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Resolution except for minimal singularities I

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Abstract

The philosophy of this article is that the desingularization invariant together with natural geometric information can be used to compute local normal forms of singularities. The idea is used in two related problems: (1) We give a proof of resolution of singularities of a variety or a divisor, except for simple normal crossings (i.e., which avoids blowing up simple normal crossings, and ends up with a variety or a divisor having only simple normal crossings singularities). (2) For more general normal crossings (in a local analytic or formal sense), such a result does not hold. We find the smallest class of singularities (in low dimension or low codimension) with which we necessarily end up if we avoid blowing up normal crossings singularities. Several of the questions studied were raised by Kollár.

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Contents

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1. Introduction

The philosophy developed in this article and in the sequel [2] is that the desingularization invariant of [3] together with natural geometric information can be used to compute local normal forms of singularities, at least when the constant locus of the invariant has low codimension. The idea is used in two related problems: (1) We give a proof of resolution of singularities of a variety or a divisor, except for simple normal crossings (i.e., which avoids blowing up simple normal crossings singularities, and ends up with a variety or a divisor having only simple normal crossings singularities). (2) For more general normal crossings (in a local analytic or formal sense), such a result does not hold. We find the smallest class of singularities (in low dimension or low codimension) with which we necessarily end up if we avoid blowing up normal crossings singularities. Several of the questions studied were raised by Kollár [10]. We have included a *Crash course on the desingularization invariant* as an Appendix, in order to make the article as self-contained as possible.

The preceding problems are interesting because normal crossings or more general "mild singularities" have to be admitted in natural geometric situations.

Example 1.1. Consider the family of projective curves X_λ,

$$z^3 + y^3 + x^3 - 3\lambda xyz = 0.$$

The curve X_λ is smooth if $\lambda^3 \neq 1$. When $\lambda = 1$, for example, the equation splits as

$$(z + y + x)(z + \epsilon y + \epsilon^2 x)(z + \epsilon^2 y + \epsilon x) = 0,$$

where ϵ denotes the cube root of unity $\epsilon = e^{2\pi i/3}$; in particular X_1 has normal crossings singularities. We cannot simultaneously resolve the singularities of a family of curves without allowing special fibres that have normal crossings singularities. (Here, for instance, because the generic and special fibres have different genera.)

As another example, resolution of singularities of an ideal or a divisor (“log-resolution” of singularities) leads to a divisor with normal crossings. In the same way, when we resolve the singularities of a singular algebraic (or analytic) variety, its total transform (or inverse image, with respect to any local embedding of the variety in a smooth space) necessarily has normal crossings singularities. From the point of view of these examples, it is reasonable to consider normal crossings singularities acceptable from the start (in any case, they can be eliminated by normalization), and to ask whether we can resolve singularities except for normal crossings. In particular, we can ask the following question.

Question 1.2. Given an algebraic variety X, can we find a proper birational morphism $\sigma : X' \to X$ such that

(1) X' has only normal crossings singularities;
(2) σ is an isomorphism over the locus of points of X having only normal crossings singularities?

An *algebraic variety* means a separated scheme of finite type over a field $\underline{k}$. Throughout this article, char $\underline{k} = 0$.

The question above is ambiguous. Roughly speaking, we say that X has normal crossings at a point a if, locally at a, every irreducible component is smooth and all intersections are transverse; in other words, locally, X can be embedded in a smooth variety Z with local coordinates $(x_1, \dots, x_n)$ at a in which X is defined by a monomial equation

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 \tag{1.1}$$

(where the α_i are nonnegative integers). The ambiguity is in the meaning of “locally” or “local coordinates”.

Definitions 1.3. Let X denote an algebraic variety over $\underline{k}$. We say that X has *simple normal crossings* (*snc*) at a point a if there is an embedding of an open neighbourhood of a in a smooth variety Z and a regular system of parameters $(x_1, \dots, x_n)$ for Z at a, with respect to which X is defined by an equation of the form (1.1).

We say that X has *normal crossings* (*nc*) at a if the same condition is satisfied, except that $(x_1, \dots, x_n)$ is a local étale coordinate system.

We will say that X has *normal crossings* (or *simple normal crossings*) *of order* k at a if precisely k exponents α_i are nonzero in (1.1).

A variety X has normal crossings at a if and only if it can be defined at a by a monomial equation with respect to formal coordinates, after a finite extension of the ground field $\underline{k}$. In the case of simple normal crossings (with reference to the definition above), each irreducible component of X containing a is given locally by $x_i = 0$, for some i. Definitions 1.3 have obvious analogues for an embedded variety X or for a divisor on a smooth variety.

Examples 1.4. The plane curve $y^2 = x^2 + x^3$ has normal crossings but not simple normal crossings at the origin. The curve $y^2 + x^2 = 0$ is nc, but is snc if and only if $\sqrt{-1} \in \underline{k}$. An embedded hypersurface defined at a point by an equation $y^2 + ux^2 = 0$, where x, y are regular coordinates and u is a unit in the local ring, is nc at a, but snc if and only if u is a square.

The answer to Question 1.2 is “yes” for snc (Theorem 1.5 following), but “no” for nc in general (Example 1.7).

Theorem 1.5. *Let X denote a reduced variety over $\underline{k}$. Let X^{snc} denote the simple normal crossings locus of X. Then there is a morphism $\sigma : X' \to X$ which is a composite of finitely many* admissible *blowings-up, such that*

(1) $X' = (X')^{\mathrm{snc}}$;
(2) *σ is an isomorphism over X^{snc}.*

An *admissible* blowing-up means a blowing-up σ with centre C which is smooth and has only simple normal crossings with respect to the exceptional divisor. I.e., with respect to a suitable local embedding of X in a smooth variety Z and the induced blowing-up sequence of Z, there are regular coordinates $(x_1, \dots, x_n)$ at any point of C, in which C is a coordinate subspace and each component of the exceptional divisor is a coordinate hyperplane $(x_i = 0)$, for some i.

Versions of Theorem 1.5 were first proved by Szabó [11] and by the authors [3, Section 12]. We give a proof in Section 3 that we sketched in a letter to Michael Temkin (2007); see [12, Theorem 2.2.11]. The theorem can be strengthened in various ways; see Section 3. For example, instead of using the snc locus, we can use the locus of points having only simple normal crossings singularities of order up to r (snc $\leq r$), for given r (Remark 3.2). Moreover, σ can be realized as a composite of smooth blowings-up

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t = X' \tag{1.2}$$

where we avoid blowing up snc singularities at every step; i.e., each centre of blowing up is disjoint from the snc locus of the corresponding total transform of X (with respect to a local embedding of X in a smooth variety); see [1]. One can also resolve singularities of pairs, preserving "semi-simple normal crossings" [6] (see [9, Problem 19]). In Theorem 1.5, we can add the following condition, considered by Kollár [9]:

(3) *the morphism σ maps the singular set* $\mathrm{Sing}\, X'$ *birationally onto the closure of* $\mathrm{Sing}\, X^{\mathrm{snc}}$.

Remark 1.6. Because of the way the invariant is used, Theorem 1.5 and the other desingularization results here are functorial. For example, Theorem 1.5 is functorial with respect to local isomorphisms (or, more generally, with respect to étale or smooth morphisms that preserve the number of irreducible components at every point). We will not always explicitly mention functoriality in the statements of the theorems. (See also Remarks 3.6 and 4.4.) Kollár gives another (non-functorial) proof of Theorem 1.5 in [9]. All the desingularization results here also have analytic versions (where the analogue of a morphism that is a finite composite of blowings-up is a morphism which can be realized by a finite blowing-up sequence over any relatively compact open set).

Our proof of Theorem 1.5 automatically provides the additional condition (3) above. Given a morphism $\sigma : X' \to X$ satisfying the conditions of Theorem 1.5, we can also get (3) by successively blowing up every component of $\mathrm{Sing}\, X'$ that does not map birationally onto a component of the closure of $\mathrm{Sing}\, X^{\mathrm{snc}}$ (although here we would have to be a little careful to preserve the condition of functoriality).

Example 1.7. The *pinch point* (*pp*) or *Whitney umbrella* $X \subset \mathbb{A}^3$ is defined by $z^2 + xy^2 = 0$. X has only nc2 singularities outside the pinch point 0. There is no birational morphism $\sigma : X' \to X$ satisfying the analogues of (1), (2) of Theorem 1.5 with nc instead of snc, according to the following argument of Kollár [9, Paragraph 8]; see also [7, Corollary 3.6.10]. At any nonzero point of the x-axis, X has two local analytic branches (over $\mathbb{C}$, say). As we go around the origin, the two branches are interchanged. This continues to hold after any birational map that is an isomorphism over the generic point of the x-axis, so we cannot eliminate the pinch point without blowing up the x-axis.

The desingularization invariant of [3] seems particularly well-suited to studying the questions above, as already evidenced by our proof of Theorem 1.5. One of our goals is to demonstrate that the invariant is a useful tool for making local computations in algebraic geometry and singularity theory.

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1.1. Minimal singularities

Because the nc-analogue of Theorem 1.5 fails, it is interesting to ask the following (a variant of a question of Kollár).

Question 1.8. Can we find the smallest class of singularities $\mathcal{S}$ with the following properties:

(1) $\mathcal{S}$ includes all nc singularities;
(2) given a reduced variety X, there exists a proper (birational) morphism $\sigma : X' \to X$ such that
  (a) $X' = (X')^{\mathcal{S}}$,
  (b) σ is an isomorphism over X^{nc}?

($X^{\mathcal{S}}$ denotes the locus of points of X having only singularities in $\mathcal{S}$, so that $X^{\mathcal{S}}$ includes all smooth points.) We can also ask: Do we get the same class of singularities $\mathcal{S}$ if, in condition (2), we require a morphism σ which is a finite composite of admissible blowings-up?

Remarks 1.9. (1) We are interested in writing *normal forms* for the singularities in $\mathcal{S}$; i.e., local models for their equivalence classes with respect to étale coordinate changes (or, equivalently, with respect to completion and finite field extension).

(2) Normal crossings singularities are singularities of hypersurfaces. We say that X is a *hypersurface* if, locally, X can be defined by a principal ideal on a smooth variety. (We say that X is an *embedded hypersurface* if $X \hookrightarrow Z$, where Z is smooth and X is defined by a principal ideal on Z.) Question 1.8 can be reduced to the case of a hypersurface using the strong desingularization algorithm of [3,5]. The algorithm involves blowing up with smooth centres in the maximum strata of the Hilbert–Samuel function. The latter determines the local embedding dimension, so the algorithm first eliminates points of embedding codimension >1 without modifying nc points. (Recall that if H is the Hilbert–Samuel function of the local ring of a variety at a given point a, then the minimal embedding dimension at a is $H(1) - 1$.)

We therefore reduce Question 1.8 to the case that, locally, $X \hookrightarrow Z$ is an embedded hypersurface, so we want to give normal forms for the singularities in $\mathcal{S}$ in terms of étale local coordinates $(x_1, \ldots, x_n)$ for Z. The table in Definition 1.10 gives normal forms for $\mathcal{S}$, for embedding dimension $n \leq 4$, and therefore answers Question 1.8 for varieties X of dimension ≤ 3 (at least with respect to morphisms that are composites of admissible blowings-up, but see also Remark 1.15).

Definition 1.10. Let $\mathcal{S}$ denote the following class of singularities in n variables, for $n \leq 4$:

$n=2$	$xy=0$	double normal crossings *nc*2
$n=3$	$xy=0$	*nc*2
	$xyz=0$	triple normal crossings *nc*3
	$z^2+xy^2=0$	pinch point *pp*
$n=4$	$xy=0$	*nc*2
	$xyz=0$	*nc*3
	$xyzw=0$	*nc*4
	$z^2+xy^2=0$	*pp*
	$z^2+(y+2x^2)(y-x^2)^2=0$	degenerate pinch point *dpp*
	$x(z^2+wy^2)=0$	product *prod*
	$z^3+wy^3+w^2x^3-3wxyz=0$	cyclic point *cp*3

Theorem 1.11. *Let X denote a reduced variety of pure dimension $n-1$, where $n=2,3,$ or 4. Then there is a morphism $\sigma : X' \to X$ given by a finite sequence of admissible blowings-up*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t = X', \tag{1.3}$$

such that

(a) $X' = (X')^{\mathcal{S}}$,

(b) *σ is an isomorphism over X^{nc}.*

Moreover, the morphism $\sigma = \sigma_X$ (or the entire blowing-up sequence (1.3)) can be realized in a way that is functorial with respect to étale morphisms.

The list of singularities in the case $n=3$ above was proposed by Kollár [10]. Theorem 1.11 for $n \le 3$ will be proved in this article (see Section 1.2). The case $n=4$ has been proved in collaboration with Pierre Lairez and is the subject of the sequel [2]. In each case, $\mathcal{S}$ is the smallest class of singularities satisfying the theorem; see Remark 1.15.

We do not have full lists of candidates for the singularities in $\mathcal{S}$, for $n \ge 5$, though we can make a few remarks: For any $n, \mathcal{S}$ will include a cyclic point singularity cp$(n-1)$ which is an irreducible limit of nc$(n-1)$ singularities along a smooth curve (see [2]). For example, cp3 above is the singularity at the origin of an irreducible hypersurface having nc3 singularities along the nonnegative w-axis. The cyclic singularity cp$k$ of order k is related to the action of the cyclic group $\mathbb{Z}_k$ of order k on $\mathbb{C}^k$ by permutation of coordinates. Cyclic singularities are higher-dimensional versions of the pinch point: pp = cp2.

For any $n, \mathcal{S}$ will include singularities that occur as limits of nc$(n-1)$, according to the way that the limit factors (i.e., according to an associated monodromy group); the reducible limits will be various products of cp$k$, $k < n-1$ (where, by convention, cp1 means a smooth point $x=0$), generalizing prod in Theorem 1.11.

Any singularity that occurs in an arbitrarily small neighbourhood of a singularity in $\mathcal{S}$ necessarily also belongs to $\mathcal{S}$. Degenerate pinch points occur along the nonnegative x-axis of cp3 (see [2, Section 2.2]). The name comes from the fact that a pinch point can be rewritten as $z^2+(y+2x)(y-x)^2=0$ after a coordinate change. The equation of a dpp has been written as in Definition 1.10 rather than in the simpler form $z^2+y^2(y+x^2)$ also to reflect the way that the normal form is determined by the desingularization invariant (see Lemma 4.2).

An optimistic reader can ask whether, in any dimension $n, \mathcal{S}$ comprises nc singularities, products of cp$k$ singularities ($k \le n-1$), and singularities that occur in small neighbourhoods of the latter.

There are many interesting variations of Question 1.8. For example, we consider the following.

Question 1.12. Can we find the smallest class of singularities $\mathcal{S}'$ with the following properties:

(1) $\mathcal{S}'$ includes all nc singularities;
(2) given a reduced variety X, there exists a proper (birational) morphism $\sigma : X' \to X$ such that
   (a) $X' = (X')^{\mathcal{S}'}$,
   (b) σ is an isomorphism over $X^{\mathcal{S}'}$?

Again we can ask: Do we get the same class of singularities if, in condition (2), we require a morphism σ which is a composite of admissible blowings-up? For either Question 1.8 or 1.12, we can also ask: If X is an embedded hypersurface, can we find the smallest class of corresponding singularities of the total transform (inverse image) of X? Are the preceding questions well-formulated—in each case, is there a (unique) smallest class of singularities satisfying the conditions stated?

Clearly, $\mathcal{S} \subset \mathcal{S}'$, for either version of Questions 1.8 and 1.12. In fact, the classes coincide for $n \leq 3$, but not in general.

Definition 1.13. If $n \leq 3$, let $\mathcal{S}' := \mathcal{S}$, where the latter is given by Definition 1.10. For $n = 4$, let $\mathcal{S}'$ be given by the singularities in $\mathcal{S}$ together with the following:

$$z^2 + y(wy + x^2)^2 = 0 \quad \text{exceptional singularity } exc. \tag{1.4}$$

Theorem 1.14. *Let X denote a reduced variety of pure dimension $n-1$, where $n = 2, 3,$ or 4. Then there is a morphism $\sigma : X' \to X$ given by a finite sequence of admissible blowings-up*

$$X = X_0 \xleftarrow{\sigma_1} X_1 \longleftarrow \cdots \xleftarrow{\sigma_t} X_t = X', \tag{1.5}$$

such that

(a) $X' = (X')^{\mathcal{S}'}$,
(b) *σ is an isomorphism over $X^{\mathcal{S}'}$.*

Moreover, the morphism $\sigma = \sigma_X$ (or the entire sequence (1.5)) can be realized in a way that is functorial with respect to étale morphisms.

Again the case $n = 4$ is proved in [2]. See Section 1.2 for the case $n = 3$. As before, $\mathcal{S}'$ is the smallest class of singularities satisfying the theorem. The exceptional singularity is a limit of dpp singularities that cannot be eliminated by blowings-up. (See Lemma 4.2 and [2, Remark 1.6].)

Remark 1.15. Resolution of singularities of an embedded hypersurface can be reformulated as "log-resolution" of singularities of a Weil divisor D on a variety Z. Stated in this way, Question 1.8 is the formulation of Kollár [9], where $\mathcal{S}$ is the smallest class of singularities that includes all normal crossings singularities and satisfies condition (2) of Question 1.8 for the support of the birational transform of D.

In the case $n = \dim Z = 3$, $\mathcal{S}$ is the *unique* smallest class of singularities satisfying this version of Question 1.8, in the following sense. If $\operatorname{Supp} D$ has a pp singularity $z^2 + xy^2 = 0$, in a coordinate chart U of Z at a point $a = 0$, then any proper birational morphism $U' \to U$ which is an isomorphism precisely over $U \setminus \{a\}$, factors through the blowing-up of $\{a\}$ (by the universal-mapping property of blowing up).

Likewise for $n = 4$. In the case of a cyclic point singularity cp3; i.e., a hypersurface $X \subset Z$ defined in local coordinates by $z^3 + wy^3 + w^2x^3 - 3wxyz = 0$, any birational morphism $Z' \to Z$, which modifies the cp3 singularity but is an isomorphism over $Z \setminus \{\text{dpp}, \text{cp3}\}$, factors through the blowing-up either of {cp3} or of $\{\text{dpp}, \text{cp3}\} = (z = y = w = 0)$. But both of these blowings-up produce a new cp3 singularity.

Notation 1.16. We write {dpp, cp3} to denote the set of points with singularities of type dpp or cp3. More generally, for any finite list of singularities $\mathcal{T}$, $\{\mathcal{T}\}$ denotes the set of points with singularities in $\mathcal{T}$.

Remark 1.17. Our proof of Theorem 1.11 also gives normal forms or local models for the singularities of the total transform of X, corresponding to $\mathcal{S}$. (Equivalently, it gives local models for the "transform" of a divisor D, where the latter is defined as the support of the birational transform plus the exceptional divisor). For example, in the case $n = 3$, the following table gives the possible (reduced) exceptional divisors.

Singularity	Exceptional divisor
$x = 0$	$(y = 0)$
	$(y = 0) + (z = 0)$
	$(x + z^2 = 0)$
$xy = 0$	$(z = 0)$
$xyz = 0$	
$z^2 + xy^2 = 0$	$(x = 0)$

The third line in the table gives the possibility of a non-transverse exceptional divisor at a smooth point of the variety. This cannot be eliminated because it occurs in a neighbourhood of the origin in the last line (pp).

Following is a theorem on resolution except for codimension one singularities which can be eliminated by normalizing. It can be considered also as a "higher-dimensional version" of Theorems 1.11 and 1.14 in the case $n = 3$.

Theorem 1.18. *Let X denote a reduced variety (in any dimension) and let X^{ncp} denote the open subset of X consisting of smooth points, double normal crossings points ($xy = 0$) and pinch points ($z^2 + xy^2 = 0$). Then there exists a morphism $\sigma : X' \to X$ which is a finite composite of admissible blowings-up, such that*

(1) $X' = (X')^{\text{ncp}}$;
(2) *σ is an isomorphism over X^{ncp};*
(3) $\operatorname{Sing} X'$ *maps birationally onto the closure of* $\operatorname{Sing} X^{\text{ncp}}$.

Again the theorem can be realized functorially, and it is easy to write local models for the singularities of the total transform. Kollár proves the assertion of Theorem 1.18 with a proper birational morphism σ [9, Theorem 16]. Note that the term "pinch point" in Theorem 1.18 means a hypersurface singularity of the form $z^2 + xy^2 = 0$ in any number of variables $x, y, z, \ldots$.

Note that, in Theorem 1.11 in the case $n = 3$, pinch points are isolated. Theorem 1.18 has an important new feature (which also occurs in the case $n = 4$ of Theorem 1.11)—any new singularities that occur as limits of pinch points can be eliminated. Of course, Theorem 1.11 for $n = 4$ suggests an analogue of Theorem 1.18 with normal crossings singularities of order up to 3; we have not yet been able to prove this.

One can ask whether there are interesting relationships between the questions above and other classification problems in singularity theory. For example, the singularities in $\mathcal{S}$ when $n = 3$ are the same as those which occur for the images of stable differentiable mappings $\varphi : M^2 \to N^3$ (between manifolds of the dimensions indicated). This question reflects a point of view towards resolution of singularities suggested to us many years ago by René Thom.

1.2. The desingularization invariant as a computational tool

Our proofs of the results in this article are based on using the desingularization invariant of [3] as a tool for computing and simplifying local normal forms. As an illustration, we will outline proofs of Theorems 1.11 and 1.14 in the case $n = 3$ in this subsection.

In the Appendix, we will try to provide a working knowledge of the desingularization algorithm and the invariant as they are used here, for a reader not necessarily familiar with a complete proof of resolution of singularities. Suppose that $X \hookrightarrow Z$ is an embedded hypersurface, where Z is smooth. Let $\mathrm{inv} = \mathrm{inv}_X$ denote the desingularization invariant for X. We recall that inv is defined iteratively on the strict transform X_{j+1} of $X = X_0$ for any finite sequence of inv-*admissible* blowings-up

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_{j+1}} Z_{j+1}. \tag{1.6}$$

(A blowing-up is inv-*admissible* if it is admissible and inv is locally constant on its centre.) In particular, $\mathrm{inv}(a)$, where $a \in X_{j+1}$ depends not only on X_{j+1} but also on the *history* of blowings-up (1.6).

Let $a \in X_j$. Then $\mathrm{inv}(a)$ has the form

$$\mathrm{inv}(a) = (\nu_1(a), s_1(a), \ldots, \nu_t(a), s_t(a), \nu_{t+1}(a)), \tag{1.7}$$

where $\nu_k(a)$ is a positive rational number ("residual multiplicity") if $k \le t$, each $s_k(a)$ is a nonnegative integer (which counts certain components of the exceptional divisor), and $\nu_{t+1}(a)$ is either 0 or ∞. The successive pairs $(\nu_k(a), s_k(a))$ are defined inductively over *maximal contact* subvarieties of increasing codimension.

Let inv_k denote the truncation of inv after the k'th pair in (1.7) ($\mathrm{inv}_k := \mathrm{inv}$ if $k > t$). Then inv_k can be defined iteratively over a sequence of inv_k-admissible blowings-up (1.6). For each k, inv_k is upper semicontinuous, and also *infinitesimally upper-semicontinuous* in the sense that inv_k can only decrease after blowing up with inv_k-admissible centre.

It is easy to see that, in *year zero* (i.e., if $j = 0$), $\mathrm{inv}(a) = (2, 0, 1, 0, \infty)$ if and only if X has a double normal crossings singularity $z^2 + y^2 = 0$ at a. Some other year-zero hypersurface examples follow:

$x = 0$	smooth	$\mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}1) := (1, 0, \infty)$
$x_1 x_2 \cdots x_k = 0$	nc$k$	$\mathrm{inv}(0) = \mathrm{inv}(\mathrm{nc}k) := (k, 0, 1, 0, \ldots, 1, 0, \infty)$
$z^2 + xy^2 = 0$	pp	$\mathrm{inv}(0) = \mathrm{inv}(\mathrm{pp}) := (2, 0, 3/2, 0, 1, 0, \infty)$

(where, for nc$k$, there are $k - 1$ pairs $(1, 0)$). For $k \ge 3$, nc$k$ is not characterized by the value of inv; for example, the singularity $x_1^k + x_2^k + \cdots + x_k^k = 0$ also has $\mathrm{inv}(0) = (k, 0, 1, 0, \ldots, 1, 0, \infty)$ with $k - 1$ pairs $(1, 0)$.

Consider the case $\dim Z = 3$ and now suppose that $a \in X_j$, *for an arbitrary year* j. Then $\mathrm{inv}(a) = \mathrm{inv}(\mathrm{nc}2) = (2, 0, 1, 0, \infty)$ if and only if X_j can be defined near a by an equation

$$z^2 + x^\alpha y^2 = 0, \tag{1.8}$$

where α is a positive integer and $(x = 0)$ is an *exceptional divisor*. If $\alpha \geq 2$ and we blow up with centre $(z = x = 0)$, then the strict transform X_{j+1} of X_j is given locally by

$$z^2 + x^{\alpha-2}y^2 = 0.$$

(The preceding blowing-up is inv_1-admissible.) After finitely many such blowings-up, we get either $\alpha = 0$ or $\alpha = 1$; i.e., we get either

$$z^2 + y^2 = 0 \quad \mathrm{nc2}$$
$$\text{or} \quad z^2 + xy^2 = 0 \quad \mathrm{pp}.$$

A crucial point is that the blowings-up we have described locally above are actually globally-defined inv_1-admissible blowings-up; the ideal (x^α) is the *monomial part* of a *coefficient marked ideal* defined on a *maximal contact hypersurface* ($(z = 0)$ at the point a above), at any point of $(\mathrm{inv}_1 = (2,0)) := \{p : \mathrm{inv}_1(p) = (2,0)\}$. The blowings-up above, to reduce α to 0 or 1, constitute "combinatorial or monomial resolution of singularities" of the monomial marked ideal. We will call this simplification of (1.8) by resolution of singularities of the monomial marked ideal a *cleaning* or an application of the *cleaning lemma* (see Section 2). Note the blowings-up involved in applying the cleaning lemma above are inv_1- but not inv-admissible. See the Appendix for details of the ideas above.

Proof of Theorems 1.11 and 1.14, case $n = 3$. We will first show that, given a reduced variety X of dimension 2, there exists a morphism $\sigma : X' \to X$ which is a finite composite of admissible blowings-up, as required in Theorem 1.11.

Singular points of type nc3 are isolated, and each have only nc2 singularities in some neighbourhood. Therefore, the points in the complement of {nc3} with $\mathrm{inv} > \mathrm{inv}(\mathrm{nc2}) = (2,0,1,0,\infty)$ form a closed set disjoint from {nc3}. So we can blow up with closed inv-admissible centres with $\mathrm{inv} > \mathrm{inv}(\mathrm{nc2}) = (2,0,1,0,\infty)$, until the maximum value of the invariant over the complement of {nc3} is $\leq \mathrm{inv}(\mathrm{nc2})$.

We now apply the cleaning lemma to blow up until every point of the stratum $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc2}))$ is either nc2 or pp. (There are now no other singular points in a neighbourhood of this stratum.) The centres of blowing up involved in using the cleaning lemma are disjoint from a neighbourhood of {nc3}.

We can now use the desingularization algorithm to resolve any singularities (i.e., to reduce to $\mathrm{ord}.X = 1$) in the complement of $(\mathrm{inv} = \mathrm{inv}(\mathrm{nc2}))$ and the original {nc3}, by admissible blowings-up. This suffices to prove Theorem 1.11 in the case $n = 3$.

To prove Theorem 1.14 in the case $n = 3$ (in particular, to show that $\mathcal{S} = \mathcal{S}'$ in this case), note that, if $n = 3$, then singular points of type pp (as well as nc3) are isolated, and each have only nc2 singularities in some neighbourhood. So we can simply repeat the proof above, changing "{nc3}" to "{nc3, pp}" wherever the former occurs. □

Remark 1.19. The argument above provides the normal forms listed in Remark 1.17 for the total transform of X at every singular point (i.e., nc3, pp, or nc2) of the total transform. If, in addition to Theorems 1.11 and 1.14 in the case $n = 3$, we want to get the normal forms for the total transform listed in Remark 1.17 at all points, we may need to make additional blowings-up of smooth points of the latter. The argument is similar to that above, but we defer it to Section 4 in order to take advantage of notions introduced in Sections 2 and 3.

1.3. Comparison with the desingularization algorithm

The following example illustrates the way the cleaning lemma is used above, in comparison with the "monomial case" of the desingularization algorithm (see [5, p. 628]).

Example 1.20. Let $X \subset \mathbb{A}^3$ denote the hypersurface $(z^2 + x^3y^2 = 0)$. We first consider the desingularization algorithm applied to X. (A reader unfamiliar with the computations below can refer to the Appendix.)

Year zero. $\mathrm{inv}(0) = (2, 0, 5/2, 0, 1, 0, \infty)$. The centre C_0 of the first blowing-up σ_1 is $\{0\}$.

Year one. The total transform of $X_0 = X$ in the *x-coordinate chart* (the chart given by substituting (x, xy, xz) in place of (x, y, z)) is

$$x^2(z^2 + x^3y^2) = 0.$$

(For simplicity of notation, we are again writing (x, y, z) for the coordinates after blowing up.) The strict transform $z^2 + x^3y^2 = 0$ has the same singularity at the origin as in year zero. Now, however, $\mathrm{inv}(0) = (2, 0, 1, 1, 1, 0, \infty)$. The centre C_1 of the blowing-up σ_2 is again $\{0\}$.

Year two. The total transform in the x-chart is given by

$$x^4(z^2 + x^3y^2) = 0.$$

The strict transform again has the same singularity at $\{0\}$! Now, however, $\mathrm{inv}(0) = (2, 0, 1, 0, \infty)$ and the next centre of blowing-up C_2 is the x-axis $(z = y = 0)$. Note that C_2 is nc2 when $x \neq 0$.

Year three. The total transform in the y-chart (given by the substitution (x, y, yz)) is

$$x^4y^2(z^2 + x^3) = 0.$$

We are now in the monomial case of the desingularization algorithm; $\mathrm{inv}(0) = (2, 0, 0)$. The next blowing-up (centre $(z = x = 0)$) resolves the singularities of the strict transform, but additional blowings-up are needed to make the total transform simple normal crossings.

By comparison, if we follow the algorithm involved in the proof of Theorem 1.11 in Appendix A.2, we would take a different route starting in year two, when $\mathrm{inv}(0) = (2, 0, 1, 0, \infty)$: We would apply the cleaning lemma, which tells us to blow up with centre $C_2' := (z = x = 0)$ (as in year three above). We then get a pinch point

$$x^6(z^2 + xy^2) = 0$$

(without blowing up nc2 points). The cleaning lemma is essentially the monomial case of resolution of singularities, but (with reference to the algorithm of [3,5]) applied at an intermediate step, rather than after reduction to the monomial case.

2. Cleaning lemma

As remarked in Section 1.2, local normal forms of singularities will be simplified or *cleaned* using resolution of singularities of certain monomial marked ideals (see Appendices A.4 and A.6). The hypothesis of the *cleaning lemma* 2.1 reflects the structure of the marked ideals which occur in the recursive definition of the centres of blowing up involved in the desingularization algorithm (see Appendix).

Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ denote a marked ideal (see Appendix A.4) and let $\mathcal{I} = \mathcal{M}(\underline{\mathcal{I}}) \cdot \mathcal{R}(\underline{\mathcal{I}})$ denote the factorization of $\mathcal{I}$ into monomial and residual parts (see Appendix A.6). The ideal

$\mathcal{M}(\mathcal{I})$ is locally generated by a monomial in components of the normal crossings divisor E, whose exponents divided by d are invariants of the equivalence class of $\underline{\mathcal{I}}$ (see Definition A.10). Set $\underline{\mathcal{M}}(\underline{\mathcal{I}}) = (\mathcal{M}(\underline{\mathcal{I}}), d)$. Then $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}) \subset \operatorname{cosupp} \underline{\mathcal{I}}$ and any admissible sequence of blowings-up of $\underline{\mathcal{M}}(\underline{\mathcal{I}})$ is admissible for $\underline{\mathcal{I}}$.

In general, however, it is not true that the transforms $\underline{\mathcal{I}}'$ and $\underline{\mathcal{M}}(\underline{\mathcal{I}})'$ by an admissible blowing-up of $\underline{\mathcal{M}}(\underline{\mathcal{I}})$ satisfy $\underline{\mathcal{M}}(\underline{\mathcal{I}}') = \underline{\mathcal{M}}(\underline{\mathcal{I}})'$ (since the exceptional divisor might factor from the pull-back of $\mathcal{R}(\underline{\mathcal{I}})$).

Lemma 2.1. *Suppose that $\underline{\mathcal{R}}(\underline{\mathcal{I}}) := (Z, N, E, \mathcal{R}(\underline{\mathcal{I}}), \operatorname{ord} \mathcal{R}(\underline{\mathcal{I}}))$ admits a maximal contact hypersurface P at some point of its cosupport (in particular, E is transverse to P; see Definitions 3.3 and A.11). Then, after transformation of $\underline{\mathcal{I}}$ by an admissible sequence of blowings-up of $\underline{\mathcal{M}}(\underline{\mathcal{I}})$, we can assume that* $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}})$ *is disjoint from the strict transform of P.*

Proof. Any non-empty intersection D of components of E is transverse to P. Therefore, if we blow up with centre $C = D \cap N$, then, on the strict transform of P, the exceptional divisor does not factor from the pull-back of $\mathcal{R}(\underline{\mathcal{I}})$ and the transforms $\underline{\mathcal{I}}'$ and $\underline{\mathcal{M}}(\underline{\mathcal{I}})'$ satisfy $\underline{\mathcal{M}}(\underline{\mathcal{I}}') = \underline{\mathcal{M}}(\underline{\mathcal{I}})'$. The result follows from desingularization of $\underline{\mathcal{M}}(\underline{\mathcal{I}})$. $\square$

Examples 2.2. Let $Z = N = \mathbb{A}^3$, $d = 2$. (1) Take $\mathcal{I} = (xy(z^2 + xy^2))$, $E = (x = 0) + (y = 0)$. Then $\underline{\mathcal{M}}(\underline{\mathcal{I}}) = ((xy), 2)$ and $\underline{\mathcal{R}}(\underline{\mathcal{I}}) = ((z^2 + xy^2), 2)$; $\underline{\mathcal{R}}(\underline{\mathcal{I}})$ has $(z = 0)$ as a maximal contact hypersurface. The blowing-up σ of $(x = y = 0)$ is admissible for $\underline{\mathcal{M}}(\underline{\mathcal{I}})$, and the exceptional divisor of σ does not factor from the pull-back of $\mathcal{R}(\underline{\mathcal{I}})$.

(2) Take $\mathcal{I} = (xz(z^2 + xy^2))$, $E = (x = 0) + (z = 0)$. Then $\underline{\mathcal{M}}(\underline{\mathcal{I}}) = ((xz), 2)$ and $\underline{\mathcal{R}}(\underline{\mathcal{I}}) = ((z^2 + xy^2), 2)$. The blowing-up of $(x = z = 0)$ is admissible for $\underline{\mathcal{M}}(\underline{\mathcal{I}})$, but the exceptional divisor factors from the pull-back of $\mathcal{R}(\underline{\mathcal{I}})$.

2.1. Cleaning

Consider the desingularization algorithm for an embedded hypersurface $X \hookrightarrow Z$, and let $a \in X_{j_0}$, for some $j = j_0$ (notation of A.2). The invariant $\operatorname{inv}(a)$ has the form

$$\operatorname{inv}(a) = (\nu_1(a), s_1(a), \dots, \nu_q(a), s_q(a), \nu_{q+1}(a))$$

(see (A.2)). Suppose that $p < q$. According to the desingularization algorithm, $(\operatorname{inv}_p \geq \operatorname{inv}_p(a))$ is (locally) the support of a marked ideal $\underline{\mathcal{I}} = \underline{\mathcal{I}}^p = (Z_{j_0}, N, E, \mathcal{I}, d)$ on a maximal contact subvariety N of codimension p in Z_{j_0}. Consider $\underline{\mathcal{M}}(\underline{\mathcal{I}}^p)$ and $\underline{\mathcal{R}}(\underline{\mathcal{I}}^p)$ as above. Then there is an inv_p-admissible sequence of blowings-up of Z_{j_0},

$$Z = Z_{j_0} \overset{\sigma_{j_0+1}}{\longleftarrow} Z_{j_0+1} \longleftarrow \cdots \overset{\sigma_{j_1}}{\longleftarrow} Z_{j_1}, \tag{2.1}$$

such that $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}^p)_{j_1} = \emptyset$, where $\underline{\mathcal{M}}(\underline{\mathcal{I}}^p)_{j_1}$ denotes the transform of $\underline{\mathcal{M}}(\underline{\mathcal{I}}^p)$ in year j_1 (by resolution of singularities of a monomial marked ideal [5, Section 5, Step II.A]). Such a blowing-up sequence will be called a *cleaning*. The centres of blowing up are invariantly defined closed subspaces of $(\operatorname{inv}_p \geq \operatorname{inv}_p(a))$ (cf. Definition A.10 and [5, *loc. cit.*]).

As remarked above, cleaning does not necessarily mean that $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}_{j_1}) = \emptyset$, though $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}_{j_1})$ will be disjoint from points where Lemma 2.1 applies.

In general, we will use cleaning to transform $\operatorname{cosupp} \underline{\mathcal{M}}(\underline{\mathcal{I}}^p)$ to $\emptyset$, successively for $p = q-1, q-2, \dots, 1$.

Example 2.3. Suppose that $s_{p+1}(a) = 0$. Then E consists of only "new" exceptional divisors for $\mathrm{inv}_{p+1/2}$ at a (see Appendices A.9 and A.10), so that $\underline{\mathcal{R}}(\underline{\mathcal{I}}^p)$ has a maximal contact hypersurface in N transverse to E.

Remark 2.4. The truncated invariant inv_p is well-defined over (2.1), and is both semicontinuous and infinitesimally semicontinuous (i.e., it can only decrease after each blowing-up). But the cleaning sequence (2.1) is not, in general, inv-admissible. The residual multiplicity ν_{p+1} (see Definition A.19) can be defined as usual over (2.1), so that $\mathrm{inv}_{p+1/2}$ is well-defined and semicontinuous (though not necessarily infinitesimally semicontinuous).

Moreover, we can extend $\mathrm{inv}_{p+1/2}$ to a modified invariant on Z_{j_1} by considering j_1 to be "year zero" for $\mathrm{inv}_{p+1/2}$, and can then follow the usual desingularization algorithm and definition of inv starting in this year (i.e., j_1 will be the year of birth for the value $\mathrm{inv}_{p+1/2}(a)$ of $\mathrm{inv}_{p+1/2}$ over a point $a \in Z_{j_1}$). In other words, all components of the exceptional divisor at a, except those counted by $s_1(a), \ldots, s_p(a)$ are counted by $s_{p+1}(a)$ (they are considered the "old" exceptional divisors for $\mathrm{inv}_{p+1/2}$ at a); then $\mathrm{inv}_{p+1} := (\mathrm{inv}_{p+1/2}, s_{p+1})$ extends to a semicontinuous invariant inv on Z_{j_1} by the construction in Appendix A.10, and we can afterwards follow the desingularization algorithm.

3. Simple normal crossings

In this section, we prove the following result (see Theorem 1.5).

Theorem 3.1. *Let X denote a reduced variety and let X^{snc} denote the locus of points of X that have only simple normal crossings singularities. Then there is a morphism $\sigma : X' \to X$ which is a composite of finitely many admissible blowings-up, such that*

(1) $X' = (X')^{\mathrm{snc}}$;
(2) σ *is an isomorphism over* X^{snc};
(3) σ *maps* $\operatorname{Sing} X'$ *birationally onto the closure of* $\operatorname{Sing} X^{\mathrm{snc}}$.

Remark 3.2. Let $X^{\mathrm{snc}\le r}$ denote the locus of points of X having only simple normal crossings singularities of orders $\le r$. There is a simple variant of Theorem 3.1 where snc is replaced by snc $\le r$. For example, we can deduce this from Theorem 3.1 by blowing up singularities of order $>r$.

Definitions 3.3. Let $X \hookrightarrow Z$, where Z is smooth of dimension n. Let E denote a finite collection of smooth hypersurfaces in Z having only simple normal crossings. We say that (X, E) is *simple normal crossings* (*snc*) at a point a if there is a regular system of parameters $(x_1, \ldots, x_n)$ at a in which each irreducible component of X is a coordinate subspace and each member of E is a coordinate hyperplane. There is an analogous notion of *normal crossings* (*nc*) at a. We say that X and E are *transverse* at a if they are nc and each component of E is transverse to X at a. We write $(X, E)^{\mathrm{snc}}$ to denote the simple normal crossings locus of (X, E).

Consider a sequence of blowings-up of Z,

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_t} Z_t. \tag{3.1}$$

Write $X_0 := X$ and $E_0 := E$, where we order the members of E_0 in an arbitrary way. Let X_{j+1} denote the strict transform of X_j, $j = 0, 1, \ldots$. We again say that the sequence (3.1) is *admissible* if, for each successive $j = 0, 1, \ldots$, the blowing-up σ_{j+1} has smooth centre $C_j \subset X_j$

such that (C_j, E_j) is snc, where, for all $j \geq 1$, E_j denotes the (ordered) collection of strict transforms of the members of E_{j-1}, together with $\sigma_j^{-1}(C_{j-1})$ added as the last element.

Theorem 3.4. *Let $X \hookrightarrow Z$ denote an embedded reduced hypersurface, where Z is smooth, and let E denote a finite collection of smooth hypersurfaces in Z having only simple normal crossings. Then there is a finite admissible sequence of blowings-up (3.1) such that*

(1) $(X_t, E_t) = (X_t, E_t)^{\mathrm{snc}}$;
(2) *the morphism σ given by the composite of the σ_j is an isomorphism over $(X, E)^{\mathrm{snc}}$;*
(3) *σ maps* $\operatorname{Sing} X_t$ *birationally onto the closure of* $\operatorname{Sing} X^{\mathrm{snc}}$.

Moreover, the theorem is functorial with respect to étale or smooth morphisms preserving the number of irreducible components of X and E at every point (cf. Remark 1.6).

The sequence of blowings-up (3.1) will be independent of an ordering of E_0. If $X \hookrightarrow Z$ is an embedded variety, then the strong desingularization algorithm of [3,5] (cf. Remarks 1.9(2)) proceeds by first blowing up non-hypersurface points and points where (the transform of) E intersects a local minimal embedding variety for (that of) X, to reduce to the case that $X \hookrightarrow Z$ is an embedded hypersurface. So we can reduce Theorem 3.1 to Theorem 3.4. On the other hand, we can reduce Theorem 3.4 to the case $E = \emptyset$, simply by replacing X by $X \cup E$.

Let $X \subset Z$ be as in Theorem 3.4 (with $E = \emptyset$). Consider the desingularization invariant $\mathrm{inv} = \mathrm{inv}_X$ and the sequence of inv-admissible blowings-up (3.1) given by the desingularization algorithm of [3,5]. Let $a \in X_j$. We will write a_i to denote the image of a in X_i, for any $i \leq j$.

Recall that, if X is nc$q$ at a point a, then $\mathrm{inv}(a) = \iota_q$, where

$$\iota_q := (q, 0, 1, 0, \dots, 1, 0, \infty)$$

with $q - 1$ pairs $(1, 0)$.

Lemma 3.5. *Let $X \subset Z$ denote an embedded hypersurface, where Z is smooth. Consider the desingularization invariant* $\mathrm{inv} = \mathrm{inv}_X$ *and the sequence of* inv*-admissible blowings-up (3.1) given by the desingularization algorithm, as above.*

(1) *Let $a \in X = X_0$. Then* $\mathrm{inv}(a) = \iota_q$ *and X has q local analytic (respectively, irreducible) components at a if and only if X is* nc$q$ *(respectively,* snc$q$*) at a.*
(2) *Let $a \in X_j$, for given j. If* $\mathrm{inv}(a) = \iota_q$ *and X_j has q local analytic (respectively, irreducible) components at a, then we can choose local analytic, i.e., étale (respectively, regular) coordinates at a,*

$$(x, u) = (x_1, \dots, x_q, u_1, \dots, u_{n-q}),$$

in which the ideal of X_j is generated by a product $f = f_1 \cdots f_q$ such that

$$\begin{aligned}
f_1 &= x_1, \\
f_2 &= x_1 + u^{\alpha^1} x_2, \\
f_3 &= x_1 + u^{\alpha^1}\left(x_2 \cdot \xi_{32} + u^{\alpha^2} x_3\right), \\
f_4 &= x_1 + u^{\alpha^1}\left(x_2 \cdot \xi_{42} + u^{\alpha^2}\left(x_3 \cdot \xi_{43} + u^{\alpha^3} x_3\right)\right), \\
&\cdots,
\end{aligned}$$

where each $u^{\alpha^k} = u_1^{\alpha_1^k} \cdots u_{n-q}^{\alpha_{n-q}^k}$, with $(u_l = 0) \in E_j$ if $\alpha_l^k > 0$.

Proof. See Appendix A.2 for the "if" direction of (1). The "only if" direction of (1) is a special case of (2). We will prove (2).

By the Weierstrass preparation theorem, the ideal of X_j at a has a generator of the form

$$f(y,z) = z^q + \sum_{i=2}^{q} b_i(y) z^{q-i} \tag{3.2}$$

in local étale coordinates $(y,z) = (y_1,\ldots,y_{n-1},z)$ at $a = 0$, where $\mathrm{ord}_a b_i \geq i$, for each i, and $(z = 0)$ is a maximal contact hypersurface. Since X_j has q components at a, we can factor (3.2) as

$$z^q + \sum_{i=2}^{q} b_i(y) z^{q-i} = \prod_{j=1}^{q} \left(z - a_j(y)\right),$$

where $\sum a_j = 0$.

Then the coefficient ideal corresponding to the maximal contact hypersurface $(z = 0)$ is equivalent to $((a_j), 1)$. (See Example A.13.)

Since $\mathrm{inv}(a) = \iota_q$, the ideal (a_j) has order 1 at a, after division by a monomial u^{α^1} in the exceptional divisor. After a change of coordinates, we can assume that $a_1 = u^{\alpha^1} y_1$, where $(z = y_1 = 0)$ is a second maximal contact subspace (i.e., maximal contact subspace of codimension 2), and that each a_j, $j \geq 2$, is of the form

$$a_j = u^{\alpha^1}\left(y_1 \cdot \eta_{j1} + c_j\right).$$

Again, the ideal (c_j) on $(z = y_1 = 0)$ has order 1 at a after division by an exceptional monomial u^{α^2}, and so on. So we can write f in the form $f = f_1 \cdots f_q$, where the first $q-1$ factors are of the form

$$\begin{aligned}
f_1 &= z + u^{\alpha^1} y_1,\\
f_2 &= z + u^{\alpha^1}\left(y_1 \cdot \eta_{21} + u^{\alpha^2} y_2\right),\\
f_3 &= z + u^{\alpha^1}\left(y_1 \cdot \eta_{31} + u^{\alpha^2}\left(y_2 \eta_{32} + u^{\alpha^3} y_3\right)\right),\\
&\cdots,
\end{aligned}$$

(recall that $\sum a_j = 0$) and the result follows, by a further coordinate change. □

Proof of Theorem 3.4. We can assume that $E = \emptyset$. Given $p \in \mathbb{N}$, let $\Sigma_p(X)$ denote the locus of points lying in at least p irreducible components of X. Let q denote the largest value of $\mathrm{ord}_a X$ at snc points of X. We blow up with inv-admissible centres following the desingularization algorithm as long as the maximum value of inv is $>\iota_q$, stopping when the maximum value $= \iota_q$, say in year j_0. Set $I_q(X, j_0) := (\mathrm{inv} = \iota_q) \subset X_{j_0}$. Then $I_q(X, j_0) \neq \emptyset$ since it includes the snc points of X (includes in the sense that all previous blowings-up are isomorphisms over such points of X).

Using the desingularization algorithm, we can blow up any component of $I_q(X, j_0)$ which is not generically snc (to decrease inv). Therefore, we can assume that every component of $I_q(X, j_0)$ is generically snc.

Let $a \in I_q(X, j_0)$. Choose coordinates at a satisfying Lemma 3.5. The locus

$$\left(x_1 = \cdots = x_{q-1} = 0\right) \bigcap \left(\mathrm{ord}\, u^{\alpha^{q-1}} \geq 1\right)$$

is the cosupport of a monomial marked ideal of order 1 on a maximal contact subvariety $N = (x_1 = \cdots = x_{q-1} = 0)$ of codimension $q-1$. According to the Cleaning Lemma 2.1, we can reduce α^{q-1} to 0 by finitely many globally-defined inv_{q-1}-admissible blowings-up.

We can repeat the preceding argument using the monomial marked ideal $\left((u^{\alpha^{q-2}}), 1\right)$ on the subspace $x_q = x_1 = \cdots x_{q-2} = 0$ to reduce α^{q-2} to 0, etc., eventually to reduce all α^k to 0.

Remark 3.6. For simplicity, we have begun in a way that ignores the problem of functoriality. In fact, if $n := \dim Z$, then, for each $p = n, n-1, \ldots, q$, we should follow the desingularization algorithm (starting as if in "year zero") until $\mathrm{inv} \le \iota_p$, even if $I_p(X, j) = \emptyset$ $(p > q)$, blow up any component of $I_p(X, j)$ which is not generically snc, and then perform cleaning as above. Globally, $(x_1 = \cdots = x_k = 0) \cap (\mathrm{ord}\, u^{\alpha^k} \ge 1)$, $k \le p-1$, is given by the cosupport of an invariantly defined monomial marked ideal $\underline{\mathcal{M}}(\underline{\mathcal{I}}^k)$ on the locus $(\mathrm{inv}_k \ge (\iota_p)_k)$ for the truncated invariant, and the cleaning procedure of Section 2.1 applies.

Suppose that we are now in year j_1. The result of our cleaning above is that (X_{j_1}, E_{j_1}) is snc at all points of $\Sigma_q(X_{j_1})$, and therefore in a neighbourhood of $\Sigma_q(X_{j_1})$.

We now apply the desingularization algorithm to (X_{j_1}, E_{j_1}) restricted to the complement of $\Sigma_q(X_{j_1})$ (where we regard j_1 as "year zero") to blow up with smooth centres over the complement of $\Sigma_q(X_{j_1})$ until the maximum value of inv is $\le \iota_{q-1}$.

However, the centres of the blowings-up involved will not necessarily be closed in X_{j_1} and its strict transforms (since, in the process, we will introduce nonzero s-terms in inv).

For example, the total transform of X at a point of $\Sigma_q(X_{j_1})$ is of the form $(u^\alpha x_1 \cdots x_q = 0)$, where $u^\alpha = u_1^{\alpha_1} \cdots u_{n-q}^{\alpha_{n-q}}$ is a monomial in exceptional divisors. The centre of the blowing up of X_{j_1} will be given near such a point of $\Sigma_q(X_{j_1})$ by

$$\bigcup_{i=1}^{q} \left(u_{l_1} = \cdots = u_{l_p} = x_1 = \cdots = \widehat{x_i} = \cdots = x_q = 0, x_i \neq 0\right), \tag{3.3}$$

for some $l_1, \ldots, l_p$ (where $\widehat{x_i}$ means that x_i is deleted from the expression).

We can simply modify the algorithm by first blowing up with centre given by

$$\left(u_{l_1} = \cdots = u_{l_p} = x_1 = \cdots = x_q = 0\right)$$

(the intersection of the closures of the components in (3.3)) to separate the components, and by then blowing up the union of these (closed) components. The two blowings-up are admissible and include no (points lying over) snc points of X.

In general, given a union of subvarieties

$$\left(u_{l_{i1}} = \cdots = u_{l_{ip_i}} = x_1 = \cdots = \widehat{x_i} = \cdots = x_q = 0\right),$$

for certain $i = 1, \ldots, q$ (where each $p_i > 0$), we can blow up finitely many times with centres of increasing dimension in $\Sigma_q(X_j)$, $j = j_1, \ldots$, to separate these varieties (before blowing them up, for example).

We thus modify each of the blowings-up of (X_{j_1}, E_{j_1}) above; we get a finite sequence of blowings-up with closed admissible centres over the complement of the snc locus of X, after which (X_j, E_j) is snc on $T_q(X_j)$, where $T_q(X_j)$ denotes the inverse image of $\Sigma_q(X_{j_1})$ in X_j, and the maximum value of inv on the complement of $T_q(X_j)$ is $\le \iota_{q-1}$, for some $j = j_1' \ge j_1$.

We then blow up any component of $I_{q-1}(X, j_1')$ which is not generically snc, and apply the cleaning lemma as above (over the complement of $T_q(X_{j_1'})$), to blow up further until we have

(X_{j_2}, E_{j_2}) (for some year j_2) snc at every point of $\Sigma_{q-1}(X_{j_2})$. (The centres of the blowings-up involved will be separated from the successive $T_q(X_j)$ because (X_j, E_j) is already snc in a neighbourhood of the latter.)

In general, suppose that, for some year j_k, (X_{j_k}, E_{j_k}) is snc on $\Sigma_{q-k+1}(X_{j_k})$. We apply the desingularization algorithm over the complement of $\Sigma_{q-k+1}(X_{j_k})$ as above, until the maximum value of inv is $\le \iota_{q-k}$. The closure of each centre of blowing up can be separated into a disjoint union of smooth subvarieties as above. Afterwards, we again blow up the components of $(\mathrm{inv} = \iota_{q-k})$ that are not generically snc, and then apply the cleaning lemma. So we get a finite sequence of blowings-up with smooth admissible centres, after which $(X_{j_{k+1}}, E_{j_{k+1}})$ has snc at every point of $\Sigma_{q-k}(X_{j_{k+1}})$.

Eventually, we get (X_j, E_j) snc on $\Sigma_1(X_j) = X_j$. We thus get the theorem with conditions (1) and (2), and condition (3) is clear from the choices of blowings-up (see also Theorem 1.5 ff.). $\square$

4. Pinch points

Our main goal in this section is to prove Theorem 1.18. In comparison with Theorem 1.11 in the case $n = 3$, the problem here is to eliminate new singularities that intervene as limits of pinch points (Section 4.1). By contrast, in Section 4.3, we will show that new singularities which occur as limits of degenerate pinch points cannot necessarily be eliminated.

Before turning to Theorem 1.18, we indicate how to get the normal forms listed in Remark 1.17 for the total transform in Theorems 1.11 and 1.14 in the case $n = 3$.

4.1. Minimal singularities in 3 variables

Let $X \subset Z$ denote an embedded hypersurface where Z is smooth and of pure dimension 3. According to the proofs of Theorems 1.11 and 1.14 in the case $n = 3$ (see Section 1.1), we have a sequence of blowings-up

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_j} Z_j$$

after which every point of X_j has only nc3, nc2 and pp singularities. Moreover, we have the normal forms listed in Remark 1.17 at every singular point of X_j (see Remark 1.19).

Remark 4.1 (*How to get the normal forms of* Remark 1.17 *at every point*)**.** Write $W := Z_j$, $Y := X_j$, and let E denote (the support of) the exceptional divisor E_j. Set $\Sigma := \mathrm{Sing}\, Y$. We apply the desingularization algorithm to (Y, E) in W, over the open subset $V = W \setminus \Sigma$. This is now "year zero" for the desingularization algorithm, so that inv will have a meaning different than before. For example, consider a pp where $Y = (z^2 + xy^2 = 0)$ and $(x = 0)$ is the exceptional divisor; then at a nearby point $z = x = 0$, $y \neq 0$, we have $\mathrm{inv} = (1, 1, 2, 0, \infty)$. There is a neighbourhood of Σ in which $Y \cap V$ has only smooth points, but (Y, E) has the following possible forms, characterized by the value of the invariant shown (in year zero).

$$\begin{array}{lll} Y : y = 0 & E : \emptyset & \mathrm{inv} = (1, 0, \infty) \\ Y : y = 0 & E : x = 0 & \mathrm{inv} = (1, 1, 1, 0, \infty) \\ Y : y = 0 & E : y + x^2 = 0 & \mathrm{inv} = (1, 1, 2, 0, \infty) \end{array}$$

We blow up with centre prescribed by the desingularization algorithm for (Y, E) restricted to V, until the maximum value of inv is $(1, 1, 2, 0, \infty)$. The centres involved are separated from Σ and its inverse images. We can also blow up any closed component of $(\mathrm{inv} = (1, 1, 2, 0, \infty))$.

Now, at a point where $\mathrm{inv} = (1, 1, 2, 0, \infty)$, the strict transform of $Y \cup E$ is given by an equation

$$y(y + u^{\alpha} x^2) = 0,$$

where $(u = 0)$ is the exceptional divisor. We can blow up using the cleaning lemma to reduce to $\alpha = 0$. (The centres of the blowings-up involved in cleaning are separated from the inverse images of Σ.) We thus reduce to the case that the (strict transforms of) Y, E are given by equations of the form $y = 0$, $y + x^2 = 0$ (respectively) at every point of the transform of $(\mathrm{inv} = (1, 1, 2, 0, \infty))$.

Let Σ' denote the union of the latter and the inverse image of Σ. We repeat the argument above to blow up (over the complement of Σ') until the maximum value of inv is $(1, 1, 1, 0, \infty)$, and then use the cleaning lemma to reduce locally to $Y = (y = 0)$, $E = (x = 0)$.

A further sequence of blowings-up over the complement of the points already considered, until the maximum value of inv becomes $(1, 0, \infty)$, completes the argument.

4.2. *Pinch points in higher dimension*

Consider a hypersurface $X \hookrightarrow Z$, Z smooth, with a pinch point singularity at a point a; in local coordinates, $z^2 + xy^2 = 0$. Then

$$\mathrm{inv}(a) = (2, 0, 3/2, 0, 1, 0, \infty)\,. \tag{4.1}$$

But (4.1) does not guarantee that a is a pp; for example, $z^2 + y^3 + x^3 = 0$ has the same value of inv but an isolated singularity at 0.

Lemma 4.2. *Let $X \hookrightarrow Z$ denote a hypersurface, Z smooth, and let $a \in X$. Then*

(1) *a is a pinch point* pp *if and only if*

$$\mathrm{inv}(a) = (2, 0, 3/2, 0, 1, 0, \infty)$$

and the singular subset of X, Sing X *has codimension* 2 *in Z at a;*

(2) *a is a degenerate pinch point* dpp *if and only if*

$$\mathrm{inv}(a) = (2, 0, 3/2, 0, 2, 0, \infty)$$

and Sing X *has codimension* 2 *at a.*

Proof. Suppose that X has order 2 at a point a. Then, in suitable étale local coordinates $(x, z) = (x_1, \ldots, x_{n-1}, z)$ at a, X is given by an equation

$$z^2 + b(x) = 0. \tag{4.2}$$

If $\mathrm{inv}_2(a) = (2, 0, 3/2, 0)$, then we can choose new coordinates $(x, y, z) = (x_1, \ldots, x_{n-2}, y, z)$ in which (4.2) becomes

$$z^2 + y^3 + B(x)y + C(x) = 0. \tag{4.3}$$

Then Sing X lies in

$$z = 0,$$
$$y^3 + B(x)y + C(x) = 0,$$
$$3y^2 + B(x) = 0.$$

If Sing X has codimension 2 at a, then the last 2 equations have a common factor, so (4.3) can be rewritten in the form

$$z^2 + (y - A(x))^2(y + 2A(x)) = 0. \tag{4.4}$$

Clearly, a is a pinch point if and only if $\mathrm{ord}_a A = 1$, and (1) follows. Likewise $\mathrm{inv}(a) = (2, 0, 3/2, 0, 2, 0, \infty)$ if and only if A is the square of a function of order 1, after an étale coordinate change, so (2) follows. $\square$

Proof of Theorem 1.18. We can reduce to the case that $X \hookrightarrow Z$ is an embedded hypersurface, Z smooth. We then divide the argument into three parts.

(I) We can blow up following the desingularization algorithm as long as the maximum value of inv is $>\mathrm{inv(pp)} := (2, 0, 3/2, 0, 1, 0, \infty)$. (The blowings-up involved do not modify pp, nc2 or smooth points of X.)

Suppose that the maximum value of inv is inv(pp) (in some year of the resolution history). Then the locus $(\mathrm{inv} = \mathrm{inv(pp)})$ is a smooth subset of X of codimension 3 in Z. Each component of this set either contains no pp or is generically pp (according as Sing X has codimension >2 or $= 2$ at the generic point). We can blow up to get rid of all components with no pp.

Then at any point a with $\mathrm{inv}(a) = \mathrm{inv(pp)}$, the strict transform of X is defined by an equation

$$z^2 + u^\alpha \left(y + u^\beta x\right)^2 \left(y - 2u^\beta x\right) = 0,$$

in suitable étale local coordinates $(u, x, y, z) = (u_1, \dots, u_{n-3}, x, y, z)$ for Z, where $u^\alpha = u_1^{\alpha_1} \cdots u_{n-3}^{\alpha_{n-3}}$ and $\alpha_i > 0$ only if $(u_i = 0)$ is a component of the exceptional divisor (and likewise for u^β).

We use the cleaning lemma first to reduce to the case $\beta = 0$ (α will increase in the process):

$$\left(z = y = 0, \mathrm{ord}\, u^\beta \geq 1\right) \subset (\mathrm{inv}_2 = (2, 0, 3/2, 0)) \tag{4.5}$$

is the cosupport of an invariantly defined monomial marked ideal with associated multiplicity 1 on a maximal contact subvariety of codimension 2; any component of this set extends to an inv_2-admissible centre of blowing up. The blowings-up involved in applying the cleaning lemma have centres given locally by components of (4.5) and its transforms.

Secondly, we use the cleaning lemma to reduce to the case $|\alpha| \leq 1$, where $|\alpha| := \alpha_1 + \cdots + \alpha_{n-3}$, using

$$\left(z = 0, \mathrm{ord}\, u^\alpha \geq 2\right) \subset (\mathrm{inv}_1 = (2, 0)).$$

If $\alpha = 0$, we have a pinch point. If $|\alpha| = 1$, then we have a singularity of the form

$$z^2 + u_1(y + x)^2(y - 2x) = 0,$$

where u_1 is an exceptional divisor. In this case, we blow up with centre given locally by

$$(z = y = x = u_1 = 0) \subset (\mathrm{inv} = (2, 0, 3/2, 0, 1, 0, \infty))$$

to get

$$z^2 + u_1^2(y + x)^2(y - 2x) = 0. \tag{4.6}$$

We now repeat the second cleaning step above to get a pinch point.

(II) Let us say we are now in year j_0. Let P denote the closure of (the inverse image in year j_0 of) the pp locus in year zero. (In the local coordinates of (4.6), P is the strict transform of

$(z = y = x = 0)$ by the blowing-up of $(z = u = 0)$.) At any point of P, we can choose étale coordinates in which X and the support of the exceptional divisor are given as

$$z^2 + xy^2 = 0 \quad \text{and} \quad \sum_{i=0}^{s}(u_i = 0) \tag{4.7}$$

(respectively), for some $s \geq 0$. At nearby nc2 singularities (when $x \neq 0$ above), we can find étale coordinates in which X and the support of the exceptional divisor are given as

$$z^2 + y^2 = 0 \quad \text{and} \quad \sum_{i=0}^{s}(u_i = 0), \tag{4.8}$$

for some $s \geq 0$.

We now apply the desingularization algorithm outside P (where we consider j_0 as "year zero") until the maximum value of inv is inv(nc2) $= (2, 0, 1, 0, \infty)$. Each component of every centre of blowing up involved is either separated from (the inverse image of) P above, or, near a point as in (4.7), of the form $z = y = 0,\ u_j = 0$, for certain j. We can handle this as in the proof of Theorem 3.4, blowing up to separate such components at P before we blow them up.

Cleaning as in the proof of Theorem 1.11, case $n = 3$ (see Section 1.1), produces nc2, or pp at special points of the stratum (inv $= (2, 0, 1, 0, \infty)$).

(III) We can now use the desingularization algorithm to resolve any singularities remaining outside {nc2, pp} (i.e., to reduce to ord $= 1$), by admissible blowings-up. This completes the proof. (Condition (3) of the theorem has also been satisfied.) □

Remark 4.3. The proof above provides normal forms analogous to those listed in Remark 1.17 for the total transform at every singular point (i.e., nc2 or pp) of the final strict transform. In order to get the appropriate normal forms also at smooth points of the latter, we need two more steps (see also Section 4.1).

(IV) We apply the desingularization algorithm to the pair given by the final strict transform and exceptional divisor, outside {nc2, pp}, until the maximum value of inv is $(1, 1, 2, 0, \infty)$. Note that different components of a centre of the blowings-up involved may meet at the pp locus, but we can separate them as in the proof of Theorem 3.4. We also blow up any closed components of (inv $= (1, 1, 2, 0, \infty)$). We can then clean the latter locus. (See Remark 4.1.)

(V) We can now apply Theorem 3.4 outside the closed set given by {nc2, pp} together with the locus cleaned up in (IV).

Remark 4.4. We have not explicitly considered functoriality in the proof of Theorem 1.18, nor in the proofs of Theorems 1.11 and 1.14 (when $n = 3$) in Section 1.2. To ensure functoriality, we have to be a little more careful, as indicated in Remark 3.6. For example, in part (I) of the proof of Theorem 1.18 above, we should really blow up until inv $\leq$ inv(pp), eliminate the components of (inv $=$ inv(pp)) which contain no pp, and then perform the cleaning blowings-up whether or not the latter is non-empty.

4.3. Limits of degenerate pinch points

Remark 4.5. Suppose we use the desingularization algorithm as in the proof of Theorem 1.18 above, to blow up until the maximum value of inv is $(2, 0, 3/2, 0, 2, 0, \infty)$. At a point a of a component of $(\mathrm{inv} = (2, 0, 3/2, 0, 2, 0, \infty))$ which is generically dpp, X is defined by an equation of the form

$$z^2 + u^\alpha \left(y + u^\beta x^2\right)^2 \left(y - 2u^\beta x^2\right) = 0,$$

where u^α and u^β are again monomials in components of the exceptional divisor.

Using the cleaning lemma as above, we can blow up avoiding dpp to reduce to the case that $\beta = 0$ and $|\alpha| = 0$ or 1. If $\alpha = 0$, then we have a dpp.

Suppose that $|\alpha| = 1$. In this case, the singularity cannot be eliminated in the way we handled a similar situation in the proof above. By a change of variables, we can rewrite the equation as

$$z^2 + uy(y + x^2)^2 = 0$$

(where u here denotes a single variable). Blowing up $(z = y = u = 0)$ results in $z^2 + y(uy + x^2) = 0$—the *exceptional singularity* in Theorem 1.11. (See [2, Section 1].)

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Appendix. Crash course on the desingularization invariant

Our purpose in this section is to provide a working knowledge of the desingularization invariant, sufficient to understand the way it is used in our main results without reading all the details of the desingularization algorithm and the invariant (for example, in [3,5]).

Resolution of singularities of a variety X can be described by an *iterative* algorithm. Desingularization can be realized, according to Hironaka [8] by a sequence of blowings-up. The desingularization invariant $\mathrm{inv} = \mathrm{inv}_X$ can be defined iteratively over a sequence of suitable blowings-up. Resolution of singularities can be realized by choosing, as each successive centre of blowing up, the maximum locus of inv; this is the approach of [3].

Every iterative algorithm can be described, in an equivalent way, by a recursive algorithm. The desingularization algorithm of [3] is presented recursively in [5] (as well as in [9,13], for the case of a hypersurface X). The recursive presentation has a certain advantage from the point of view of formal clarity, but hides the explicit calculations involved in computing the invariant and its maximal loci, as needed for this article. The brief presentation below mixes the iterative and recursive aspects.

We restrict our attention to the case of a hypersurface. Throughout this appendix, $X \subset Z$ denotes an *embedded hypersurface* defined over a field $\underline{k}$ of characteristic zero (i.e., Z is a smooth variety and X is a subvariety of pure codimension 1, usually reduced).

A.1. Resolution of singularities

Theorem A.1. *There is a sequence of blowings-up*

$$Z = Z_0 \xleftarrow{\sigma_1} Z_1 \longleftarrow \cdots \xleftarrow{\sigma_t} Z_t, \tag{A.1}$$

where each σ_{j+1} has smooth centre C_j, such that if $X_0 = X$, $E_0 = E := 0$ and, for each $j = 0, 1, \ldots,$

(i) X_{j+1} *denotes the strict transform of* X_j,
(ii) E_{j+1} *denotes the exceptional divisor of* $\sigma_1 \circ \cdots \circ \sigma_{j+1}$,

then, for each j,

(1) C_j *and* E_j *have only simple normal crossings,*
(2) $C_j \subset \operatorname{Sing} X_j$ *or* X_j *is smooth and* $C_j \subset X_j \cap E_j$,
(3) C_j *is the maximum locus of an invariant* $\operatorname{inv}_X(\cdot)$ *(see Remark A.2);*
(4) X_t *is smooth and* X_t, E_t *are* snc.

Note that (1) implies E_{j+1} is snc. The support of each exceptional divisor E_{j+1} has ordered components $H_1^j, \ldots, H_{j+1}^{j+1}$ (not necessarily irreducible), where $H_{j+1}^{j+1} := \sigma_{j+1}^{-1}(C_j)$ and each $H_i^{j+1}, i < j+1$ denotes the strict transform in Z_{j+1} of H_i^i. We will denote each H_i^j by H_i, for short. The "invariant" inv_X is invariant with respect to étale (or smooth) morphisms of Z and ground-field extensions.

A.2. The desingularization invariant

The *desingularization invariant* $\operatorname{inv} = \operatorname{inv}_X$ can be defined inductively over any suitable sequence of blowings-up (A.1). More precisely, for each $j = 0, 1, \ldots,$ we define inv on Z_{j+1} assuming that it is defined on $Z_0, \ldots, Z_j$ and each blowing-up $\sigma_{i+1}, i \le j$ is inv-*admissible* in the sense that

(1) the centre $C_i \subset Z_i$ of σ_{i+1} is smooth and simple normal crossings with E_i, where E_i is the exceptional divisor of $\sigma_1 \circ \cdots \circ \sigma_i$;
(2) inv is constant on every component of C_i.

Write $X_0 := X$. For each $j \ge 0$, let $X_{j+1} \subset Z_{j+1}$ denote the strict transform of X_j by σ_{j+1}. If $a \in Z_j$, then $\operatorname{inv}_X(a)$ depends on the previous blowings-up. (A functorial algorithm for resolution of singularities necessarily has some historical memory; cf. [4, Example 1.9], [9, Section 3.6].) In fact, if $a \in Z_j$, then $\operatorname{inv}_X(a)$ depends only on X_j and certain subblocks of the set of components of E_j, which we describe below.

Let $a \in Z_j$. Then $\operatorname{inv}(a)$ has the form

$$\operatorname{inv}(a) = (\nu_1(a), s_1(a), \ldots, \nu_q(a), s_q(a), \nu_{q+1}(a)), \tag{A.2}$$

where $\nu_k(a)$ is a positive rational number if $k \le q$, each $s_k(a)$ is a nonnegative integer, and $\nu_{q+1}(a)$ is either 0 (the order of an ideal generated by a unit) or ∞ (the order of the zero ideal). The successive pairs $(\nu_k(a), s_k(a))$ are defined inductively over *maximal contact* subvarieties of increasing codimension. $\operatorname{inv}(a) = (0)$ if and only if $a \in Z_j \setminus X_j$.

We order finite sequences of the form (A.2) lexicographically. Then $\operatorname{inv}(\cdot)$ is upper-semicontinuous on each Z_j, and *infinitesimally upper-semicontinuous* in the sense that, if $a \in Z_j$, then $\operatorname{inv}(\cdot) \le \operatorname{inv}(a)$ on $\sigma_{j+1}^{-1}(a)$.

Remark A.2. In Theorem A.1, consider $a \in X_j$ in the maximum locus of inv. If $\operatorname{inv}(a) = (\ldots, \infty)$, then C_j = maximum locus of inv is smooth and $\operatorname{inv} < \operatorname{inv}(a)$ on $\sigma_{j+1}^{-1}(a)$. If $\operatorname{inv}(a) = (\ldots, 0)$, then the maximum locus of inv in fact may have several smooth components – it is given by the intersection of a smooth subspace of Z_j with a normal crossings divisor – and inv decreases after finitely many "monomial" or "combinatorial" blowings-up (centre given by any component of the maximum locus). See Remark A.17.

We also introduce truncations of inv. Let $\mathrm{inv}_{k+1}(a)$ denote the truncation of $\mathrm{inv}(a)$ after $s_{k+1}(a)$ (i.e., after the $(k+1)$st pair), and let $\mathrm{inv}_{k+1/2}(a)$ denote the truncation of $\mathrm{inv}(a)$ after $\nu_{k+1}(a)$. ($\mathrm{inv}_{k+1/2}(a) := \mathrm{inv}(a) =: \mathrm{inv}_{k+1}(a)$ if $k \geq q$ in (A.2).)

Given $a \in Z_j$, let a_i denote the image of a in Z_i, $i \leq j$. (We will speak of *year i* in the history of blowings-up.) The *year of birth* of $\mathrm{inv}_{k+1/2}(a)$ (or $\mathrm{inv}_{k+1}(a)$) denotes the smallest i such that $\mathrm{inv}_{k+1/2}(a) = \mathrm{inv}_{k+1/2}(a_i)$ (respectively, $\mathrm{inv}_{k+1}(a) = \mathrm{inv}_{k+1}(a_i)$).

Let $a \in Z_j$. Let $E(a)$ denote the set of components of E_j which pass through a. The entries $s_k(a)$ of $\mathrm{inv}(a)$ are the sizes of certain subblocks of $E(a)$, as follows. Let i denote the birth-year of $\mathrm{inv}_{1/2}(a) = \nu_1(a)$, and let $E^1(a)$ denote the collection of elements of $E(a)$ that are strict transforms of components of E_i (i.e., strict transforms of elements of $E(a_i)$). Set $s_1(a) := \#E^1(a)$. We define $s_{k+1}(a)$, in general, by induction on k: Let i denote the year of birth of $\mathrm{inv}_{k+1/2}(a)$ and let $E^{k+1}(a)$ denote the set of elements of $E(a) \setminus \big(E^1(a) \cup \cdots \cup E^k(a)\big)$ that are strict transforms of components of E_i. Set $s_{k+1}(a) := \#E^{k+1}(a)$.

Clearly, all $s_k(a) = 0$ in year zero (i.e., if $a \in Z$). We will be interested in $\mathrm{inv}(a)$ often in the case that all $s_k(a) = 0$, in some given year j.

Given $a \in Z_j$, $\nu_1(a)$ means $\mathrm{ord}_a X_j$. The entries $\nu_k(a)$ of $\mathrm{inv}(a)$ in general are *residual orders* that we define in general in Appendix A.10. We will first consider the invariant in year zero,

$$\mathrm{inv}(a) = (\nu_1(a), 0, \ldots, \nu_q(a), 0, \nu_{q+1}(a)),$$

where the $\nu_k(a)$ are simpler (Appendix A.7). In year zero, $\nu_{q+1}(a) = \infty$. (In general, $\nu_{q+1}(a) = 0$ only if $E(a) \setminus \big(E^1(a) \cup \cdots \cup E^q(a)\big) \neq \emptyset$.)

A.3. Maximal contact

Let $a \in X$ and let $d := \nu_1(a) = \mathrm{ord}_a X$. Let f denote a local generator of $\mathcal{I}_X$ in a neighbourhood U of a in Z such that $\nu_1(x) \leq d$, $x \in U$. We will write $\mathrm{cosupp}\,(f, d)$ or $\mathrm{cosupp}\,(\mathcal{I}_X, d)$ for the locus of points of order d of f in U. Say that a (local) blowing-up $\sigma : Z' \to U \subset Z$ with smooth centre $C \subset U$ is ord-*admissible* if $C \subset \mathrm{cosupp}\,(f, d)$.

Let X' denote the strict transform of X by an ord-admissible blowing-up $\sigma : Z' \to U \subset Z$ with centre C. At a point of Z', $\mathcal{I}_{X'}$ is generated by $f' := y_{\mathrm{exc}}^{-d} f \circ \sigma$, where y_{exc} denotes a local generator of the ideal of the exceptional divisor $\sigma^{-1}(C)$. We will use the same notation $X' \subset Z'$ for the strict transform of X by a sequence of ord-admissible local blowings-up.

A *maximal contact hypersurface* for $\mathcal{I}_X$ at a denotes a hypersurface $N = V(z)$, where z is a regular function on a neighbourhood U as above, such that $\mathrm{ord}_a z = 1$, with the property that $\mathrm{cosupp}\,(\mathcal{I}_{X'}, d) \subset N'$ after any sequence of ord-admissible local blowings-up. ($N' = V(z')$ denotes the strict transform of N. See the formal Definition A.11.)

Example A.3. Suppose that $\mathcal{I}_X$ has a local generator f which can be written as a Weierstrass polynomial in local coordinates $(x_1, \ldots, x_n)$ at $a = 0$,

$$f(x) = x_n^d + c_{d-1}(\tilde{x})x_n^{d-1} + \cdots + c_0(\tilde{x}), \tag{A.3}$$

where the coefficients c_i are regular (or analytic) functions in $\tilde{x} = (x_1, \ldots, x_{n-1})$ such that $\mathrm{ord}_a c_i \geq d - i$. After a coordinate change $x_n' = x_n - c_{d-1}(\tilde{x})/d$, we can assume that $c_{d-1} = 0$. We claim that $z := x_n$ defines a maximal contact hypersurface.

Clearly, $\mathrm{ord}_x f = d$ if and only if $z = 0$ and $\mathrm{ord}_{\tilde{x}} c_i \geq d - i$, for all i. (Note that c_i can be identified with the restriction to $N = V(z)$ of the partial derivative $\partial_z^i f := \partial^i f/\partial z^i$.)

Let $\sigma : Z' \to U \subset Z$ be an ord-admissible local blowing-up with smooth centre C. We can assume that $C = \{x_r = \cdots = x_n = 0\}$, after a transformation of the $\tilde{x}$-coordinates. Then Z' can be covered by coordinate charts U_{x_j}, $j = r, \ldots, n$, where the "x_j-coordinate chart" U_{x_j} has coordinates $(y_1, \ldots, y_n)$ given by $y_k = x_k/x_j$ if $k = r, \ldots, n$, $k \neq j$, and $y_k = x_k$ otherwise. The strict transform X' lies in the union of the charts U_{x_j}, $j = r, \ldots, n-1$.

Consider, for example, the chart U_{x_r} with coordinates

$$y_j = x_j, \quad j \le r, \qquad y_j = x_j/x_r, \quad j > r.$$

In this chart, the strict transform is given by $f'(y) = 0$, where

$$\begin{aligned} f'(y) &= y_r^{-d} f \circ \sigma \\ &= y_n^d + c'_{d-2}(\tilde{y}) y_n^{d-2} + \cdots + c'_0(\tilde{y}), \end{aligned}$$

and each

$$c'_i(\tilde{y}) = y_r^{-(d-i)} c_i \circ \tilde{\sigma}. \tag{A.4}$$

The strict transform f' has the same form as our original function f; in particular, $\mathrm{ord}_y f' = d$ if and only if $y_n = 0$ and $\mathrm{ord}_{\tilde{y}} c'_i \ge d - i$, for all i. Moreover $y_n = z' := y_r^{-1} z \circ \sigma$; i.e., $N' = V(y_n)$ is the strict transform of N. Our claim follows.

Example A.4. Suppose that $\mathcal{I}_X$ has a local generator f of the form $f = z \cdot g$, where $\mathrm{ord}_a z = 1$. Clearly, in a neighbourhood of a, $\mathrm{ord}_x f = d$ if and only if $z = 0$ and $\mathrm{ord}_x g = d - 1$. Consider the transforms $f' := y_{\mathrm{exc}}^{-d} f \circ \sigma$, $z' := y_{\mathrm{exc}}^{-1} z \circ \sigma$ and $g' := y_{\mathrm{exc}}^{-(d-1)} g \circ \sigma$ by an ord-admissible local blowing-up σ. Then $f' = z' \cdot g'$, and $\mathrm{ord}_y f' = d$ if and only if $z' = 0$ and $\mathrm{ord}_y g' = d - 1$. It follows that $N = V(z)$ is a maximal contact hypersurface.

In general, if $N = V(z)$ is a maximal contact hypersurface for $\mathcal{I}_X$ at a, then, in a neighbourhood of a, $\mathrm{ord}_x f = d$ if and only if $x \in N$ and $\mathrm{ord}_x \partial_z^i f|_N \ge d - i$, $i = 0, \ldots, d-1$ (likewise for the transforms by an admissible blowing-up). Moreover, the transformation formula $f' := y_{\mathrm{exc}}^{-d} f \circ \sigma$ for an ord-admissible blowing-up σ implies the following transformation rules for the partial derivatives $\partial_z^i f$:

$$\partial_{z'}^i f' = y_{\mathrm{exc}}^{-(d-i)} \partial_z^i f \circ \sigma, \quad i = 0, \ldots, d-1.$$

It therefore makes sense to regard the data given by (f, d) on Z as "equivalent" to those given on N by $(c_i, d - i) := \left(\partial_z^i f|_N, d - i\right)$, $i = 0, \ldots, d-1$, with respect to the corresponding transformation rules. Since $\dim N = \dim Z - 1$, this idea of equivalence is a basis for induction on dimension.

Note, however, that we might have $\mathrm{ord}_a c_i > d - i$, for all i. We define

$$\nu_2(a) := \min_{0 \le i \le d-1} \frac{\mathrm{ord}_a c_i}{d - i}. \tag{A.5}$$

To continue an inductive definition of the invariant, we need to work not only with data of the form (f, d), where $\mathrm{ord}_a f = d$, but also, more generally, with a "marked ideal" $\underline{\mathcal{I}} = (\mathcal{I}, d)$, where $\mathrm{ord}_a \mathcal{I} \ge d$.

We will return to the invariant in year zero below, but it is convenient to first formalize the ideas of *marked ideal* and *equivalence* in a general setting.

A.4. Marked ideals

Definitions A.5. A *marked ideal* $\underline{\mathcal{I}}$ is a quintuple $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, where

(1) $Z \supset N$ are smooth varieties,
(2) $E = \sum_{i=1}^{s} H_i$ is a simple normal crossings divisor on Z which is transverse to N and *ordered* (the H_i are smooth hypersurfaces in Z, not necessarily irreducible, with ordered index set as indicated),
(3) $\mathcal{I} \subset \mathcal{O}_N$ is an ideal,
(4) $d \in \mathbb{N}$.

The *cosupport* of $\underline{\mathcal{I}}$,

$$\operatorname{cosupp}\underline{\mathcal{I}} := \{x \in N : \operatorname{ord}_x \mathcal{I} \geq d\}.$$

We say that $\underline{\mathcal{I}}$ is of *maximal order* if $d = \max\{\operatorname{ord}_x \mathcal{I} : x \in \operatorname{cosupp}\underline{\mathcal{I}}\}$. The *dimension* $\dim \underline{\mathcal{I}}$ denotes $\dim N$.

A blowing-up $\sigma : Z' \to Z$ (with smooth centre C) is $\underline{\mathcal{I}}$-*admissible* (or simply *admissible*) if $C \subset \operatorname{cosupp}\underline{\mathcal{I}}$, and C, E have only normal crossings. The *(controlled) transform* of $\underline{\mathcal{I}}$ by an admissible blowing-up $\sigma : Z' \to Z$ is the marked ideal $\underline{\mathcal{I}}' = (Z', N', E', \mathcal{I}', d' = d)$, where

(1) N' is the strict transform of N by σ,
(2) $E' = \sum_{i=1}^{s+1} H_i'$ (where H_i' denotes the strict transform of H_i, for each $i = 1, \ldots, s$, and $H_{s+1}' := \sigma^{-1}(C)$—the exceptional divisor of σ, introduced as the last member of E'),
(3) $\mathcal{I}' := \mathcal{I}_{\sigma^{-1}(C)}^{-d} \cdot \sigma^*(\mathcal{I})$ (where $\mathcal{I}_{\sigma^{-1}(C)} \subset \mathcal{O}_{N'}$ denotes the ideal of $\sigma^{-1}(C)$).

In this definition, note that $\sigma^*(\mathcal{I})$ is divisible by $\mathcal{I}_{\sigma^{-1}(C)}^{d}$ and E' is a normal crossings divisor transverse to N', because σ is admissible. We likewise define the transform by a sequence of admissible blowings-up.

We say that two marked ideals $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ (with the same ambient variety Z and the same normal crossings divisor E) are *equivalent* if they have the same sequences of *test transformations* (i.e., every test sequence for one is a test sequence for the other). *Test transformations* are transformations of a marked ideal by morphisms of three possible kinds: admissible blowings-up, projections from products with an affine line, and *exceptional blowings-up* [5, Definition 2.5]. In particular, if $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$ are equivalent, then they have the same cosupport and their transforms by any sequence of admissible blowings-up have the same cosupport. The remaining two types of test transformations are used to prove functoriality properties of the desingularization invariant and algorithm. We refer the reader to [5, Section 2] for definitions; we do not need these notions explicitly here.

In particular, equivalent marked ideals have the same resolution sequences.

Definition A.6. A *resolution of singularities* of a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ is a sequence of admissible blowings-up (A.1) after which $\operatorname{cosupp}\underline{\mathcal{I}}' = \emptyset$.

Example A.7. Given a hypersurface $X \hookrightarrow Z$ as above, we introduce the marked ideal

$$\underline{\mathcal{I}}_X := (Z, Z, \emptyset, \mathcal{I}_X, 1). \tag{A.6}$$

Then a resolution of singularities of $\underline{\mathcal{I}}_X$ (which is functorial with respect to étale morphisms) provides a resolution of singularities of X, *before the last blowing up* for $\underline{\mathcal{I}}_X$. Consider the resolution sequence for $\underline{\mathcal{I}}_X$. Each centre of blowing-up is smooth and snc with respect to the exceptional divisor. The last blowing-up leads to empty cosupport, and the centre of the last blowing-up includes all smooth points of X. It follows that strict transform of X coincides with the centre at this step. So we have resolved the singularities of X.

To interpret the data $\{(c_i, d-i)\}$ on N of Appendix A.3 as a marked ideal, it is convenient to define sums of marked ideals. In general, we will shorten the notation $(Z, N, E, \mathcal{I}, d)$ to $(E, \mathcal{I}, d)$ or $(\mathcal{I}, d)$ when the remaining entries are unambiguous.

Definition A.8. Consider marked ideals $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d) = (\mathcal{I}, d)$ and $\underline{\mathcal{J}} = (Z, N, E, \mathcal{J}, d) = (\mathcal{J}, d)$. Define $\underline{\mathcal{I}} + \underline{\mathcal{J}}$ as $(\mathcal{I}^{l/d} + \mathcal{J}^{l/e}, l)$, where $l = \operatorname{lcm}(d, e)$. Likewise, for any finite sum.

It is easy to see:

(1) $\operatorname{cosupp}(\underline{\mathcal{I}} + \underline{\mathcal{J}}) = \operatorname{cosupp}\underline{\mathcal{I}} \cap \operatorname{cosupp}\underline{\mathcal{J}}$;
(2) a blowing-up $\sigma : Z' \to Z$ is admissible for $\underline{\mathcal{I}} + \underline{\mathcal{J}}$ if and only if σ is admissible for both $\underline{\mathcal{I}}$ and $\underline{\mathcal{J}}$, and the transforms satisfy $\underline{\mathcal{I}}' + \underline{\mathcal{J}}' = (\underline{\mathcal{I}} + \underline{\mathcal{J}})'$.

Addition is not associative, but $\underline{\mathcal{I}} + \underline{\mathcal{J}}$ is equivalent to $(\underline{\mathcal{I}}^e + \underline{\mathcal{J}}^d, de)$, and addition is associative up to equivalence.

Example A.9. In the notation of Appendix A.3, let $\underline{\mathcal{J}}$ denote the marked ideal $(Z, Z, \emptyset, \mathcal{I}_X, d)$, so that $\underline{\mathcal{J}}|_U = ((f), d)$. Define the *coefficient marked ideal* $\underline{\mathcal{C}}_U(\underline{\mathcal{J}}) = (U, N, \emptyset, \mathcal{C}, d_{\mathcal{C}})$ as the sum of the marked ideals $((c_i), d-i) = (U, N, \emptyset, (c_i), d-i)$. Then $\underline{\mathcal{J}}|_U$ is equivalent to $\underline{\mathcal{C}}_U(\underline{\mathcal{J}})$.

Definition A.10 (*Invariants of a Marked Ideal*)**.** Given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ and a point $a \in \operatorname{cosupp}\underline{\mathcal{I}}$, we set

$$\mu_a(\underline{\mathcal{I}}) := \frac{\operatorname{ord}_a \mathcal{I}}{d} \quad \text{and} \quad \mu_{H,a}(\underline{\mathcal{I}}) := \frac{\operatorname{ord}_{H,a} \mathcal{I}}{d}, \quad H \in E \tag{A.7}$$

($\operatorname{ord}_{H,a}\mathcal{I}$ denotes the *order* of $\mathcal{I} \subset \mathcal{O}_N$ *along* $H|_N$ at a; i.e., the largest $\mu \in \mathbb{N}$ such that $\mathcal{I}_a \subset \mathcal{I}^{\mu}_{H|_N, a}$.)

Both $\mu_a(\underline{\mathcal{I}})$ and $\mu_{H,a}(\underline{\mathcal{I}})$ depend only on the equivalence class of $\underline{\mathcal{I}}$ and $\dim N$ [5, Section 6].

Definition A.11 (*Maximal Contact*)**.** Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d) = (\mathcal{I}, d)$ be a marked ideal and let $a \in N$. Let z denote a regular function on a neighbourhood of a in N such that $\operatorname{ord}_a = 1$. Then $P := V(z)$ is a *maximal contact hypersurface* for $\underline{\mathcal{I}}$ at a if P is transverse to E and $(\mathcal{I}, d) + ((z), 1)$ is equivalent to $(\mathcal{I}, d)$ on a neighbourhood of a in Z.

Lemma A.12. *A marked ideal* $\underline{\mathcal{I}} = (Z, N, \emptyset, \mathcal{I}, d)$ *admits a maximal contact hypersurface at* $a \in N$ *if and only if* $\operatorname{ord}_a \mathcal{I} = d$ *(i.e.,* $\underline{\mathcal{I}}$ *is of maximal order on a sufficiently small neighbourhood of* a*).*

Proof. The "only if" direction is consequence of invariance of $\mu_a(\underline{\mathcal{I}})$. In the other direction, if $\operatorname{ord}_a \mathcal{I} = d$, then there is a local section f of $\mathcal{I}$ at a and a partial derivative $\partial^\alpha := \partial^\alpha / \partial x^\alpha$ of order $d-1$, with respect to local coordinates of N, such that $z := \partial^\alpha f$ has order 1 at a. Then $P := V(z) \subset N$ is a maximal contact hypersurface at a [5, Section 4]. □

A.5. Coefficient ideals

We now formalize the coefficient data $\{(c_i, d-i)\}$ of Appendix A.3 as a marked ideal. Let $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d) = (\mathcal{I}, d)$ be a marked ideal of maximal order, and let $a \in \operatorname{cosupp} \underline{\mathcal{I}}$. Suppose that $P = V(z)$ is a maximal contact hypersurface for $\underline{\mathcal{I}}$, in some neighbourhood U of a. In a suitable U, we can find a system of local coordinates $(x_1, \ldots, x_n)$ for N such that $x_n = z$ and the components of E are given by $x_i = 0$, $i = 1, \ldots, r < n$. Let $\mathcal{D}_z(\mathcal{I})$ denote the ideal generated by f, $\partial f/\partial z$, for all $f \in \mathcal{I}$, and let $\underline{\mathcal{D}_z}(\underline{\mathcal{I}})$ denote the marked ideal $(\mathcal{D}_z(\mathcal{I}), d-1)$. For $j \geq 2$, we inductively set $\mathcal{D}_z^j(\mathcal{I}) := \mathcal{D}_z\left(\mathcal{D}_z^{j-1}(\mathcal{I})\right)$, and we define marked ideals

$$\underline{\mathcal{D}_z^j}(\underline{\mathcal{I}}) := \left(\mathcal{D}_z^j(\mathcal{I}), d-j\right), \quad j = 0, \ldots, d-1,$$

$$\begin{aligned} \underline{\mathcal{C}_z^{d-1}}(\underline{\mathcal{I}}) &:= \sum_{j=0}^{d-1} \underline{\mathcal{D}_z^j}(\underline{\mathcal{I}}) \\ &= \left(\mathcal{C}_z^{d-1}(\underline{\mathcal{I}}), d_{\underline{\mathcal{C}}}\right), \quad \text{say.} \end{aligned}$$

We define the *coefficient (marked) ideal* $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ as the restriction of the latter to the maximal contact hypersurface P; i.e.,

$$\underline{\mathcal{C}_z}(\underline{\mathcal{I}}) := \left(U, P, E, \mathcal{C}_z^{d-1}(\underline{\mathcal{I}})|_P, d_{\underline{\mathcal{C}}}\right).$$

Then $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{I}}$ (in the chart U), essentially by the calculations in Appendix A.3 (see [5, Section 4]).

Example A.13. Suppose that $E = \emptyset$ and $\mathcal{I}$ is a principal ideal generated by $f(x)$ as in (A.3). Assume that $c_{d-1} = 0$. Set $z := x_n$. Then $P = V(z)$ is a maximal contact hypersurface, and the coefficient ideal $\underline{\mathcal{C}_z}(\underline{\mathcal{I}}) = \sum_{i=0}^{d-2}((c_i), d-i)$.

Suppose that $f(x)$ splits; i.e.,

$$z^d + c_{d-2}(\tilde{x})z^{d-2} + \cdots + c_0(\tilde{x}) = (z - b_1(\tilde{x})) \cdots (z - b_d(\tilde{x})).$$

Then $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ is equivalent to the marked ideal $\sum_{j=1}^{d}((b_j), 1)$. This follows from the fact the $\operatorname{ord}_a b_j \geq k$, for all j, if and only if $\operatorname{ord}_a \sigma_i \geq ki$, for all i, where σ_i denotes the ith elementary symmetric function of the b_j.

Remark A.14. In general, since the coefficient ideal $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ is equivalent to $\underline{\mathcal{I}}$ (in a chart U as above), any resolution of singularities of $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ is a resolution of singularities of $\underline{\mathcal{I}}$ over U (as in Definitions A.5). Since $\dim \underline{\mathcal{C}_z}(\underline{\mathcal{I}}) < \dim \underline{\mathcal{I}}$, the idea is to use the coefficient ideal as a basis for induction on dimension. There are two main problems involved in carrying out this idea.

(1) Passage from $\underline{\mathcal{I}}$ to $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ requires that $\underline{\mathcal{I}}$ be of maximal order, so that it admits a maximal contact hypersurface (according to Lemma A.12). But $\underline{\mathcal{C}_z}(\underline{\mathcal{I}})$ is not necessarily of maximal order, so we cannot *a priori* repeat the construction inductively.

Moreover, maximal contact is not unique. Local centres of blowing up chosen by an inductive construction as above need not *a priori* glue together to give a global centre of blowing up. This gluing problem can be resolved by iterating a suitable inductive construction in decreasing dimension to define a desingularization invariant (or, as in [5], by using functoriality properties of equivalence classes of marked ideals to make a stronger inductive assumption that guarantees gluing).

(2) In general, a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$ of maximal order admits a maximal contact hypersurface $P = V(z)$, according to Lemma A.12, only provided that $E = \emptyset$ (for example, in year zero).

Item (1) of the Remark is treated using the constructions in Appendices A.6 and A.8 and item (2) using Appendix A.9.

A.6. Monomial and residual ideals

In general, given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d) = (\mathcal{I}, d)$, we can factor $\mathcal{I}$ as

$$\mathcal{I} = \mathcal{M}(\underline{\mathcal{I}}) \cdot \mathcal{R}(\underline{\mathcal{I}}),$$

where $\mathcal{M}(\underline{\mathcal{I}})$ is a product of the ideals $\mathcal{I}_H$ of the components H of E, and $\mathcal{R}(\underline{\mathcal{I}})$ is divisible by no such exceptional divisor. We call $\mathcal{M}(\underline{\mathcal{I}})$ the *monomial* or *divisorial part* and $\mathcal{R}(\underline{\mathcal{I}})$ the *residual* or *nonmonomial part* of $\mathcal{I}$.

We define the *residual multiplicity* of $\underline{\mathcal{I}}$ at a point $a \in \operatorname{cosupp} \underline{\mathcal{I}}$,

$$\nu_{\underline{\mathcal{I}}}(a) := \frac{\operatorname{ord}_a \mathcal{R}(\underline{\mathcal{I}})}{d}.$$

Then

$$\nu_{\underline{\mathcal{I}}}(a) = \mu_a(\underline{\mathcal{I}}) - \sum_{H \in E} \mu_{H,a}(\underline{\mathcal{I}})$$

(cf. Definition A.10), so that $\nu_{\underline{\mathcal{I}}}(a)$ depends only on the equivalence class of $\underline{\mathcal{I}}$.

We use the residual multiplicity to define the term $\nu_2(a)$ in inv, and inductively to define $\nu_j(a)$, $j \geq 2$. (See Appendix A.7 and Definition A.19.)

Let $\operatorname{ord} \mathcal{R}(\underline{\mathcal{I}})$ denote the maximum order of $\mathcal{R}(\underline{\mathcal{I}})$ on $\operatorname{cosupp} \underline{\mathcal{I}}$. Then the *residual (marked) ideal*

$$\underline{\mathcal{R}}(\underline{\mathcal{I}}) := (\mathcal{R}(\underline{\mathcal{I}}), \operatorname{ord} \mathcal{R}(\underline{\mathcal{I}})) = (Z, N, E, \mathcal{R}(\underline{\mathcal{I}}), \operatorname{ord} \mathcal{R}(\underline{\mathcal{I}}))$$

is a marked ideal of maximal order.

In general, a blowing-up that is admissible for $\underline{\mathcal{R}}(\underline{\mathcal{I}})$ need not be admissible for $\underline{\mathcal{I}}$. If $\mathcal{M}(\underline{\mathcal{I}}) = 1$, however (for example, in year zero), then $\underline{\mathcal{R}}(\underline{\mathcal{I}}) = \underline{\mathcal{I}}$ and any blowing-up that is $\underline{\mathcal{R}}(\underline{\mathcal{I}})$-admissible will also be $\underline{\mathcal{I}}$-admissible. This is enough to define the invariant in year zero.

Remark A.15. In order to calculate the resolution invariant at a point a in any year of the resolution history, we make the construction above locally at a. In particular, we can identify E with the set $E(a)$ of components of E at a, and $\operatorname{ord} \mathcal{R}(\underline{\mathcal{I}}) = \operatorname{ord}_a \mathcal{R}(\underline{\mathcal{I}})$. This localization of the construction will be assumed in the computation below.

A.7. The invariant in year zero

All $s_i = 0$. Let $\underline{\mathcal{I}}^0 := \underline{\mathcal{I}}_X$ (see Example A.7). Then $\mathcal{R}(\underline{\mathcal{I}}^0) = \mathcal{I}^0$. Consider $a \in \operatorname{cosupp} \underline{\mathcal{I}}^0$. Then $\nu_{\underline{\mathcal{I}}^0}(a) = \operatorname{ord}_a \mathcal{I}_X = \nu_1(a)$. We set:

$\underline{\mathcal{J}}^0 := \underline{\mathcal{R}}(\underline{\mathcal{I}}^0)$. Then $\underline{\mathcal{J}}^0$ is of maximal order. Let $P = V(z)$ be a maximal contact hypersurface for $\underline{\mathcal{J}}^0$ at a.

$\underline{\mathcal{I}}^1 :=$ the coefficient ideal $\underline{\mathcal{C}}_z(\underline{\mathcal{J}}^0) = \big(Z, P, \emptyset, \mathcal{C}(\underline{\mathcal{J}}^0), d_{\mathcal{C}}\big)$.

We define

$$\nu_2(a) := \nu_{\underline{\mathcal{I}}_1}(a) = \mathrm{ord}_a \mathcal{R}(\underline{\mathcal{C}}_z(\underline{\mathcal{J}}^0))/d_{\underline{\mathcal{C}}}$$

(of course, here in year zero, $\mathcal{R}(\underline{\mathcal{C}}_z(\underline{\mathcal{J}}^0)) = \mathcal{C}(\underline{\mathcal{J}}^0)$), and we iterate the preceding construction: Set $\underline{\mathcal{J}}^1 := \underline{\mathcal{R}}(\underline{\mathcal{I}}^1)$. Then $\underline{\mathcal{J}}^1$ is of maximal order, so it admits a maximal contact hypersurface $Q = V(w)$ in P; Q is of the form $V(z, w)$ in a coordinate chart of Z—a "codimension two maximal contact subspace", etc. We thus define $\nu_3(a), \ldots$. At a certain step, the coefficient ideal $\underline{\mathcal{I}}^q = \underline{\mathcal{C}}_{\cdot}(\underline{\mathcal{J}}^{q-1})$ becomes zero (e.g., we might run out of variables). Then we put $\nu_{q+1}(a) := \infty$ and $\mathrm{inv}(a) = (\nu_1(a), 0, \nu_2(a), 0, \ldots, 0, \nu_{q+1}(a))$. The locus of points $(\mathrm{inv} = \mathrm{inv}(a))$ (the locus of points where $\mathrm{inv} = \mathrm{inv}(a)$) is (locally) the last maximal contact subspace, of codimension q.

Example A.16. Let X denote the hypersurface $(z^2 + xy^2 = 0)$ in $Z = \mathbb{A}^3$. We show that (in year zero), $\mathrm{inv}(0) = (2, 0, 3/2, 0, 1, 0, \infty)$ and $(\mathrm{inv} = \mathrm{inv}(0))$ is $C_0 = \{0\}$; this will be the first centre of blowing-up in the resolution algorithm. The calculations needed to compute $\mathrm{inv}(0)$ according to the preceding definition are presented in the following table. The marked ideal $\underline{\mathcal{I}}^{i+1}$ in each row $i + 1$ of the table lives on the maximal contact subspace (of codimension $i + 1$) in row i. Each $\underline{\mathcal{I}}^{i+1}$ is the coefficient ideal of $\underline{\mathcal{J}}^i$. It is clear that $(\mathrm{inv} = \mathrm{inv}(0))$ is the last maximal contact subspace $(z = y = x = 0)$.

Codim i	Marked ideal $\underline{\mathcal{I}}^i$	Residual ideal $\underline{\mathcal{J}}^i$	Maximal contact
0	$(z^2 + xy^2, 1)$	$(z^2 + xy^2, 2)$	$(z = 0)$
1	$(xy^2, 2)$	$(xy^2, 3)$	$(z = y = 0)$
2	$(x, 1)$	$(x, 1)$	$(z = y = x = 0)$
3	0		

A.8. Companion ideals

We use the notation of Appendix A.6. Recall that, in general, a blowing-up that is admissible for $\underline{\mathcal{R}}(\underline{\mathcal{I}})$ need not be admissible for $\underline{\mathcal{I}}$. We define the *companion ideal* $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ as

$$\underline{\mathcal{G}}(\underline{\mathcal{I}}) := \begin{cases} (\mathcal{R}(\underline{\mathcal{I}}), \mathrm{ord}\,\mathcal{R}(\underline{\mathcal{I}})) + (\mathcal{M}(\underline{\mathcal{I}}), d - \mathrm{ord}\,\mathcal{R}(\underline{\mathcal{I}})), & \mathrm{ord}\,\mathcal{R}(\underline{\mathcal{I}}) < d \\ (\mathcal{R}(\underline{\mathcal{I}}), \mathrm{ord}\,\mathcal{R}(\underline{\mathcal{I}})), & \mathrm{ord}\,\mathcal{R}(\underline{\mathcal{I}}) \geq d. \end{cases}$$

It is not difficult to see that $\mathrm{cosupp}\,\underline{\mathcal{G}}(\underline{\mathcal{I}}) = \mathrm{cosupp}\,\underline{\mathcal{R}}(\underline{\mathcal{I}}) \cap \mathrm{cosupp}\,\underline{\mathcal{I}}$ and thus that $\underline{\mathcal{G}}(\underline{\mathcal{I}})$-admissible blowings-up are also $\underline{\mathcal{I}}$-admissible. Moreover, the equivalence class of $\underline{\mathcal{G}}(\underline{\mathcal{I}})$ depends only on the equivalence class of $\underline{\mathcal{I}}$; this is a consequence of the same property for the invariants (A.7) (see [5, Corollary 5.3]).

This is enough to define the invariant at a point *a in any year of the blowing-up history*, provided that all $s_i(a) = 0$. We simply use the year zero definition of Appendix A.7 with one change: For each i, we take $\underline{\mathcal{J}}_i := \underline{\mathcal{G}}(\underline{\mathcal{I}}_i)$.

Remark A.17. In the preceding definition, note that each $\nu_{i+1}(a) := \nu_{\underline{\mathcal{I}}_i}(a)$, where the latter is still the residual multiplicity as defined in Appendix A.7. But the change in the definition of the $\underline{\mathcal{J}}^i$ may result in a change in $\nu_i(a)$, $i \geq 2$, and it might result in a change in the last term $\nu_{q+1}(a)$ of $\mathrm{inv}(a)$:

In the current situation, we will arrive at a certain step q where either $\mathcal{I}^q = 0$ or $\mathcal{I}^q = \mathcal{M}(\underline{\mathcal{I}}^q)$. In the former case, we put $\nu_{q+1}(a) := \infty$, as in Appendix A.7. In the latter case, we put

$\nu_{q+1}(a) := 0$ (the order of $\mathcal{R}(\underline{\mathcal{I}}^q)$). This is the *monomial case* of resolution of singularities; see [5, Section 5, Step II, Case A]. We do not need the invariant in the case that $\nu_{q+1}(a) = 0$ in this article, but monomial resolution intervenes in the cleaning lemma (Section 2).

A.9. Coefficient ideals with boundary

The construction of this subsection is needed to treat the terms $s_i(a)$, in general. Let $\underline{\mathcal{J}} = (Z, N, E, \mathcal{J}, d)$ denote a marked ideal of maximal order. We call E the *boundary* of $\underline{\mathcal{J}}$. Set $\underline{\mathcal{J}}_\emptyset := (Z, N, \emptyset, \mathcal{J}, d)$. Then, locally, $\underline{\mathcal{J}}_\emptyset$ admits a maximal contact hypersurface $P = V(z)$, by Lemma A.12. However, P need not be snc with respect to E.

We "add the boundary to the coefficient ideal" (see (A.8)) to ensure that the centre of blowing up will lie in all components of the boundary, so will automatically be snc with respect to the boundary divisor.

At any point a of cosupp $\underline{\mathcal{J}}$, the boundary determines a marked ideal $\sum(\mathcal{I}_H, 1)$, where the sum is over all components H of E such that $a \in H$. At a, the coefficient ideal plus boundary is given by

$$\underline{\mathcal{I}}' := \underline{\mathcal{C}}_z(\underline{\mathcal{J}}_\emptyset) + \sum(\mathcal{I}_H|_{(z=0)}, 1). \tag{A.8}$$

Note that $\underline{\mathcal{I}}'$ itself has empty boundary. Resolution of singularities of $\underline{\mathcal{I}}'$ involves centres in the maximal contact hypersurface $(z = 0)$ and its successive strict transforms. During the resolution process, the *new* exceptional divisors that accumulate are automatically transverse to (the strict transform of) $(z = 0)$, and the *old* exceptional divisors (the boundary above) will be moved away.

Remark A.18. Given a marked ideal $\underline{\mathcal{I}} = (Z, N, E, \mathcal{I}, d)$, set $E(\underline{\mathcal{I}}) := E$.

Again consider $\underline{\mathcal{I}} = \underline{\mathcal{I}}_X$. Let a denote a point in year zero. Write $E^1(a) = E(a)$. Resolution of singularities of the companion ideal $\underline{\mathcal{J}} = \underline{\mathcal{G}}(\underline{\mathcal{I}})$ at a provides a sequence of admissible blowings-up for $\underline{\mathcal{I}}$ over a. Consider a point b over a, in any year of the resolution history for $\underline{\mathcal{J}}$.

Suppose that $b \in$ cosupp $\underline{\mathcal{J}}$. Then $\nu_1(b) = \nu_1(a)$. Let $E^1(b)$ denote the set of transforms of elements of $E(a)$ at b (the "old exceptional divisors"). Note also that $\underline{\mathcal{J}}_\emptyset$ has accumulated a set of "new exceptional divisors" $E(b) \setminus E^1(b)$ at b. Moreover, $\underline{\mathcal{J}}_\emptyset$ has a maximal contact hypersurface at b, transformed from year zero, so transverse to the new exceptional divisors.

On the other hand, suppose that $b \notin$ cosupp $\underline{\mathcal{J}}$. Then $\nu_1(b) < \nu_1(a)$. When the order first drops (the "year of birth" of $\mathrm{inv}_{1/2} = \mathrm{inv}_{1/2}(b)$), we choose a new companion ideal $\underline{\mathcal{J}}$ and a new maximal contact hypersurface for $\underline{\mathcal{J}}_\emptyset$ at b, which need not be transverse to $E(b)$. Then we set $E^1(b) := E(b)$ and repeat the process above.

Then, at a point c in any year of the resolution history for $\underline{\mathcal{I}}$, the boundary in (A.8) is $E^1(c)$ and the coefficient ideal plus boundary is

$$\underline{\mathcal{C}}_z(\underline{\mathcal{J}}_\emptyset) + \sum_{H \in E^1(c)} (\mathcal{I}_H|_{(z=0)}, 1), \tag{A.9}$$

with $E(\underline{\mathcal{C}}_z(\underline{\mathcal{J}}_\emptyset)) = E(c) \setminus E^1(c)$. The companion ideal $\underline{\mathcal{J}}$ involved here is the transform of that which occurs first in the year of birth of $\mathrm{inv}_{1/2}(c) = \nu_1(c)$. The marked ideal $\underline{\mathcal{J}}_\emptyset$ at c is obtained simply by replacing $E(c)$ in $\underline{\mathcal{J}}$ by $E(c) \setminus E^1(c)$.

In the iterative construction of the invariant, the boundary phenomenon occurs on maximal contact subspaces of every codimension i. The boundary components added to the coefficient ideal on a maximal contact subspace of codimension i at a point a are the elements of $E^i(a)$; i.e., the components of the exceptional divisor counted by $s_i(a)$ (see Appendix A.2 and Remark A.21).

A.10. The desingularization invariant and an example computation

We begin with a definition of inv, in the general case.

Definition A.19 (*The desingularization invariant*)**.** We repeat the iterative scheme in Appendices A.7 and A.8 above, with the changes need to accommodate the boundary terms.

As in Appendix A.2, we assume, by induction, that inv has been defined up to year j (so that blowings-up have been determined, up to $\sigma_{j+1} : Z_{j+1} \to Z_j$). Let $\underline{\mathcal{I}}^0$ denote the transform in year $j+1$ of $\underline{\mathcal{I}}_X$ (see Example A.7). Consider $a \in \operatorname{cosupp} \underline{\mathcal{I}}^0$. Then $\nu_{\underline{\mathcal{I}}^0}(a) = \operatorname{ord}_a \mathcal{I}_X = \nu_1(a)$. We define $E^1(a)$ as in Appendix A.2 or Appendix A.9 and set $s_1(a) = \#E^1(a)$. We take:

$\underline{\mathcal{J}}^0 := \underline{\mathcal{G}}(\underline{\mathcal{I}}^0)$. Then $\underline{\mathcal{J}}^0$ is of maximal order. Let $P = V(z)$ be a maximal contact hypersurface for $\underline{\mathcal{J}}^0_\emptyset$ at a (see Remark A.18).

$\underline{\mathcal{I}}^1 :=$ the coefficient ideal plus boundary, i.e.,

$$\underline{\mathcal{I}}^1 := \underline{\mathcal{C}}_z(\underline{\mathcal{J}}^0_\emptyset) + \sum_{H \in E^1(a)} (\mathcal{I}_H|_{(z=0)}, 1),$$

as in (A.9).

We define

$$\nu_2(a) := \nu_{\underline{\mathcal{I}}_1}(a), \qquad s_2(a) = \#E^2(a),$$

with $E^2(a)$ as in Appendix A.2, and iterate the construction.

We finish when $\nu_{q+1}(a) = 0$ or ∞, as in Remark A.17.

Remark A.20. If $\nu_{q+1}(a) = \infty$, then the locus $\mathrm{inv} = \mathrm{inv}(a)$ is the maximal contact subspace of codimension q. The latter is simple normal crossings with the exceptional divisor (transverse if all $s_i(a) = 0$).

Remark A.21. In practical terms, $\underline{\mathcal{I}}^i$ lives on a maximal contact subspace of codimension i. To pass from $\underline{\mathcal{I}}^i$ to the companion ideal $\underline{\mathcal{J}}^i$, we use the factorization $\mathcal{I}^i = \mathcal{M}(\underline{\mathcal{I}}^i)\mathcal{R}(\underline{\mathcal{I}}^i)$ of Appendix A.6. At a point a, $\mathcal{M}(\underline{\mathcal{I}}^i)$ is a monomial in the exceptional divisors in $E(a) \setminus (E^1(a) \cup \cdots \cup E^i(a))$, which are transverse to N^i (the "new" exceptional divisors in codimension i). The "old" exceptional divisors in $E^i(a)$ are transformed from the year of birth of $\mathrm{inv}_{i-1/2} = \mathrm{inv}_{i-1/2}(a)$. They are counted by $s_i(a)$ rather than considered elements of $E(\underline{\mathcal{I}}^i)$.

Example A.22. We compute the blowings-up given by the desingularization algorithm for the pinch-point singularity, after the first blowing-up given in Example A.16. The following table provides the computations needed to find the invariant and the centre C of the blowing-up at the origins of the charts corresponding to the coordinate substitutions indicated. Note that the pinch-point singularity persists to year two. The strict transform of the pinch-point hypersurface in the year-one chart exhibited lies in the union of the two year-two charts shown. The calculations at a given point provide the next centre of blowing up over a neighbourhood of that point; globally, the maximum locus of the invariant will be blown up first.

In each subtable, the passage from $\underline{\mathcal{J}}^i$ to $\underline{\mathcal{I}}^{i+1}$ is given by taking the coefficient ideal plus boundary, on the maximal contact subspace of codimension $i+1$.

Codim i	Marked ideal $\underline{\mathcal{I}}^i$	Companion ideal $\underline{\mathcal{J}}^i = \underline{\mathcal{G}}(\underline{\mathcal{I}}^i)$	Maximal contact	Boundary E^i
Year one. Coordinate chart (x, xy, xz)				
0	$(x(z^2+xy^2), 1)$	$(z^2+xy^2, 2)$	$(z=0)$	
1	$(xy^2, 2)$	$(y^2, 2)$	$(z=y=0)$	$(x=0)$
2	$(x, 1)$	$(x, 1)$	$(z=y=x=0)$	
3	0			
$\mathrm{inv}(0)=(2,0,1,1,1,0,\infty)$, $C_1=\{0\}$				
Year two. Coordinate chart (x, xy, xz)				
0	$(x^2(z^2+xy^2), 1)$	$(z^2+xy^2, 2)$	$(z=0)$	
1	$(xy^2, 2)$	$(y^2, 2)$	$(z=y=0)$	
2	0			
$\mathrm{inv}(0)=(2,0,1,0,\infty)$, $C_2=(z=y=0)$				
Year two. Coordinate chart (xy, y, yz)				
0	$(xy^2(z^2+xy), 1)$	$(z^2+xy, 2)$	$(z=0)$	
1	$(xy, 2)$			
$\mathrm{inv}(0)=(2,0,0)$, $C_2=\{0\}$				

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