however permit to pass to the limit in (19) except when $\underline{a}$ is linear in its second argument. A pointwise convergence of Du_n is needed.

Assumption (5) plays a central role in proving such a convergence. Specifically the following result holds true.

> Let $\underline{a}$ satisfy (2)–(5), and (f_n) be a sequence of $W^{-1,p'}(\Omega) \cap L^1(\Omega)$.
> Let u_n be the solution of (9) with $f = f_n$.
> If (f_n) is bounded in $L^1(\Omega)$, then (u_n) is relatively compact in $W_0^{1,q}(\Omega)$ for every q in $[1, (N/(N-1))(p-1))$. (21)

As in Subsection 4, let $\psi \in C(R, R)$ be such that, for $\varepsilon > 0$ fixed, $\psi(s) = \varepsilon$ if $s > \varepsilon$, $\psi(s) = s$ if $-\varepsilon \leqq s \leqq \varepsilon$, $\psi(s) = -\varepsilon$ if $s < -\varepsilon$. Using (9) with $f = f_n$ and f_m, $u = u_n$ and u_m, and $v = \psi(u_n - u_m)$ we obtain

$$\begin{aligned} &\int_\Omega \psi'(u_n - u_m)(\underline{a}(x, Du_n) - \underline{a}(x, Du_m))(Du_n - Du_m) \\ &\quad = \int_\Omega (f_n - f_m)\, \psi(u_n - u_m). \end{aligned} \tag{22}$$

Since $\|f_n\|_{L^1(\Omega)} \leqq B$, (5) and (22) imply the estimate

$$\begin{gathered} \int_{D_{n,m,\varepsilon}} \frac{1}{\beta(x, Du_n, Du_m)} |Du_n - Du_m|^s \leqq 2\varepsilon B \\ D_{n,m,\varepsilon} = \{x \in \Omega,\ |u_n(x) - u_m(x)| \leqq \varepsilon\}. \end{gathered} \tag{23}$$

Estimate (23) and Hölder's inequality give

$$\int_{D_{n,m,\varepsilon}} |Du_n - Du_m| \leqq \varepsilon^{1/s} c_1 \left(\int_{D_{n,m,\varepsilon}} (\beta(x, Du_n, Du_m))^{s'/s} \right)^{1/s'},$$

where $c_1 = (2B)^{1/s}$.

The above inequality, assumption (5), and the $W_0^{1,q}$-estimate on u_n yield

$$\int_{D_{n,m,\varepsilon}} |Du_n - Du_m| \leqq c_2 \varepsilon^{1/s}. \tag{24}$$

Estimate (24) is used to prove that (Du_n) is a Cauchy sequence in $L^1(\Omega)$. We have

$$\int_\Omega |D(u_n - u_m)| = \int_{D_{n,m,\varepsilon}} |D(u_n - u_m)| + \int_{\Omega \setminus D_{n,m,\varepsilon}} |D(u_n - u_m)|,$$

so that, by (24),

$$\int_\Omega |D(u_n - u_m)| \leqslant c_2 \varepsilon^{1/s} + c_3 \operatorname{meas}\{x \in \Omega \colon |u_n(x) - u_m(x)| > \varepsilon\}^{1 - 1/q} \tag{25}$$

for some q in $(1, (N/(N-1))(p-1))$.

Since u_n is a Cauchy sequence in measure (in fact u_n is even a Cauchy sequence in $L^1(\Omega)$), (25) implies that for some $n_0(\varepsilon)$ depending on ε

$$\int_\Omega |D(u_n - u_m)| \leqq c_2 \varepsilon^{1/s} + \varepsilon, \qquad \text{for} \quad n, m \geqq n_0(\varepsilon),$$

which proves that (Du_n) is a Cauchy sequence in $L^1(\Omega)$ and thus that

$$Du_n \to Du \qquad \text{in } L^1(\Omega).$$

By virtue of (20), we also obtain the convergence statement

$$Du_n \to Du \qquad \text{in } L^q(\Omega), \text{ for every } q \text{ in} \left[1, \frac{N}{N-1}(p-1)\right). \tag{26}$$

Assertion (21) is proved. Assumption (4) together with Vitali's theorem imply that

$$\underline{a}(\cdot, Du_n) \to \underline{a}(\cdot, Du) \qquad \text{in } L^r(\Omega) \text{ for every } r \text{ in} \left[1, \frac{N}{N-1}\right). \tag{27}$$

It is now possible to pass to the limit in (19). We conclude that

$$-\operatorname{div}(\underline{a}(\cdot, Du)) = f \qquad \text{in the distribution sense.}$$

Thus, u is a weak solution of (7) (that is, it satisfies (8)). Theorem 1 is proved.

Remark 6. The conclusion is stronger than (8). Indeed, since u belongs to $W_0^{1,q}(\Omega)$ for all q in $[1, (N/(N-1))(p-1))$, $\underline{a}(\cdot, Du)$ belongs to $L^r(\Omega)$, for all r in $[1, N/(N-1))$ and

$$\int_\Omega \underline{a}(x, Du)\, Dv = \langle f, v\rangle_{M(\Omega), C(\bar{\Omega})}, \qquad \text{for every } v \text{ in } \bigcup_{r'>N} W_0^{1,r'}(\Omega).$$

Remark 7. The method used in this section does not allow us to prove Theorem 1 in the case of a general "Leray–Lions" operator, for example, when $\underline{a}$ has a non-linear dependence on Du, together with a dependence on u.

Remark 8. Under the assumptions of Theorem 1, the uniqueness of the solution of (7) in the sense of (8) is false. Indeed there exists, in the linear case ($p=2$), an example of non-uniqueness due to J. Serrin (see [SE]). This example gives non-uniqueness in the space $W_0^{1,q}(\Omega)$ for $q = N/(N-1+\varepsilon)$ and an arbitrary $\varepsilon > 0$.

Remark 9. After completion of this work, we learned that S. Kichenassamy has obtained a result of existence and uniqueness of solution for (7) in the particular case $A = -\Delta_p$, $\Omega = \mathbb{R}^N$, and $f = \sum_{i=1}^m \gamma_i \delta(\cdot - a_i)$, with $1 < p < \infty$, $a_i \in \mathbb{R}^N$, $m \geqq 1$, $\gamma_i \in R$, $\sum_{i=1}^m \gamma_i = 0$ (see [K]).

6. Our goal in this subsection is to obtain the appropriate functional space for a weak solution of (7) when f is in $L^m(\Omega)$ with $m > 1$.

Let $\underline{a}$ satisfy (2)–(5), and p be given by (3). We set $\bar{m} = Np/(Np - N + p)$.

If $p = N$, then $\bar{m} = 1$, and if f is in $L^m(\Omega)$, then $m > \bar{m} = 1$ and (7) is known to have a weak solution in $W_0^{1,p}(\Omega)$ (which is the solution of (9), given by [LL], since $f \in W^{-1,p'}(\Omega)$).

Let us now assume that $p < N$. Then $\bar{m} > 1$ and if f is in $L^m(\Omega)$, $m \geqq \bar{m}$, (7) is known to have a weak solution in $W_0^{1,p}(\Omega)$ (since $f \in W^{-1,p'}(\Omega)$). The only case of interest is when f is in $L^m(\Omega)$, $1 < m < \bar{m}$, and we prove the following

PROPOSITION 1. *Let $\underline{a}$ satisfy (2)–(5) and $p < N$ (p given by (3)). Let $1 < m < \bar{m} = Np/(Np - N + p)$ and f be in $L^m(\Omega)$. Then (7) has a weak solution u in $W_0^{1,q}(\Omega)$ for all $1 \leqq q < (p-1)m^*$ (recall that $m^* = mN/(N-m)$).*

Remark 10. Note that, when $m = 1$, $(p-1)m^* = (N/(N-1))(p-1)$ and, when $m = \bar{m}$, $(p-1)m^* = p$. In both cases we obtain the optimal q.

Proof of Proposition 1. Proposition 1 will be proved if an estimate in $W_0^{1,q}(\Omega)$, $q < (p-1)m^*$, for the solution u of (9) is obtained when

$f \in W_0^{-1,p'}(\Omega)$ and f is an arbitrary element of a bounded set of $L^m(\Omega)$. It suffices to prove that

$$\text{for every } q \text{ in } [1, (p-1)m^*), \text{ for every } B>0, \text{ there exists } C>0 \text{ (depending only on } m, q, \underline{a}, \Omega, \text{ and } B) \text{ such that if } f \text{ lies in } W^{-1,p'}(\Omega) \cap L^m(\Omega) \text{ and } u \text{ is a solution of (9) then } \|u\|_{W^{1,q}} \leqq C, \text{ whenever } \|f\|_{L^m} \leqq B. \tag{28}$$

We prove (28) by a method very similar to that of Subsection 4.

Let f be an element of $W^{-1,p'}(\Omega) \cap L^m(\Omega)$, and u be the solution of (9), and assume that $\|f\|_{L^m} \leqq B$. We denote various constants (depending only on m, q, $\underline{a}$, Ω, and B) by $c_1, c_2 \cdots$.

We now follow step by step the proof of the estimate of u in Subsection 4.

Setting, for an integer n

$$B_n = \{x \in \Omega, n \leqq |u(x)| \leqq n+1, |Du(x)| \geqslant M\}$$

$$E_n = \{x \in \Omega, n < |u(x)|\}$$

and taking the same ψ as in the proof of (14) in Subsection 4, we obtain

$$\alpha \int_{B_n} |Du|^p \leqq \int_{E_n} f\psi(u).$$

Then

$$\int_{B_n} |Du|^p \leqq \frac{\|f\|_{L^m}}{\alpha} (\text{meas}(E_n))^{1/m'}, \qquad \left(\text{with } m' = \frac{m}{m-1}\right). \tag{29}$$

Let q be strictly less than p. Hölder's inequality yields

$$\int_{B_n} |Du|^q \leqq \left(\int_{B_n} |Du|^p\right)^{q/p} (\text{meas}(B_n))^{(p-q)/p}.$$

Since

$$\text{meas}(B_n) \leqq \frac{1}{n^{q^*}} \int_{B_n} |u|^{q^*}$$

and

$$\text{meas}(E_n) \leqq \frac{1}{n^{q^*}} \int_{\Omega} |u|^{q^*},$$

we deduce with the help of (29) that

$$\int_{B_n} |Du|^q \leqq c_1 \, \|u\|_{L^{q^*}}^{q^*(q/pm')} \left(\int_{B_n} |u|^{q^*} \right)^{(p-q)/p} \frac{1}{n^{(q^*/p)(q/m'+p-q)}}.$$

Repeated use of Hölder's inequality implies that

$$\sum_{n=1}^{\infty} \int_{B_n} |Du|^q \leqq c_1 \, \|u\|_{L^{q^*}}^{q^*(q/pm')} \left(\sum_{n=1}^{\infty} \int_{B_n} |u|^{q^*} \right)^{(p-q)/p} \left(\sum_{n=1}^{\infty} \frac{1}{n^{(q^*/q)(p-q/m)}} \right)^{q/p}.$$

Then, recalling (12), we obtain

$$\int_{\Omega} |Du|^q \leqq c_2 + c_1 \, \|u\|_{L^{q^*}}^{q^*(q/pm'+(p-q)/p)} \left(\sum_{n=1}^{\infty} \frac{1}{n^{(q^*/q)(p-q/m)}} \right)^{q/p}. \tag{30}$$

Using Sobolev Imbedding Theorem, as in Subsection 4, we derive an estimate on u in $L^{q^*}(\Omega)$ and, by (30), in $W_0^{1,q}(\Omega)$, provided

$$\begin{aligned} q^* \left(\frac{q}{pm'} + \frac{p-q}{p} \right) &< q, \\ \frac{q^*}{q} \left(p - \frac{q}{m} \right) &> 1. \end{aligned} \tag{31}$$

The second part of (31) is true if q is strictly less than $(p-1)m^*$. Note that $1 < (p-1)m^* < p$, since $1 < m < \bar{m}$.

The first part of (31) is true since $1 < p < N$ and $m < \bar{m} = Np/(Np - N + p)$.

We have thus proved (28), and therefore Proposition 1.

Remark 11. Estimates on Du are also obtained in [Ta] by rearrangement techniques.

III. Lower Order Perturbations

7. Our goal in this section is to show how the method described in Section II enables us to generalize certain results for semilinear elliptic equations (see [BS, GM2, G]) to the case of operators with a non-linear principal part as in Section II.

Consider for instance the equation

$$\begin{aligned} Au + g(\cdot, u) &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{32}$$

where $Au = -\mathrm{div}(\underline{a}(\cdot, Du))$, $\underline{a}$ satisfies (2)–(5), f lies in $M(\Omega)$, and g satisfies:

$$g(x, s) \text{ measurable in } x \in \Omega, \text{ for all } s \in R \text{ and continuous in } s \in R, \text{ a.e. in } x \in \Omega; \tag{33}$$

$$g(x, s)s \geqq 0 \quad \forall s \in R, \text{ a.e. in } x \in \Omega; \tag{34}$$

$$\mathrm{Sup}\{|g(x, s)|, |s| \leqq t\} \in L^1_{\mathrm{loc}}(\Omega), \forall t \in R^+. \tag{35}$$

We say that u is a weak solution of (32) if

$$u \in W_0^{1,1}(\Omega), \qquad \underline{a}(\cdot, Du) \in L^1_{\mathrm{loc}}(\Omega), \qquad g(\cdot, u) \in L^1_{\mathrm{loc}}(\Omega),$$

$$\int_\Omega \underline{a}(x, Du)\, D\psi + \int_\Omega g(x, u)\psi = \langle f, \psi \rangle, \qquad \forall \psi \in C_0^\infty(\Omega). \tag{36}$$

The following theorem holds.

THEOREM 2. *Let $\underline{a}$ satisfy* (2)–(5), *g satisfy* (33)–(35), *and f be an element of $L^1(\Omega)$.*

Then there exists a weak solution u of (32).

It is known that it is not possible to replace $f \in L^1(\Omega)$ by $f \in M(\Omega)$ in Theorem 2, even when $A = -\Delta$. For instance, if $N = 3$, $0 \in \Omega$, $f = \delta$, $A = -\Delta$, $g(\cdot, u) = u^3$, (32) has no weak solution (see [B], or more generally [BV, BP], for the problem of "removable singularities"). It would be interesting to characterize the measures f for which (32) has a weak solution (as it is done in [GM1] for $A = -\Delta$, see also [BP]). An easy result is that we can assume f in $M(\Omega)$ in Theorem 2 if g does not grow too rapidly at infinity with respect to its second argument. For instance, we can suppose f in $M(\Omega)$ if we replace the very weak hypothesis (35) by the following stronger hypothesis:

$$\text{there exist } b_1, b_2, \delta \text{ with } b_1 \in L^1_{\mathrm{loc}}(\Omega),\ b_2 \in L^\infty_{\mathrm{loc}}(\Omega),\ \delta < N(p-1)/(N-p), \text{ such that } |g(x, s)| \leqq b_1(x) + b_2(x)\,|s|^\delta \text{ a.e. in } x, \text{ for every real number } s. \tag{37}$$

In fact we will prove the following theorem.

THEOREM 3. *Let $\underline{a}$ satisfy* (2)–(5), *g satisfy* (33), (34), *and* (37), *and f lies in $M(\Omega)$. Then there exists a weak solution u of* (32) (*that is, a u that satisfies* (36)).

Remark 12. If for example, $p = 2$, $A = -\Delta$, $g(u) = |u|^{\delta-1}u$, then the bound $\delta < N/(N-2)$ is optimal for $N \geqq 3$. If $\delta \geqq N/(N-2)$, there exists some f in $M(\Omega)$ for which (32) has no weak solution. It suffices to take for f a Dirac mass at y for any $y \in \Omega$.

Remark 13. It is also possible to have some dependence of g on Du in Theorems 2 and 3. In the case of Theorem 2, this dependence is possible if $g(x, s, \xi)$ grows at infinity in ξ less than $|\xi|^{p-\varepsilon}$ for some $\varepsilon > 0$. The argument is developed in [G] for $p = 2$ in the case of a linear operator A.

The proof of Theorems 2 and 3 are performed by solving an approximate problem (Subsection 8). Estimates on the solutions of the approximate problem are obtained (Subsection 9) and the limit process is the object of Subsection 10. The proof of these Theorems is very similar to that of [GM2, G].

8. In this subsection the function $\underline{a}$ satisfies (2)–(5) and g satisfies (33), (34). We define, for $n \in N$, the function g_n by truncation of g, that is,

$$\begin{aligned} &g_n(x, s) = g(x, s) && \text{if } \ |g(x, s)| \leqq n, x \in \Omega, s \in R \\ &g_n(x, s) = n && \text{if } \ g(x, s) > n, x \in \Omega, s \in R \\ &g_n(x, s) = -n && \text{if } \ g(x, s) < -n, x \in \Omega, s \in R. \end{aligned} \tag{38}$$

Note that (33), (34) are satisfied with g_n in place of g.

Let f_n be an element of $W^{-1,p'}(\Omega)$, with $p' = p/(p-1)$ and p given in (3).

It is known [LL] that there exists a weak solution u_n of (32) with $g = g_n$ and $f = f_n$ which satisfies

$$\begin{gathered} u_n \in W_0^{1,p}(\Omega) \\ \int_\Omega a(x, Du_n)\, Dv + \int_\Omega g_n(x, u_n) v = \langle f_n, v \rangle, \qquad \forall v \in W_0^{1,p}(\Omega). \end{gathered} \tag{39}$$

9. This subsection is devoted to the derivation of estimates on the solution u_n of (39), when (f_n) is bounded in $L^1(\Omega)$. More precisely we assume that the hypotheses of Subsection 8 are satisfied and we assume that

there exists a positive constant B, with $\|f_n\|_{L^1} \leqq B$, $\forall n \in \mathbb{N}$. (40)

We are going to establish the following two assertions.

$\int_{\{|u_n| > t\}} |g(x, u_n)| \leqq \int_{\{|u_n| > t\}} |f_n|$, for every integer n and every t in R^+, where $\{|u_n| > t\} = \{x \in \Omega \colon |u_n(x)| > t\}$. (41)

The sequence (u_n) is relatively compact in $W_0^{1,q}(\Omega)$ for all q in $[1, (N/(N-1))(p-1))$, where p is given in (3). (42)

Proof of (41). Let (ψ_i) be a sequence of real smooth increasing functions. The choice of $\psi_i(u_n)$ as a test function in (39) yields

$$\int_\Omega g_n(x, u_n)\,\psi_i(u_n) \leqq \int_\Omega f_n \psi_i(u_n). \tag{43}$$

If $\psi_i(s)$ converges to the function $\psi(s)$ defined by

$$\begin{aligned} \psi(s) &= 1 && \text{if } s > t \\ \psi(s) &= 0 && \text{if } -t \leqq s \leqq t \\ \psi(s) &= -1 && \text{if } s < -t, \end{aligned}$$

we obtain estimate (41).

Proof of (42). Letting $t = 0$ in (41) yields

$$\| g_n(\cdot, u_n)\|_{L^1} \leqq \|f_n\|_{L^1}.$$

Recalling (40) and setting $h_n = f_n - g_n(\cdot, u_n)$, we deduce that

$$\|h_n\|_{L^1} \leqq 2B,\ h_n \in W^{-1,p'}(\Omega) \cap L^1(\Omega). \tag{44}$$

Note that u_n is the solution of (9) with $f = h_n$.

From (44) and Subsection 4 of Section II (see (10)), we then deduce that (u_n) is bounded in $W_0^{1,q}$ for $1 \leqq q < (N/(N-1))(p-1)$ and (21) in Subsection 5 of Section II implies (42).

10. This subsection is devoted to the proof of Theorems 2 and 3. Let $\underline{a}$ satisfy (2)–(5), g satisfy (33), (34), and (f_n) be a sequence of $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that there exists a positive constant B, with $\|f_n\|_{L^1} \leqq B$.

Let g_n be defined by (38).

There exists $u_n \in W_0^{1,p}(\Omega)$ such that (cf. Subsection 8)

$$\int_\Omega \underline{a}(x, Du_n)\, Dv + \int_\Omega g_n(x, u_n) v = \int_\Omega f_n v, \qquad \forall v \in C_0^\infty(\Omega). \tag{45}$$

With the help of the results of Subsection 9, the sequence (u_n) is relatively compact in $W_0^{1,q}(\Omega)$ for $1 \leqq q < (N/(N-1))(p-1)$. Then we can assume (after extraction of a subsequence, still denoted by (u_n))

$$\begin{aligned} u_n &\to u && \text{in } W_0^{1,q}(\Omega),\ 1 \leqq q < \frac{N}{N-1}(p-1), \\ u_n &\to u && \text{a.e.} \\ \underline{a}(\cdot, Du_n) &\to \underline{a}(\cdot, Du) && \text{in } L^r(\Omega),\ 1 \leqq r < \frac{N}{N-1}. \end{aligned} \tag{46}$$

The last assertion in (46) is a direct consequence of hypothesis (4) as in Subsection 5 of Section II.

Proof of Theorem 3. We can take $B = \|f\|_{M(\Omega)}$ and we assume that f_n converges to f in the distribution sense. The continuous imbedding of $W_0^{1,q}(\Omega)$ in $L^{q^*}(\Omega)$ for $1 \leqq q < N$ implies that

$$u_n \to u \qquad \text{in } L^r(\Omega),\ 1 \leqq r < \frac{N(p-1)}{N-p}. \tag{47}$$

Note that $q^* < N(p-1)/(N-p)$ because $q < N(p-1)/(N-1)$.
Then assumption (37) coupled with (47) yields

$$g(\cdot, u_n) \to g(\cdot, u) \qquad \text{in } L^1_{\text{loc}}(\Omega). \tag{48}$$

By (46), (48), and the fact that f_n converges to f in the distribution sense, we can pass to the limit in (45) and we obtain (36). This completes the proof of Theorem 3. In fact by Fatou's lemma and estimate (41) (with $t = 0$) we also have $g(\cdot, u) \in L^1(\Omega)$ (and $\|g(\cdot, u)\|_{L^1} \leqq B = \|f\|_{M(\Omega)}$). We can thus say that

$$\begin{aligned} &u \in W_0^{1,q}(\Omega), && \text{for all } 1 \leqq q < \frac{N}{N-1}(p-1), \\ &\underline{a}(\cdot, Du) \in L^r(\Omega), && \text{for all } 1 \leqq r < \frac{N}{N-1}, \\ &g(\cdot, u) \in L^1(\Omega) && \\ &\int_\Omega \underline{a}(x, Du)\, Dv + \int_\Omega g(x, u) v = \langle f, v \rangle, && \\ & && \text{for any } v \text{ in } \bigcup_{r' > N} W_0^{1,r'}(\Omega). \end{aligned} \tag{49}$$

Proof of Theorem 2. In the case of Theorem 2 it is not so easy to pass to the limit in the second term of the left hand side of (45) (g does not satisfy (37) but only (35)). We proceed as in [GM2]. We can take $B = \|f\|_{L^1}$ and assume that f_n converges to f in $L^1(\Omega)$.
By (46) we have

$$g_n(\cdot, u_n) \to g(\cdot, u) \qquad \text{a.e.} \tag{50}$$

In order to prove that $g_n(\cdot, u_n)$ converges to $g(\cdot, u)$ in $L^1_{\text{loc}}(\Omega)$, it suffices to prove that

$$g_n(\cdot, u_n) \text{ is equiintegrable on } K \text{ for all } K \subset \Omega,\ K \text{ compact}. \tag{51}$$

We omit the proof of (51), which is the same as the corresponding result in [GM2]. We remark that the sequence (f_n) is equiintegrable on Ω, and

$\text{meas}\{|u_n| > t\}$ converges to zero, *uniformly* with respect to n, when t goes to $+\infty$. Then we use the estimate (41) and the hypothesis (35).

By (50) and (51), we deduce that

$$g_n(\cdot, u_n) \to g(\cdot u) \qquad \text{in } L^1_{\text{loc}}(\Omega). \tag{52}$$

As in the proof of Theorem 3 we then conclude (from (45), (46), (52), and $f_n \to f$ in $L^1(\Omega)$) that u satisfies (36). This proves Theorem 2. In fact, we have $g(\cdot, u) \in L^1(\Omega)$ and $\|g(\cdot u)\|_{L^1} \leqq B = \|f\|_{L^1}$, by Fatou's lemma, (50), and (41) with $t = 0$. Thus u satisfies (49), that is,

$$\begin{aligned}
&u \in W_0^{1,q}(\Omega), && \text{for all } 1 \leqq q < \frac{N}{N-1}(p-1),\\
&\underline{a}(\cdot, Du) \in L^r(\Omega), && \text{for all } 1 \leqq r < \frac{N}{N-1},\\
&g(\cdot, u) \in L^1(\Omega)\\
&\int_\Omega \underline{a}(\cdot, Du)\, Dv + \int_\Omega g(x, u) v = \int_\Omega fv,\\
& && \text{for any } v \text{ in } \bigcup_{r' > N} W_0^{1,r'}(\Omega).
\end{aligned} \tag{53}$$

IV. Parabolic Case

In this section we show how the method of Section II allows us to extend the previous existence results to the parabolic case.

Let $Q = \Omega \times (0, T)$, T a real positive number and P the differential operator

$$P(v) = \frac{\partial v}{\partial t} - \text{div}(\underline{a}(x, t, Dv)), \tag{54}$$

where $\underline{a}\colon \Omega \times (0, T) \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following hypotheses:

$\underline{a}$ is measurable in (x, t), for all ξ in $\mathbb{R}^N$ and continuous in $\xi \in \mathbb{R}^N$ for a.e. (x, t) in Q; (55)

there exists three constants p, M, α, with $p \in (2 - 1/(N+1), \infty)$, $M \geqq 0$, $\alpha > 0$, such that for any ξ in $\mathbb{R}^N$ with $|\xi| > M$
$\underline{a}(x, t, \xi)\xi \geqq \alpha |\xi|^p$ for a.e. (x, t) in Q,
$\underline{a}(x, t, 0) = 0$; (56)

there existst a function b in $L^{p'}(Q)$ and a constant $K \geqq 0$ such that, for any ξ in $\mathbb{R}^N$,
$|\underline{a}(x, t, \xi)| \leq K(b(x, t) + |\xi|^{p-1})$ for a.e. (x, t) in Q; (57)

there exists three constants s, γ, k and a function d such that $s \geqq 2$, $d \in L^1(Q)$, $k \geqq 0$, $\gamma < (s-1)[(p(N+1)-N)/(N+1)]$,
$(\underline{a}(x, t, \xi) - \underline{a}(x, t, \eta))(\xi - \eta) \geqq (1/\beta(x, t, \xi, \eta)) |\xi - \eta|^s$ for a.e. (x, t) in Q and any ξ, η in $\mathbb{R}^N$,
$0 < \beta(x, t, \xi, \eta) \leqq k(d(x, t)^{s-1} + |\xi|^{\gamma} + |\eta|^{\gamma})$ for a.e. (x, t) in Q and any ξ, η in $\mathbb{R}^N$. (58)

We consider the following Cauchy problem

$$\begin{gathered} P(u) = f \\ u \in L^q(0, T, W_0^{1,q}(\Omega)) \qquad \text{for} \quad q < \frac{p(N+1)-N}{N+1} \\ u(x, 0) = u_0(x), \end{gathered} \tag{59}$$

where

$$f \text{ is an element of } M(Q) \tag{60}$$

$$u_0 \text{ is an element of } M(\Omega). \tag{61}$$

In (59) the initial condition, $u(x, 0) = u_0(x)$, is to be taken in a classical sense, since we will show that $u \in C([0, T], H^{-s}(\Omega))$ for s large enough.

We will prove the following theorem

THEOREM 4. *Under the hypotheses* (54)–(58) *and* (60), (61), *there exists a solution u of the equation* (59).

Proof. We sketch the proof, which is similar to the one of Theorem 1. We define an "aproximate" equation to (59) for which we know the existence of a solution (see [L]). We choose a sequence $(f_n) \subset C_0^{\infty}(Q)$ such that $\|f\|_{L^1(Q)} \leqq B = \|f\|_{M(Q)}$ and a sequence $(u_0^u) \subset C_0^{\infty}(\Omega)$ such that $\|u_0^n\|_{L^1(\Omega)} \leqq \|u_0\|_{M(\Omega)} = C$. The sequences f_n and u_0^n converge respectively to f and u_0 in the distribution sense.

Let u_n be the solution of the Cauchy–Dirichlet problem

$$\begin{aligned} &P(u_n) = f_n \\ &u_n \in L^p(0, T; W_0^{1,p}) \\ &u_n(x, 0) = u_0^n(x). \end{aligned} \tag{62}$$

Let ψ be the real function

$$\begin{aligned} \psi(s)&=1 && \text{if} \quad s>1 \\ \psi(s)&=s && \text{if} \quad -1\leqq s\leqq 1 \\ \psi(s)&=-1 && \text{if} \quad s<-1. \end{aligned}$$

Taking $\psi(u_n(t)\,\chi_{(0,t)}$ as test function in (62) we have

$$\int_\Omega |u_n(t)| - \int_\Omega \Phi(u_0^n) \leqq B + \frac{1}{2}\,\text{meas}\,\Omega,$$

where $\Phi(s)=\int_0^s \psi(\sigma)\,d\sigma$.

By virtue of the previous inequality we have

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leqq c_1, \tag{63}$$

because

$$\int_\Omega \Phi(u_0^n) \leqq \int_\Omega |u_0^n| \leqq c_2.$$

Let n be a fixed integer and define ψ by

$$\begin{aligned} \psi(s)&=1 && \text{if} \quad s>n+1 \\ \psi(s)&=s-n && \text{if} \quad +n\leqq s\leqq n+1 \\ \psi(s)&=s+n && \text{if} \quad -n-1<s<-n \\ \psi(s)&=-1 && \text{if} \quad s\leqq -n-1 \\ \psi(s)&=0 && \text{if} \quad -n<s<n. \end{aligned}$$

The choice of $\psi(u_n)$ as a test function yields

$$\int_{B_m} |Du_n|^p \leqq \frac{B+C}{\alpha} \tag{64}$$

with

$$B_m=\{(x,t)\in Q;\, m\leqq |u_n(x,t)|\leqq m+1,\, |Du_n(x,t)|\geqq M\}. \tag{65}$$

Now let $q<(p(N+1)-N)/(N+1)$, $r=((N+1)/N)q$. We have

$$\int_{B_m} |Du_n|^q \leqq c_3(\text{meas}\,B_m)^{1-q/p} \leqq c_3\left(\int_{B_m} |u_n|^r\right)^{(p-q)/p} \frac{1}{m^{r(p-q)/p}}.$$

So that

$$\int_Q |Du_n|^q \leqq c_4(n_0) + c_3 \sum_{m=n_0}^{\infty} \left(\int_{B_m} |u_n|^r \right)^{(p-q)/p} \frac{1}{m^{r(p-q)/p}}$$

$$\leqq c_4(n_0) + c_3 \left(\int_Q |u_n|^r \right)^{(p-q)/p} \left(\sum_{m=n_0}^{\infty} \frac{1}{m^{r(p-q)/q}} \right)^{q/p}. \tag{66}$$

Applying Hölder's inequality yields

$$\|u_n\|_{L^r(\Omega)} \leqq \|u_n\|_{L^1(\Omega)}^{\theta} \|u_n\|_{L^{q^*}(\Omega)}^{1-\theta} \leqq c_5 \|u_n\|_{L^{q^*}(\Omega)}^{1-\theta},$$

where $1-\theta = ((1-r)/(1-q^*)) \cdot (q^*/r)$.

The above inequality leads to

$$\|u_n\|_{L^r(0,T;L^r)}^r \leqq c_6 \int_0^T \|u_n\|_{L^{q^*}(\Omega)}^{q^*(1-r)/(1-q^*)}$$

$$= c_6 \|u_n\|_{L^q(0,T;L^{q^*})}^q$$

if r is such that $q^*(1-r)/(1-q^*) = q$, that is, $r = ((N+1)/N)q$.

Sobolev imbedding Theorem implies that

$$\|u_n\|_{L^q(0,T;L^{q^*})}^q = \int_0^T \left(\int_\Omega |u_n|^{q^*} \right)^{q/q^*} dt$$

$$\leqq c_7 \int_0^T \left(\int_\Omega Du_n|^q \right) dt$$

$$\leqq c_8(n_0) + c_9 \|u_n\|_{L^q(0,T;L^{q^*})}^{q(p-q)/p}$$

$$\times \left(\sum_{m=n_0}^{\infty} \frac{1}{m^{(N+1)(p-q)/N}} \right)^{q/p}.$$

From the previous bound on q we have the a priori estimate

$$\|u_n\|_{L^q(0,T;L^{q^*})} \leqq c_{10}, \tag{67}$$

and then the estimate

$$\|u_n\|_{L^q(0,T;W_0^{1,q})} \leqq c_{11} \tag{68}$$

follows as in Theorem 1.

From the previous a priori estimates we deduce that (u_n') is a sequence bounded in the space $L^1(0,T;W^{-1,s}) + L^1(0,T;L^1)$, with $s = (p(N+1)-N)/(N+1)(p-1)$. So the sequence (u_n) is relatively compact in $L^1(Q)$ by a compactness lemma of Aubin's type. Such a lemma can be found, for example, in [Si, Te]. Finally we can prove the convergence in $L^1(Q)$ of Du_n as in Theorem 1 (using (58)) and we deduce that u_n converges to u in $L^q(0, T, W_0^{1,q})$ for all $q < (p(N+1) - N)/(N+1)$. Thus u is a solution of equation (59).

Note that for s large enough $\partial u_n/\partial t$ converges strongly to $\partial u/\partial t$ in $L^1(0, T; H^{-s}(\Omega))$. Thus u_n converges strongly to u in $C([0, T], H^{-s}(\Omega))$, and $u_n(\cdot, 0)$ converges to $u(\cdot, 0)$ in $H^{-s}(\Omega)$. Since $u_n(\cdot, 0) = u_0^n$ and u_0^n converges to u_0 in the distribution sense, we deduce that $u_0 = u(\cdot, 0)$.

ACKNOWLEDGMENT

This work has been partially supported by MPI, CNR (L.B.), Université de Savoie (L.B., T.G.).

REFERENCES

[S] G. STAMPACCHIA, Equations elliptiques du second ordre à coefficients discontinus, *in* "Séminaire de Mathématiques Supérieures, Université de Montréal, 1965."

[BS] H. BREZIS AND W. STRAUSS, Seminilear elliptic equations in L^1, *J. Math. Soc. Japan* **25** (1973), 565–590.

[BBC] PH. BENILAN, H. BREZIS AND M. CRANDALL, A semilinear equation in L^1, *Ann. Scuola Norm. Sup. Pisa* **2** (1975), 523–555.

[BCP] PH. BENILAN, M. CRANDALL, AND A. PAZY, Evolution equations governed by accretive operators, in preparation.

[GM1] T. GALLOUET AND J. M. MOREL, Resolution of a semilinear equation in L^1, *Proc. Roy. Soc. Edinburgh Sect. A* **96** (1984), 275–288.

[GM2] T. GALLOUET AND J. M. MOREL, On some semilinear problems in L^1, *Boll. Un. Mat. Ital. A* (6) **4** (1985), 121–131.

[G] T. GALLOUET, Equations elliptiques semilinéaires avec, pour la non linéarité, une condition de signe et une dépendance sous-quadratique par rapport au gradient, *Ann. Fac. Sc. Univ. Toulouse* **9**, No. 2 (1988), 161–169.

[BP] P. BARAS AND M. PIERRE, Critère d'existence de solutions positives pour des équations semi-linéaires non monotones, *Ann. Inst. H. Poincaré* **3** (1985), 185–212.

[LL] J. LERAY AND J. L. LIONS, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty et Browder, *Bull. Soc. Math. France* **93** (1965), 97–107.

[SE] J. SERRIN, Pathological solutions of elliptic differential equations, *Ann. Scuola Norm. Sup. Pisa* (1964), 385–387.

[B] H. BREZIS, Some variational problems of the Thomas Fermi type, *in* "Variational Inequalities" (Cottle, Gianessi, Lions, Eds.), pp. 53–73, Wiley, New York, 1980.

[BV] H. BREZIS AND L. VERON, Removable singularities for some nonlinear elliptic equations, *Arch. Rational Mech. Anal.* **75** (1980), 1–6.

[L] J. L. LIONS, "Quelques méthodes de rèsolution des problèmes aux limites non linéaires," Dunod, Paris, 1968.

[Si] J. SIMON, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl.* **196** (1987), 65–96.

[Te] R. TEMAM, "Navier–Stokes Equations," North-Holland, Amsterdam, 1977.

[Ta] G. TALENTI, Non-linear elliptic equations, rearrangements of functions and Orlicz spaces, *Ann. Mat. Pura Appl.* (4) **120** (1979), 159–184.

[K] S. KICHENASSAMY, Quasilinear problems with singularities, *Manuscripta Math.* **57** (1987), 281–313.