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A Description of the Brauer–Severi Scheme of Trace Rings of Generic Matrices

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Here we use invariant theory to describe the Brauer–Severi scheme of the fibers of trace rings of generic matrices with an algebraically closed base field of characteristic zero when the trace ring is viewed as a sheaf of algebras over the variety of matrix invariants. Using this approach, we first prove that the Brauer–Severi scheme of a trace ring is isomorphic to Proj Q, for a graded ring Q whose generators we describe in the first section. This description also has a relevant interpretation over base fields of arbitrary characteristic. In the second section of this paper we show that the Brauer–Severi scheme of the fiber of a trace ring over a point that is not too degenerate will have smooth irreducible components meeting transversally and describe these irreducible components as Brauer–Severi schemes of certain algebras. © 1996 Academic Press, Inc.

The Brauer–Severi variety has been useful in the study of central simple algebras. For example, Amitsur proved that the function field of the Brauer–Severi variety of a central simple algebra is a generic splitting field for that algebra and that if two central simple algebras have isomorphic Brauer–Severi varieties, their Brauer classes generate the same subgroup of the Brauer group [1]. Ideally, one could extend the concept of the Brauer–Severi variety to a more general context. One attempt to do so has been the definition of a Brauer–Severi scheme given by van den Bergh in [17]. Our present goal is to obtain a better understanding of the concepts that van den Bergh introduces in [17] by using invariant theory to describe some of these schemes. We will work over an algebraically closed base field, k.

The first section of this paper parallels the results given in [17], but these results are viewed from a slightly different perspective. We concen-

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trate on describing the Brauer–Severi scheme associated to n-dimensional representations (n a positive integer) of a finitely generated free noncommutative algebra over k, which we will denote by $F_m = k\{\mathscr{Y}_1, \dots, \mathscr{Y}_m\}$ when we have m generators. What is new in this first section is a description of graded k-algebras $Q_{m,n}$ such that the Brauer–Severi scheme associated to the n-dimensional representations of F_m is isomorphic to $\mathrm{Proj}(Q_{m,n})$.

In the second section, we start examining the local structure of the Brauer–Severi scheme. Here we assume that k is an algebraically closed field of characteristic zero and work with trace rings of generic matrices. We then specialize the trace rings and consider the Brauer–Severi schemes of these specializations. We call these specialized schemes local Brauer–Severi schemes. Then we are able to use the results of the first section as well as the Luna Slice Theorem [9, p. 97] to say something about the étale local structure of the local Brauer–Severi schemes associated to $\mathrm{Proj}(Q_{m,n})$. As the Luna Slice Theorem requires working over an algebraically closed field of characteristic zero, this section depends heavily on this assumption for k. Our major result in this section is Theorem 2.7 which says that under most conditions, the local Brauer–Severi scheme will have smooth irreducible components that meet transversally. Under these conditions, these irreducible components are described as Brauer–Severi schemes in their own right. In general, we give an upper bound on how many irreducible components a local Brauer–Severi scheme can have.

In the third section, we use an example to illustrate some of the concepts developed in Section 2. In particular, we see that when $m = 2$ and $n = 2$ the local structure of the Brauer–Severi scheme can degenerate from an irreducible, smooth conic in $\mathbb{P}^2_k$, to two projective lines intersecting transversally, then to a nonreduced structure on a single projective line. We show how this degeneration corresponds to the degeneration of two-dimensional representations of F_2.

We start by summarizing some of the definitions and results found in [17]. Let R be a commutative Noetherian k-algebra and let A be an R algebra that is not necessarily commutative. Let $B_n(A, R)$ denote the set of pairs (ϕ, P) such that P is a left A-module that is projective of rank n as an R-module and ϕ: $A \to P$ is a surjective left A-module homomorphism. Two pairs (ϕ, P) and (ψ, Q) in $B_n(A, R)$ are *equivalent* if there exists an R-module isomorphism u: $P \to Q$ such that $u \circ \phi = \psi$. If such an isomorphism exists, we write $(\phi, P) \sim (\psi, Q)$.

Let $Bsev_n(A, R)$ denote the set of equivalence classes of $\sim$ in $B_n(A, R)$. If S is any commutative R-algebra, $S \mapsto Bsev_n(A \otimes_R S, S)$ defines a functor from commutative R-algebras to sets which naturally extends to a functor on R-schemes. By [17, Prop. 2], since this functor is a closed subfunctor of the Grassmann functor, it is representable by an R-scheme. Therefore, we define the Brauer−Severi scheme of A over R of degree n (denoted $\mathrm{Bsev}_n(A, R)$) to be the R-scheme representing the functor $S \mapsto Bsev_n(A \otimes_R S, S)$.

In [17], van den Bergh provides us with an alternate characterization of the T-points of $\mathrm{Bsev}_n(A, R)$ for any k-algebra T. We find this characterization more conducive to the application of invariant theory and so we include this characterization below.

Let T be any k-algebra and let V, V' be locally free T-modules of rank n. Let $\phi \in \mathrm{Hom}_k(A, \mathrm{End}_T(V))$, $\phi' \in \mathrm{Hom}_k(A, \mathrm{End}_T(V'))$ with $\phi(R) \subseteq T \subseteq \mathrm{End}_T(V)$, $\phi'(R) \subseteq T \subseteq \mathrm{End}_T(V')$, and assume there exist $v \in V$ and $v' \in V'$ such that $\phi(A)Tv = V$ and $\phi'(A)Tv' = V'$. Then we say that the triples (ϕ, v, V) and (ϕ', v', V') are *equivalent* if there exists a k-module isomorphism $\alpha\colon V \to V'$ such that $\alpha(v) = v'$ and $\alpha \circ \phi(a) \circ \alpha^{-1} = \phi'(a)$ for all $a \in A$.

LEMMA 0.1 [17, Lemma 3]. *Let T be a k-algebra. Then the T-points of* $\mathrm{Bsev}_n(A, R)$ *are in one-to-one correspondence with equivalence classes of triples (ϕ, v, V) where V is a locally free T-module of rank n, $\phi \in$* $\mathrm{Hom}_k(A, \mathrm{End}_T(V))$ *such that $\phi(R) \subseteq T$, and $v \in V$ is such that $\phi(A)Tv = V$.*

By [17, Prop. 5], if A is any k-algebra that can be generated by m elements, then $\mathrm{Bsev}_n(A, k)$ can be embedded as a closed subscheme of $\mathrm{Bsev}_n(F_m, k)$ by fixing a surjection $F_m \to A$. Therefore, the study of $\mathrm{Bsev}_n(F_m, k)$ will help us develop a context in which to study other Brauer−Severi schemes.

Fix $m \geq 2$ and $n \geq 2$ and let $V = k^n$ with standard basis $\{e_1, \dots, e_n\}$ (so e_j is the vector with a jth component of 1 and all other components are zero). As all projective k-modules of rank n are isomorphic to V, it follows from Lemma 0.1 that the k-points of $\mathrm{Bsev}_n(F_m, k)$ are in one-to-one correspondence with equivalence classes of pairs (ϕ, v) where $\phi\colon F_m \to M_n(k)$ is a k-algebra homomorphism and $v \in V$ is such that $\phi(F_m)v = V$. We say a pair (ϕ, v) is *Brauer stable* if $\phi(F_m)v = V$. Note that Brauer stability is a property of equivalence classes in that if (ϕ, v) is Brauer stable, then so is any pair that is equivalent to (ϕ, v).

To classify these equivalence classes of Brauer stable pairs, we turn to invariant theory. Let $X_{m,n} = M_n(k) \oplus \cdots \oplus M_n(k)$ be the affine mn^2-dimensional space of m-tuples of $n \times n$ matrices over k. Then we can identify the representation $\phi\colon F_m \to M_n(k)$ with the m-tuple $(\phi(\mathscr{Y}_1), \dots, \phi(\mathscr{Y}_m)) \in X_{m,n}$. We define a $GL_n(k)$ action on $X_{m,n}$ by letting ϕ^g be the representation corresponding to $(g\phi(\mathscr{Y}_1)g^{-1}, \dots, g\phi(\mathscr{Y}_m)g^{-1})$

for any $\phi \in X_{m,n}$ and any $g \in GL_n(k)$. Then two Brauer stable pairs (ϕ, v), $(\psi, w) \in X_{m,n} \times V$ are equivalent if and only if there exists a $g \in GL_n(k)$ such that $(\phi^g, gv) = (\psi, w)$.

Now we categorize $\mathrm{Bsev}_n(F_m, k)$ as an appropriate $GL_n(k)$-quotient. One difficulty in creating such a quotient is that the orbits of the Brauer stable points are not closed in $X_{m,n} \times V$. Note that if (ϕ, v) is Brauer stable, then the point $(\phi, 0)$ is not Brauer stable yet it is in the closure of the $GL_n(k)$ orbit of (ϕ, v). Indeed, $\lim_{\lambda \to 0} \lambda 1_n \cdot (\phi, v) = (\phi, 0)$ for any $(\phi, v) \in X_{m,n} \times V$. Therefore if $f \in k[X_{m,n} \times V]$ is a $GL_n(k)$-invariant (i.e., f is constant on $GL_n(k)$ orbits), $f(\phi, v) = f(\phi, 0)$ for any $(\phi, v) \in X_{m,n} \times V$, since f is continuous in both the Zariski and analytic topologies. Therefore, we get an isomorphism between $k[X_{m,n}]^{GL_n(k)}$, the subring of $GL_n(k)$-invariants in $k[X_{m,n}]$, and $k[X_{m,n} \times V]^{GL_n(k)}$, the subring of $GL_n(k)$-invariants in $k[X_{m,n} \times V]$. By a theorem of Nagata's (e.g., [11, Theorem 3.4]), these invariant rings are affine so we get an isomorphism of varieties $(X_{m,n} \times V)//GL_n(k) = \mathrm{Spec}(k[X_{m,n} \times V]^{GL_n(k)}) \cong \mathrm{Spec}(k[X_{m,n}]^{GL_n(k)}) = X_{m,n}//GL_n(k)$, the latter quotient being the variety of invariants of m-tuples of $n \times n$ matrices.

In some sense, the above paragraph exemplifies the only difficulty in forming an appropriate quotient of Brauer stable points. We can get around this difficulty by projectivizing V and considering a $PGL_n(k) = GL_n(k)/(k^\times \cdot 1_n)$ action on the resulting Cartesian product. In particular, if λ is in the center of $GL_n(k)$, then $\lambda \cdot (\phi, v) = (\phi, \lambda \cdot v)$ for all $(\phi, v) \in X_{m,n} \times V$. Therefore, under the induced action of $GL_n(k)$, the center of $GL_n(k)$ acts trivially on $Y_{m,n} = X_{m,n} \times \mathbb{P}(V)$ where we use $\mathbb{P}(V)$ to denote the projective $n-1$ space formed by V. So a $PGL_n(k)$ action is well defined on $Y_{m,n}$. We will say that $(\phi, kv) \in Y_{m,n}$ is Brauer stable if $(\phi, v) \in X_{m,n} \times V$ is Brauer stable. Let $B_{m,n}$ denote the set of Brauer stable points in $Y_{m,n}$. As before, $\mathrm{Bsev}_n(F_m, k)$ is the scheme of all $PGL_n(k)$ orbits of Brauer stable points in $Y_{m,n}$.

To form an appropriate $PGL_n(k)$ quotient of $Y_{m,n}$, we use the theory developed in [6]. Let $S_{m,n} = k[x_{i,j}^{(l)} | 1 \le i, j \le n, 1 \le l \le m]$ be the affine coordinate ring of $X_{m,n}$ and let $k[V] = k[y_1, \ldots, y_n]$ denote the affine coordinate ring of V. Let $\Sigma_{m,n} = (S_{m,n})[y_1, \ldots, y_n]$ be the affine coordinate ring of $X_{m,n} \times V$ and grade $\Sigma_{m,n}$ according to the degrees of the y_i's. Then we can write $X_{m,n} \times \mathbb{P}(V) = \mathrm{Proj}(\Sigma_{m,n})$. As the $GL_n(k)$ action induced on $\Sigma_{m,n}$ preserves degree, a natural quotient variety to look at would be $\mathrm{Proj}((\Sigma_{m,n})^\sigma)$, where $(\Sigma_{m,n})^\sigma$ denotes the subring of $\Sigma_{m,n}$ generated by the semi-invariant functions of $\Sigma_{m,n}$. In this context, we call a function $f \in \Sigma_{m,n}$ a semi-invariant if f is homogeneous and its zero set in $Y_{m,n}$ is $PGL_n(k)$ stable. This is equivalent to saying that $f \in \Sigma_{m,n}$ is semi-invariant if f is homogeneous and for all $(\phi, v) \in X_{m,n} \times V$, $f(\phi^g, gv) = \lambda f(\phi, v)$ for all $g \in GL_n(k)$ and for some $\lambda \in k^\times$. We will see that when k is a field of characteristic zero, $\mathrm{Proj}((\Sigma_{m,n})^{\sigma})$ is a quotient of Brauer stable points and is isomorphic to $\mathrm{Bsev}_n(F_m, k)$. In order to generalize to nonzero characteristics, it is necessary to consider the Proj of a subring of $(\Sigma_{m,n})^{\sigma}$.

We conclude our introduction with a formulation of Cramer's Rule that we will need later on. First, given a set of n vectors $v_1, \ldots, v_n \in V$ we define their *bracket*, $[v_1, \ldots, v_n]$, to be the determinant of the $n \times n$ matrix whose ith column is v_i. More explicitly, $[v_1, \ldots, v_n] = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) v^{(1)}_{\sigma(1)} \cdots v^{(n)}_{\sigma(n)}$, where S_n denotes the symmetric group on n letters and $v^{(j)}_i$ is the ith coordinate of v_j. We state the following version of Cramer's Rule in terms of these brackets.

LEMMA 0.2. *Let $\{v_1, \ldots, v_n\}$ be a basis for V over k. Then $y = \sum_{i=1}^{n} \alpha_i v_i$ if and only if $[v_1, \ldots, v_{j-1}, y - \alpha_j v_j, v_{j+1}, \ldots, v_n] = 0$ for all $1 \leq j \leq n$.*

Proof. If $y = \sum_{i=1}^{n} \alpha_i v_i$, then $y - \alpha_j v_j \in \mathrm{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$. Therefore $[v_1, \ldots, v_{j-1}, y - \alpha_j v_j, v_{j+1}, \ldots, v_n] = 0$ for all $1 \leq j \leq n$.

Conversely, assume that $[v_1, \ldots, v_{j-1}, y - \alpha_j v_j, v_{j+1}, \ldots, v_n] = 0$ for all $1 \leq j \leq n$. Then $y - \alpha_j v_j \in \mathrm{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$ for all $1 \leq j \leq n$. Now, by a simple induction argument, $y - \sum_{j=1}^{n} \alpha_j v_j \in \bigcap_{j=1}^{n} \mathrm{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} = \{0\}$. Therefore, our result is proven. Q.E.D.

1. A CHARACTERIZATION OF $\mathrm{Bsev}_n(F_m, k)$

We have seen that the $GL_n(k)$-action defined on $X_{m,n} = M_n(k) \oplus \cdots \oplus M_n(k)$ induces a $PGL_n(k)$-action on $X_{m,n}$ as the center of $GL_n(k)$ acts trivially on $X_{m,n}$. This induces a corresponding action on $S_{m,n} = k[X_{m,n}]$ and we will denote the ring of $PGL_n(k)$ invariant functions under this action by $C = (S_{m,n})^{PGL_n(k)}$. Let $\mathbb{T}_{m,n}$ denote the set of all $PGL_n(k)$ equivariant polynomial functions $f\colon X_{m,n} \to M_n(k)$ where $PGL_n(k)$ acts on $M_n(k)$ via conjugation by a $GL_n(k)$ representative. The ring $\mathbb{T}_{m,n}$ is called the *trace ring of m generic $n \times n$ matrices*.

We make the following observations about $\mathbb{T}_{m,n}$. First, it is easy to see that C is the center of $\mathbb{T}_{m,n}$. Next, $\mathbb{T}_{m,n}$ contains the projection functions $X_l\colon X_{m,n} \to M_n(k)$ given by $X_l(A_1, \ldots, A_m) = A_l$. By identifying $\mathrm{Mor}_k(X_{m,n}, M_n(k))$ with $M_n(S_{m,n})$, we can view $\mathbb{T}_{m,n}$ as the subring of invariants of the induced $PGL_n(k)$-action on $M_n(S_{m,n})$. If $P = [p_{i,j}] \in M_n(S_{m,n})$ and $g \in GL_n(k)$ is a representative of $\bar{g} \in PGL_n(k)$, the induced action on $M_n(S_{m,n})$ is given by $P^{\bar{g}} = g[\bar{g}^{-1} \cdot p_{i,j}]g^{-1}$, where for any $1 \leq i, j \leq n$, $\bar{g} \cdot p_{i,j}$ denotes the image of $p_{i,j}$ in $S_{m,n}$ under the action of $\bar{g}$. Under this identification, X_l is the element of $M_n(S_{m,n})$ whose i, j entry is $x_{i,j}^{(l)}$. We call X_l the *lth generic* $n \times n$ *matrix* and the k subalgebra generated by the set $\{X_l | 1 \le l \le m\}$ is the *ring of m generic* $n \times n$ *matrices*, which we denote as $\mathbb{G}_{m,n}$.

Now any homomorphism ϕ: $F_m \to M_n(k)$ induces a unique homomorphism φ: $S_{m,n} \to k$ given by sending $x_{i,j}^{(l)}$ to the i, jth entry of $\phi(\mathscr{Y}_l)$. Note that $\ker(\varphi)$ is the maximal ideal of $S_{m,n}$ corresponding to $\phi \in X_{m,n}$. Then there exists a unique homomorphism $M_n(\varphi)$: $M_n(S_{m,n}) \to M_n(k)$ that restricts to a homomorphism $\tilde{\phi}$: $\mathbb{T}_{m,n} \to M_n(k)$ such that $\tilde{\phi}(X_l) = \phi(\mathscr{Y}_l)$ for all l. This last statement implies that the map $\phi \mapsto \tilde{\phi}$ is injective. For ease of notation, we will then identify ϕ with its corresponding $\tilde{\phi}$ on $\mathbb{T}_{m,n}$.

A more precise relationship between $\tilde{\phi}$ and ϕ is known when we restrict k to be a field of characteristic zero. In this case, [13, Theorem 2.1] tells us that $\mathbb{T}_{m,n}$ is actually generated as an algebra by C and the X_l's. It then follows from [12] that $\mathbb{T}_{m,n}$ is a universal object in a subcategory of algebras with trace. We will exploit this further a little later on.

DEFINITION 1.1. For any $W_1, \ldots, W_n \in \mathbb{T}_{m,n}$, define $[W_1, \ldots, W_n] \in \Sigma_{m,n} = k[X_{m,n} \times V]$ to be the function

$$\begin{aligned} [W_1, \ldots, W_n]:\ & X_{m,n} \times V \to k \\ & (\phi, v) \mapsto [\phi(W_1)v, \ldots, \phi(W_n)v]. \end{aligned}$$

Note that for any $W_1, \ldots, W_n \in \mathbb{T}_{m,n}$, the function $[W_1, \ldots, W_n]$ is a homogeneous element of $\Sigma_{m,n}$ that is a semi-invariant. Indeed,

$$[W_1, \ldots, W_n](\phi^g, gv) = \det(g)[W_1, \ldots, W_n](\phi, v) \tag{1}$$

for any $g \in GL_n(k)$, $(\phi, v) \in X_{m,n} \times V$.

Also, the pair (ϕ, kv) is Brauer stable if and only if there exist $W_1, \ldots, W_n \in \mathbb{T}_{m,n}$ such that $[W_1, \ldots, W_n](\phi, v) \neq 0$. Indeed, if $[W_1, \ldots, W_n](\phi, v) \neq 0$, then $\{\phi(W_1)v, \ldots, \phi(W_n)v\}$ is a basis for V over k, so $\phi(F_m)v = V$. The converse is essentially the reverse of the above argument. Therefore, Brauer stability is an open condition and so $B_{m,n}$ is an open subvariety of $Y_{m,n}$.

LEMMA 1.2. *Let (ϕ, kv) be a closed point in $B_{m,n}$, the open subscheme of Brauer stable points in $Y_{m,n} = \mathrm{Proj}(\Sigma_{m,n})$. Then the stabilizer of (ϕ, kv) is trivial and the $PGL_n(k)$-orbit of (ϕ, kv) is closed in $B_{m,n}$.*

Proof. Let $(\phi, kv) \in B_{m,n}$ and assume $(\phi^g, kgv) = (\phi, kv)$ for some $g \in GL_n(k)$. Then for any $w \in V$, there exists an $r \in \mathbb{T}_{m,n}$ such that $\phi(r)v = w$. So $gw = g\phi(r)v = \phi(r)gv$ as $\phi^g = \phi$. But $gv = \lambda v$ for some $\lambda \in k^\times$, so $gw = \phi(r)gv = \lambda\phi(r)v = \lambda w$. Hence $g = \lambda 1_n$ and thus represents the trivial element in $PGL_n(k)$. Therefore $PGL_n(k)$ acts freely on $B_{m,n}$.

Since the $PGL_n(k)$-stabilizer of all $(\phi, kv) \in B_{m,n}$ is trivial, the dimension of any $PGL_n(k)$-orbit in $B_{m,n}$ is $n^2 - 1$. Therefore, by [11, Lemma 3.7], the $PGL_n(k)$-orbit of any $(\phi, kv) \in B_{m,n}$ is closed in $B_{m,n}$. Q.E.D.

Since the orbits of the Brauer stable points have closed orbits in $B_{m,n}$, we have some hope that the orbits could be distinguishable by semi-invariant functions. The following theorem indicates that we only need the semi-invariants of the form given in 1.1 to distinguish these orbits.

THEOREM 1.3. *Assume that* (ϕ, kv) *and* (ϕ', kw) *are Brauer stable. Then the following are equivalent*:

A. (ϕ, kv) *is equivalent to* (ϕ', kw);

B. $[W_1, \dots, W_n](\phi, kv) = 0$ *if and only if* $[W_1, \dots, W_n](\phi', kw) = 0$ *for all* $W_1, \dots, W_n \in \mathbb{T}_{m,n}$.

Proof. Let (ϕ, kv) and (ϕ', kw) be Brauer stable points of $Y_{m,n}$. Note that (A) implies (B) follows directly from Eq. (1).

Next, we assume condition (B). As (ϕ, v) is Brauer stable, then there exists $H_1, \dots, H_n \in \mathbb{T}_{m,n}$ such that $\{\phi(H_1)v, \dots, \phi(H_n)v\}$ is a basis of V and so $[H_1, \dots, H_n](\phi, kv) \neq 0$. By assumption, this implies that $[H_1, \dots, H_n](\phi', w) \neq 0$ and thus $\{\phi'(H_1)w, \dots, \phi'(H_n)w\}$ is also a basis for V over k.

Let $v_i = \phi(H_i)v$, $w_i = \phi'(H_i)w$ for all $1 \leq i \leq n$, and let $a\colon V \to V$ be the automorphism defined by $a(v_i) = w_i$ for all $1 \leq i \leq n$. Then we claim that $\phi^a = \phi'$ and $a(v) = w$.

First we show that for any $r \in \mathbb{T}_{m,n}$ that $a\phi(r)a^{-1} = \phi'(r)$. Given $r \in \mathbb{T}_{m,n}$, for any $1 \leq j \leq n$ there exist $\alpha_1, \dots, \alpha_n \in k$ such that $\phi(r)v_j = \sum_{i=1}^n \alpha_i v_i$. So, by Lemma 0.2, $[v_1, \dots, v_{t-1}, \phi(r)v_j - \alpha_t v_t, v_{t+1}, \dots, v_n] = 0$ for all $1 \leq t \leq n$. By assumption, this implies $[w_1, \dots, w_{t-1}, \phi'(r)w_j - \alpha_t w_t, w_{t+1}, \dots, w_n] = 0$ for all $1 \leq t \leq n$. Thus, by Lemma 0.2, $\phi'(r)w_j = \sum_{i=1}^n \alpha_i w_i$. Also, $a\phi(r)a^{-1}w_j = a\phi(r)v_j = a(\sum_{i=1}^n \alpha_i v_i) = \sum_{i=1}^n \alpha_i w_i$. Therefore, for any $r \in \mathbb{T}_{m,n}$, the transformations $\phi'(r)$ and $a\phi(r)a^{-1}$ agree on each of the basis elements, w_j, and so are equal. In other words, $\phi^a = \phi'$.

To complete our proof of the above claim, it is sufficient to show that $av = w$. If we write $v = \sum_{i=1}^n \beta_i v_i$, then $[v_1, \dots, v_{j-1}, v - \beta_j v_j, v_{j+1}, \dots, v_n] = 0$ for all $1 \leq j \leq n$. By assumption, $[w_1, \dots, w_{j-1}, w - \beta_j w_j, w_{j+1}, \dots, w_n] = 0$ for all $1 \leq j \leq n$. So $w = \sum_{i=1}^n \beta_i w_i$ and $av = \sum_{i=1}^n \beta_i a v_i = \sum_{i=1}^n \beta_i w_i = w$. Therefore we have proven our claim and so it follows that $a \cdot (\phi, kv) = (\phi', kw)$. Q.E.D.

So the semi-invariants functions $[W_1, \dots, W_n]$ are sufficient to separate the $PGL_n(k)$-orbits of Brauer stable points. Let Q denote the C-subalgebra of $\Sigma_{m,n}$ generated by the $[W_1, \dots, W_n]$. As elements of $\Sigma_{m,n}$, the semi-invariant functions $[W_1, \dots, W_n]$ are homogeneous of degree n. As Q

is generated by the $[W_1, \ldots, W_n]$, any element of Q that is homogeneous in $\Sigma_{m,n}$ will have a degree in $\Sigma_{m,n}$ that is an integer multiple of n. To simplify our later work, we define a grading on Q that eliminates this extra factor of n. More specifically, we say an element of Q is homogeneous of degree q if the element is homogeneous of degree nq in $\Sigma_{m,n}$.

THEOREM 1.4 [13, Theorem 12.1]. *Let* $\mathrm{char}(k) = 0$. *Then the ring of* $SL_n(k)$ *invariant functions of* $\Sigma_{m,n} = k[X_{m,n} \times V]$ *is the C-algebra generated by the functions* $[W_1, \ldots, W_n]$ *where* $W_1, \ldots, W_n \in \mathbb{T}_{m,n}$. *Hence* $(\Sigma_{m,n})^{SL_n(k)} = Q$.

Note. The theorem we state here is a special case of the one stated in [13]. The original statement gives the generators of the invariants of m matrices, p vectors, and q covectors. Here, we are only interested in the case when $p = 1$ and $q = 0$.

COROLLARY 1.5. *If* $\mathrm{char}(k) = 0$ *then* $Q = (\Sigma_{m,n})^{\sigma}$.

Proof. By definition of Q, it is clear that $Q \subseteq (\Sigma_{m,n})^{\sigma}$. Let $f \in (\Sigma_{m,n})^{\sigma}$. Then for any $y \in Y_{m,n}$ such that $f(y) \neq 0$, we have $g \mapsto f(gy)/f(y)$ defining a homomorphism from $SL_n(k)$ to $k^{\times}$. Since $SL_n(k)$ has no nontrivial characters, this map must be trivial. Therefore $f(gy) = f(y)$ for all $y \in Y_{m,n}$ and for all $g \in SL_n(k)$. Hence, $f \in Q = (\Sigma_{m,n})^{SL_n(k)}$ where $Q = (\Sigma_{m,n})^{SL_n(k)}$ by Theorem 1.4. So $Q = (\Sigma_{m,n})^{\sigma}$ as claimed. Q.E.D.

When k has positive characteristic it is unclear to me whether this equality between Q and $(\Sigma_{m,n})^{\sigma}$ still holds. But for the calculation of Brauer–Severi schemes, this equality is not necessary.

In [15, Theorem 2.2], Saltman proves that the function field of the Brauer–Severi variety associated to the central quotient ring of $\mathbb{T}_{m,n}$ is a rational extension of k. In [17], van den Bergh gives another proof of Saltman's result by proving the equivalent statement that $\mathrm{Bsev}_n(F_m, k)$ is rational. Van den Bergh does this by showing that $\mathrm{Bsev}_n(F_m, k)$ is covered by a finite number of open affine sets, each open set being isomorphic to an affine space of dimension $(m-1)n^2 + n$ [17]. Using the techniques of Van den Bergh's proof we are able to show that $\mathrm{Bsev}_n(F_m, k) = \mathrm{Proj}(Q)$ where Q is graded as above.

We start by defining special sequences and their associated functions as introduced in [17]. Let M be a sequence of ordered integer pairs (α_j, β_j) for $2 \leq j \leq n$. We say that M is an m, n-*special sequence* if $1 \leq \beta_j \leq m$, $1 \leq \alpha_j < j$ for all $2 \leq j \leq n$ and if $j \neq j'$ then $(\alpha_j, \beta_j) \neq (\alpha_{j'}, \beta_{j'})$. When m and n are understood, we just say the sequence is special. Note that for a given m and n, the set of (m, n)-special sequences are finite as any such sequence forms a subset of order $n - 1$ of the finite set $\{(\alpha, \beta) \in \mathbb{N}^2 | 1 \leq \alpha \leq n, 1 \leq \beta \leq m\}$.

Let $M = \{(\alpha_j, \beta_j)\}$ be a special sequence. Then we can inductively define an associated sequence of monomials (of the generic matrices X_l) $H_1^{(M)}, \dots, H_n^{(M)} \in \mathbb{T}_{m,n}$ by letting $H_1^{(M)} = 1$ and $H_j^{(M)} = X_{\beta_j} H_{\alpha_j}^{(M)}$ for $2 \le j \le n$. Let $h_M = [H_1^{(M)}, \dots, H_n^{(M)}] \in Q$. Then h_M is a function of the type given in Definition 1.1, thus is homogeneous of degree one in Q.

Let U_M be the affine open subvariety $\operatorname{Spec}((\Sigma_{m,n})_{((h_M))})$ of $Y_{m,n} = \operatorname{Proj}(\Sigma_{m,n})$, where we use $(\Sigma_{m,n})_{((h_M))}$ to denote the elements of degree zero in the one element localization $(\Sigma_{m,n})_{h_M}$. Then U_M is the subvariety of $B_{m,n}$ given by $\{(\phi, kv) | h_M(\phi, kv) \neq 0\}$. As h_M is a semi-invariant, it is clear that U_M is a $PGL_n(k)$-stable subvariety of $B_{m,n}$. Note that U_M is nonempty as $h_M(\phi, ke_1) \neq 0$ if the α_jth column of $\phi(X_{\beta_j})$ is e_j for all j.

The following lemma is stated in [17] without proof, so we include a proof for the reader's convenience. The proof of this lemma provides some motivation for the nature of the definition of a special sequence.

LEMMA 1.6 [17, p. 336]. *The set $\{U_M | M$ is (m,n)-special$\}$ forms a finite affine open cover of $B_{m,n}$.*

Proof. From our above discussion it is clear that the set $\{U_M | M$ is (m,n)-special$\}$ is a finite collection of open affine subsets of $B_{m,n}$. So it suffices to show that the U_M cover $B_{m,n}$.

Let $(\phi, kv) \in B_{m,n}$, then $\phi(F_m)v = V$ so $v \neq 0 \in V$. If $\phi(X_l) \in kv$ for all $1 \le l \le m$, then $\dim_k(\phi(F_m)v) = 1$, contradicting the condition of Brauer stability. So there exists a β_2 such that $\phi(X_{\beta_2})v \notin \operatorname{span}\{v\}$. Let $\alpha_2 = 1$. (Note that for any special sequence $\alpha_2 = 1$ by definition.)

Now assume that for a given $3 \le j \le n$ that there exists an $(m, j-1)$-special sequence $(\alpha_2, \beta_2), \dots, (\alpha_{j-1}, \beta_{j-1})$ satisfying the following condition:

> If $H_1 = 1$ and $H_i = X_{\beta_i} H_{\alpha_i}$ for all $1 \le i \le j-1$, then the set $\{\phi(H_1)v, \dots, \phi(H_{j-1})v\}$ is an independent subset of V.

If $\phi(X_l)\phi(H_i)v \in \operatorname{span}\{v = \phi(H_1)v, \phi(H_2)v, \dots, \phi(H_{j-1})v\}$ for all $1 \le i \le j-1$ and for all $1 \le l \le m$, then $\phi(F_m)v \subseteq \operatorname{span}\{\phi(H_1)v, \dots, \phi(H_{j-1})v\}$, contradicting the Brauer stability of (ϕ, kv). Therefore, there exists a $1 \le \alpha_j \le j-1$ and a $1 \le \beta_j \le m$ such that $\phi(X_{\beta_j})\phi(H_{\alpha_j})v \notin \operatorname{span}\{\phi(H_1)v, \dots, \phi(H_{j-1})v\}$. It then follows that $(\alpha_j, \beta_j) \neq (\alpha_i, \beta_i)$ for any $i < j$ and so we have constructed an (m,j)-special sequence such that $\{\phi(H_1)v, \dots, \phi(H_j)v\}$ is a linearly independent subset of V, where $H_j = X_{\beta_j} H_{\alpha_j}$. Therefore, we can construct an (m,n)-special sequence M such that $\{\phi(H_1^{(M)})v, \dots, \phi(H_n^{(M)})v\}$ is an independent subset of V and thus a basis of V. So

$$0 \neq \left[H_1^{(M)}, \dots, H_n^{(M)}\right](\phi, kv) = h_M(\phi, kv) \tag{2}$$

which implies $(\phi, kv) \in U_M$. Q.E.D.

Consider the quotient variety for each M given by $V_M = \mathrm{Spec}(((\Sigma_{m,n})_{((h_M))})^{\sigma}) = \mathrm{Spec}(((\Sigma_{m,n})_{((h_M))})^{PGL_n(k)})$. (Note that if $f \in (\Sigma_{m,n})_{((h_M))}$, f is homogeneous of degree zero so $f^{\lambda 1_n} = f$ for all $\lambda \in k$. Therefore, the $GL_n(k)$-action on $\Sigma_{m,n}$ induces a $PGL_n(k)$ action on $(\Sigma_{m,n})_{((h_M))}$. Furthermore, since $PGL_n(k)$ has no nontrivial characters, any semi-invariant of $(\Sigma_{m,n})_{((h_M))}$ must also be a $PGL_n(k)$ invariant.) Each of these quotients is endowed with a quotient morphism $\pi_M \colon U_M \to V_M$ induced by the inclusion $((\Sigma_{m,n})_{((h_M))})^{PGL_n(k)} \hookrightarrow (\Sigma_{m,n})_{((h_M))}$. Therefore, we can glue these quotient morphisms together and form a $PGL_n(k)$ quotient of $B_{m,n}$, which we will denote by $BS_{m,n}$.

THEOREM 1.7 [17, Theorem 6]. *If M is a special sequence, then V_M is an affine space of dimension $(m-1)n^2 + n$. Furthermore, the set $\{V_M | M$ is special$\}$ is a finite open affine covering of* $\mathrm{Bsev}_n(F_m, k)$, *hence* $\mathrm{Bsev}_n(F_m, k) \cong BS_{m,n}$.

COROLLARY 1.8. *The k-scheme $BS_{m,n}$ is a smooth, rational quotient of a free $PGL_n(k)$ action on $B_{m,n}$.*

Proof. The rationality and smoothness of $BS_{m,n}$ follow directly from Theorem 1.7 and the freeness of the $PGL_n(k)$ action follows from Lemma 1.2. Q.E.D.

Although we do not know in general (i.e., in non-zero characteristic) whether Q and $(\Sigma_{m,n})^{\sigma}$ are equal, we can show that $Q_{((h_M))} = (\Sigma_{m,n})^{\sigma}_{((h_M))}$ for every special sequence M. This allows us to show that $\mathrm{Proj}(Q) = BS_{m,n}$ by embedding the V_M as open subvarieties of $\mathrm{Proj}(Q)$ and thus showing equality locally. So let M be a special sequence and let

$$t_{i,j}^{(l)}[M] = \left[H_1^{(M)}, \dots, H_{i-1}^{(M)}, X_l H_j^{(M)}, H_{i+1}^{(M)}, \dots, H_n^{(M)}\right]. \qquad (3)$$

Let $T(M) = \{t_{i,j}^{(l)}[M] \| 1 \le i,\ j \le n,\ 1 \le l \le m,\ (j,l) \ne (\alpha_r, \beta_r)$ for any $2 \le r \le n\}$. We can use the arguments on [17, p. 336] almost verbatim to prove the following theorem, which in turn proves the claim in Theorem 1.7 that the V_M are affine spaces of the appropriate dimension. In the proof we present here, we translate these arguments into the language we have developed in this paper and include a little more detail when we thought it might clarify the proof.

THEOREM 1.9. *For any special sequence M, the set $T(M) \cup \{h_M\}$ is an algebraically independent subset of Q, hence $k[V_M] = k[t h_M^{-1} | t \in T(M)] = Q_{((h_M))}$.*

Proof. Let $M = \{(\alpha_j, \beta_j)\}_{j=2}^{n}$ be a special sequence and let $(\phi, kv) \in U_M$. Then $0 \ne h_M(\phi, v) = [\phi(H_1^{(M)})v, \dots, \phi(H_n^{(M)})v]$ implies that $\{\phi(H_1^{(M)})v, \dots, \phi(H_n^{(M)})v\}$ forms a basis of V. Hence there exists a unique

$g \in GL_n(k)$ such that $(\phi^g(H_j^{(M)}))(gv) = g\phi(H_j^{(M)})v = e_j$ for all $1 \le j \le n$, where e_j denotes the jth standard basis element of $V = k^n$ given by zeros in all components except for a 1 in the jth component. For a given $(\phi, v) \in U_M$, let us denote this unique $g \in GL_n(k)$ by $g_{\phi, v}$. If $(\phi, kv) = (\phi, kw)$, then $g_{\phi, v} = \lambda g_{\phi, w}$ for some nonzero $\lambda \in k$. Therefore, for every element $(\phi, kv) \in U_M$, we can associate a unique $\bar{g}_{\phi, kv} \in PGL_n(k)$.

Let $V'_M = \{(\phi, kv) \in U_M | \bar{g}_{\phi, kv} = 1\} = \{(\phi, kv)|$ there exists a $\lambda \in k^\times$ such that $\phi(H_j^{(M)})v = \lambda e_j$ for all $j\}$. Then V'_M is defined as a closed subscheme of U_M by the radical of the ideal in $(\Sigma_{m,n})_{((h_M))}$ generated by the set $\{x_{i,\alpha_r}^{(\beta_r)} - \delta_{i,r} | 2 \le r \le n,\ 1 \le i \le n\} \cup \{y_1^{a_1} \cdots y_n^{a_n} h_M^{-1} | a_1 + \cdots + a_n = n,\ a_i \ge 0$ for all $1 \le i \le n,\ a_1 < n\}$. Hence V'_M is the subvariety of U_M whose closed points are given by $\{(\phi, kv) | kv = ke_1$ and $\phi(X_{\beta_j})e_{\alpha_j} = e_j$ for all $2 \le j \le n\}$.

Now define a morphism $\Psi\colon U_M \to V'_M$ given by $\Psi(\phi, kv) = (\phi_\psi, ke_1) = \bar{g}_{\phi, kv}(\phi, kv)$ where $\phi_\psi(X_l)$ is the $n \times n$ matrix whose i, j entry is given by $t_{i,j}^{(l)}[M](\phi, v)h_M(\phi, v)^{-1}$. Note that $(\phi_\psi, ke_1) \in V'_M$ since $t_{i,\alpha_j}^{\beta_j}[M](\phi, kv)$ $h_M(\phi, kv)^{-1}$ is zero whenever $i \ne j$ and is one when $i = j$. Therefore, $\phi_\psi(X_{\beta_j})e_{\alpha_j} = e_j$ for all j.

So $U_M \cong V'_M \times PGL_n(k)$ where the isomorphism is given by the map $(\phi, kv) \mapsto ((\phi_\psi, ke_1), \bar{g}_{\phi, kv})$. Note this is a $PGL_n(k)$ equivariant map where $PGL_n(k)$ acts on $V'_M \times PGL_n(k)$ by acting trivially on the first component and both transitively and freely on the second component. Hence, $V_M \cong V'_M$.

We can embed V'_M into V^{mn+1} by

$$(\phi, kv) \mapsto (\phi(X_1)e_1, \ldots, \phi(X_1)e_n, \ldots, \phi(X_m)e_1, \ldots, \phi(X_m)e_n). \quad (4)$$

Then V'_M is isomorphic to the affine subvariety of V^{mn} given by $\{(v_1, \ldots, v_{mn}) | v_{n(\beta_j - 1) + \alpha_j} = e_j$ for all $j\}$. Therefore, V'_M is an affine space whose affine coordinate ring is a polynomial ring in the coordinates given by th_M^{-1} for $t \in T(M)$. Hence, $k[V_M]$ is a polynomial ring in the corresponding functions. Since $th_M^{-1} \in Q_{((h_M))}$ for all $t \in T(M)$, we get $k[V_M] \subseteq Q_{((h_M))} \subseteq ((\Sigma_{m,n})_{((h_M))})^{PGL_n(k)} = k[V_M]$.

Finally, since $\{th_M^{-1} | t \in T(M)\}$ is algebraically independent in $k[V_M]$, it follows that $T(M) \cup \{h_M\}$ is algebraically independent in Q. As the dimension of $BS_{m,n}$ is $(m-1)n^2 + n$, the set $T(M) \cup \{h_M\}$ is a maximal algebraically independent set. Q.E.D.

Note that we have also shown in the proof of Theorem 1.9 the following corollary.

COROLLARY 1.10. *For any special sequence M, U_M is $PGL_n(k)$ isomorphic to $V'_M \times PGL_n(k)$ where V'_M is an affine space of dimension $(m-1)n^2 + n$ and $PGL_n(k)$ acts on $PGL_n(k) \times V'_M$ by acting trivially on V'_M and by left translation on the $PGL_n(k)$ factor.*

Finally, we get the equality we claimed in our discussion preceding Theorem 1.9.

COROLLARY 1.11. $\operatorname{Proj}(Q) = BS_{m,n} \cong \operatorname{Bsev}_n(F_m, k)$.

Proof. First we note that since the set $\{U_M | M \text{ is special}\}$ forms an affine open cover of $B_{m,n}$, it follows that $\{V_M | M \text{ is special}\}$ is an affine open cover of $\operatorname{Proj}(Q)$. Then the corollary follows directly from Theorem 1.9, the definition of $BS_{m,n}$ and Theorem 1.7. Q.E.D.

2. BRAUER–SEVERI SCHEMES OF SPECIALIZATIONS OF TRACE RINGS

For the rest of this paper we will assume that k is a field of characteristic zero. Under this restriction on k our results become more complete. As mentioned earlier, in this case the trace ring $\mathbb{T}_{m,n}$ is generated by its center C and the ring of m generic $n \times n$ matrices $\mathbb{G}_{m,n}$ as a k algebra. Then [12, Theorem 3] tells us that $\mathbb{T}_{m,n}$ is a free object in $\mathscr{T}_n$, the category of k-algebras with trace satisfying the nth Cayley–Hamilton identity. We refer the reader to [12] for the formal definition of $\mathscr{T}_n$. If A is an object in $\mathscr{T}_n$, then any k-algebra homomorphism ϕ: $F_m \to A$ induces a unique trace-preserving algebra homomorphism $\bar{\phi}$: $\mathbb{T}_{m,n} \to A$ such that $\phi(\mathscr{Y}_i) = \bar{\phi}(X_i)$ for all $1 \le i \le m$.

When working with $\mathbb{T}_{m,n}$, we will usually work in the category $\mathscr{T}_n$. In this category, there is a corresponding notion of a Brauer–Severi scheme (also introduced in [17]). Let R be a commutative Noetherian k-algebra and let A be an R algebra with trace function t: $A \to R$. Let $Bsev_n(A, R, t)$ be the set of trace isomorphism classes of triples (ϕ, p, P) where P is a projective R-module of rank n, ϕ: $A \to \operatorname{End}_R(P)$ is a trace preserving k-algebra homomorphism when $\operatorname{End}_R(P)$ is equipped with the reduced trace function, and $p \in P$ is such that $\phi(A)p = P$. As above, the assignment of commutative R-algebras to sets given by $R' \mapsto Bsev_n(A \otimes_R R', R', t \otimes id_{R'})$ defines a functor on R-schemes that is representable by an R-scheme, which we denote by $\operatorname{Bsev}_n(A, R, t)$ (see [17]).

If $\phi \in X_{m,n}$, then the unique trace-preserving homomorphism $\bar{\phi}$: $\mathbb{T}_{m,n} \to M_n(k)$ induced by the universal property of $\mathbb{T}_{m,n}$ in $\mathscr{T}_n$ agrees with our $\tilde{\phi}$: $\mathbb{T}_{m,n} \to M_n(k)$ that was induced from the homomorphism φ: $S_{m,n} \to k$ corresponding to $\phi \in X_{m,n}$ as defined in the discussion above Definition 1.1. As before, we will identify ϕ, $\tilde{\phi}$, and $\bar{\phi}$. Under this identification, [17, Prop. 12] tells us that $\operatorname{Bsev}_n(\mathbb{T}_{m,n}, C, tr) = \operatorname{Bsev}_n(F_m, k)$ as k-schemes. Therefore, our study of $\operatorname{Bsev}_n(F_m, k)$ yields results pertaining to what seems to be the most natural definition of the Brauer–Severi scheme for the trace ring of generic matrices.

If we consider $\mathbb{T}_{m,n}$ as a sheaf over $V_{m,n} = \operatorname{Spec}(C)$, we can discuss the local structure of $\mathbb{T}_{m,n}$ by looking at its fibers over the closed points of $V_{m,n}$. Let us denote the fiber of $\mathbb{T}_{m,n}$ at the closed point $\xi \in V_{m,n}$ by $\mathbb{T}_\xi$. Then $\mathbb{T}_\xi$ can be described algebraically as $(\mathbb{T}_{m,n}) \otimes_C \kappa(\xi)$, where $\kappa(\xi)$ denotes the residue field of C at ξ. We can then discuss the local Brauer–Severi scheme of $\mathbb{T}_{m,n}$ at ξ given by $\operatorname{Bsev}_n(\mathbb{T}_\xi, k, \operatorname{tr})$. It follows from the definition of these schemes that $\operatorname{Bsev}_n(\mathbb{T}_\xi, k, \operatorname{tr})$ is the scheme of equivalence classes of pairs (ψ, v) where $\psi\colon \mathbb{T}_\xi \to M_n(k)$ is a trace-preserving algebra homomorphism and $v \in V$ is such that $\psi(\mathbb{T}_\xi)v = V$. Here (ψ, v) is *equivalent* to (ψ', w) if there exists a $g \in GL_n(k)$ such that $g\psi(r)g^{-1} = \psi'(r)$ for all $r \in \mathbb{T}_\xi$ and $gv = w$.

LEMMA 2.1. *Let $\xi \in V_{m,n}$ and let $\mathbf{m}_\xi$ be the maximal ideal of C defining ξ. Then* $\operatorname{Bsev}_n(\mathbb{T}_\xi, k, \operatorname{tr}) \cong \operatorname{Proj}(Q/Q\mathbf{m}_\xi)$.

Proof. From [17, Prop. 11], $\operatorname{Bsev}_n(\mathbb{T}_\xi, k, \operatorname{tr}) \cong \operatorname{Bsev}_n(\mathbb{T}_{m,n}, C, \operatorname{tr}) \times_{\operatorname{Spec}(C)} \operatorname{Spec}(\kappa(\xi)) = \operatorname{Proj}(Q) \times_{\operatorname{Spec}(C)} \operatorname{Spec}(\kappa(\xi))$ by Corollary 1.11. As $\operatorname{Proj}(Q) \times_{\operatorname{Spec}(C)} \operatorname{Spec}(\kappa(\xi)) = \operatorname{Proj}(Q \otimes_C \kappa(\xi))$ and $Q \otimes_C \kappa(\xi) = Q/Q\mathbf{m}_\xi$, the lemma follows. Q.E.D.

Let BS_ξ be the closed subscheme of $BS_{m,n} = \operatorname{Proj}(Q)$ defined by the homogeneous ideal $\mathbf{m}_\xi Q$ and let B_ξ be the intersection of $B_{m,n}$, the open subscheme of $Y_{m,n} = \operatorname{Proj}(\Sigma_{m,n})$ of Brauer–stable points, and the closed subscheme of $Y_{m,n}$ defined by the homogeneous ideal $\mathbf{m}_\xi \Sigma_{m,n}$. If $\pi_X\colon X_{m,n} \to V_{m,n} \cong X_{m,n}//PGL_n(k)$ is the canonical quotient morphism of the $PGL_n(k)$-action defined on $X_{m,n} = \operatorname{Spec}(S_{m,n})$, then we observe that $B_\xi = [\pi_X^{-1}(\xi) \times \mathbb{P}(V)] \cap B_{m,n}$ and $BS_\xi = B_\xi//PGL_n(k)$. Therefore we can use the techniques of [8] and [9] to study $\pi_X^{-1}(\xi)$, then use these results to study B_ξ and BS_ξ.

Note that any $\phi \in X_{m,n}$ induces an F_m-module structure on $V = k^n$. We will say that ϕ is a semi-simple representation if the induced module structure on V makes V a semi-simple F_m-module. Let V_ϕ denote the F_m-module structure induced by $\phi \in X_{m,n}$ on V. Let $\xi \in V_{m,n}$ be a closed point. Then $\pi_X^{-1}(\xi) \subseteq X_{m,n}$ contains a unique closed $PGL_n(k)$ orbit by [11, Lemma 3.6] and this must be the orbit of a semisimple representation by [3, 12.6]. It also follows from [3, Section 12] that the unique closed orbit is the only orbit in $\pi_X^{-1}(\xi)$ containing a semisimple representation.

Let $\phi \in \pi_X^{-1}(\xi) \subseteq X_{m,n}$ be a semisimple representation of F_m and let H_ϕ be the $PGL_n(k)$-stabilizer of ϕ in $G = PGL_n(k)$. Then H_ϕ is reductive (e.g., [9, Proposition 2]) and H_ϕ will act on the tangent space $T_\phi(X_{m,n}) \cong X_{m,n}$ to $X_{m,n}$ at ϕ by simultaneous conjugation. Let N_ϕ be a H_ϕ-stable k-vector space complement of $T_\phi(G \cdot \phi)$ in $T_\phi(X_{m,n}) \cong X_{m,n}$ (so $T_\phi(X_{m,n})$

$= N_\phi \oplus T_\phi(G \cdot \phi)$). Then we can define an H_ϕ-action on $G \times N_\phi$ given by

$$\begin{gathered} H_\phi \times (G \times N_\phi) \to G \times N_\phi \\ (h, (g, x)) \mapsto (gh^{-1}, h \cdot x). \end{gathered} \tag{5}$$

By applying the Luna Slice Theorem [9, p. 97] and [9, Remark 2, p. 98], since $X_{m,n}$ is smooth we get an étale isomorphism

$$\pi_X^{-1}(\xi) = \pi_X^{-1}(\pi_X(\phi)) \cong \left[G \times \pi_{N_\phi}^{-1}\left(\pi_{N_\phi}(0)\right)\right]//H \tag{6}$$

where $\pi_N \colon N_\phi \to N_\phi//H$ is the canonical quotient morphism. Therefore we can determine the structure of $\pi_X^{-1}(\xi)$ by looking at the (usually) simpler scheme $\pi_{N_\phi}^{-1}(\pi_{N_\phi}(0))$ and the stabilizer H.

As a corollary to the Luna Slice Theorem, the variety $V_{m,n} = X_{m,n}//PGL_n(k)$ can be stratified into locally closed, smooth, irreducible subvarieties (see [8, Theorem II.1.1(a)]). These strata can be classified and given a partial ordering with respect to the representation types of the $\xi \in V_{m,n}$ that they contain. We will say that a closed point $\xi \in V_{m,n}$ is of representation type $\rho = (n_1\_a_1, \dots, n_r\_a_r)$ if there exists a semisimple representation $\phi \in \pi_X^{-1}(\xi)$ such that V_ϕ has r distinct simple components E_i of dimension a_i and multiplicity n_i. Note that the representation type of ξ is well defined up to a re-ordering of the pairs $n_j\_a_j$ as $\pi_X^{-1}(\xi)$ contains exactly one equivalence class of semisimple representations in $X_{m,n}$ (i.e., $\pi_X^{-1}(\xi)$ contains exactly one $PGL_n(k)$ orbit of semisimple representations). Also, if $\phi \in \pi_X^{-1}(\xi)$ is semisimple, then by [8, p. 157] its $PGL_n(k)$-stabilizer H is $PGL_n(k)$-conjugate to

$$H_{(\rho)} = \left[\left(GL_{n_1}(k) \otimes 1_{a_1}\right) \times \cdots \times \left(GL_{n_r}(k) \otimes 1_{a_r}\right)\right]/k^\times \subseteq PGL_n(k). \tag{7}$$

Note that we can therefore always choose a semisimple $\phi \in \pi_X^{-1}(\xi)$ that has stabilizer equal to $H_{(\rho)}$. If we let $V_{(\rho)} = \{\xi \in V_{m,n} | \xi$ has representation type $\rho\}$, then the $V_{(\rho)}$ define the locally closed strata mentioned above.

If ρ and ρ' are representation types of $V_{m,n}$, then we will say that ρ' is a *refinement* of ρ if there exists a $g \in PGL_n(k)$ such that $gH_{(\rho)}g^{-1} \subseteq H_{(\rho')}$. Then by [8, Theorem II.1.1(b)], $V_{(\rho)} \subseteq \mathrm{cl}(V_{(\rho')})$ if and only if ρ is a refinement of ρ'. Here we use cl( ) to denote the Zariski closure of the indicated scheme. Therefore $V_{(1\_n)}$ is an open dense subvariety of $V_{m,n}$, while $V_{(n\_1)}$ is contained in the closure of every $V_{(\rho)}$.

PROPOSITION 2.2. *Let $\xi \in V_{m,n}$ have representation type ρ and let $\phi \in \pi_X^{-1}(\xi)$ be a semisimple representation chosen such that $H_\phi = H_{(\rho)}$. Let $H = H_\phi$ and let $N = N_\phi$ be chosen as above. If $\pi_N \colon N \to N//H$ is the* *canonical quotient morphism, then the reduced induced structure on* $\pi_N^{-1}(\pi_N(0))$ *is isomorphic to* $H \cdot Y$ *where* $Y \subseteq N$ *is a finite union of affine* k*-subspaces of* N *and* $H \cdot Y$ *is the image of* $H \times Y \subseteq H \times N$ *under the given action* $H \times N \to N$.

Proof. Let $\mathbf{G}_m$ denote the algebraic group Spec $k[t, t^{-1}]$. By the Hilbert–Mumford Theorem (see, for example, [14, Theorem 2.1]), given a closed point $y \in N$, $y \in \pi_N^{-1}(\pi_N(0))$ if and only if there exists a one parameter subgroup (1-PSG) $\lambda: \mathbf{G}_m \to H$ such that $\lim_{t \to 0}(\lambda(t) \cdot y) = 0$. By [7, Section 15], given a 1-PSG λ, there exists a maximal torus $T \subseteq H$ such that the image of λ is contained in T. So let us first look at a specific maximal torus of H.

Let $\rho = (n_1\_a_1, \dots, n_r\_a_r)$ and let

$$T_{(\rho)} = \left[D_{n_1} \otimes 1_{a_1} \times \cdots \times D_{n_r} \otimes 1_{a_r}\right]/k^{\times} \subseteq H \tag{8}$$

where D_j is the set of invertible diagonal $j \times j$ matrices for any positive integer j. Then $T_{(\rho)}$ is a maximal torus of $H_{(\rho)}$. Let $n_0 = \sum_{i=1}^{r} n_i$ and let $(w_1; \cdots; w_{n_0})$ denote the element of $T_{(\rho)}$ given by the image of

$$\begin{bmatrix} \operatorname{diag}(w_1, \dots, w_{n_1}) \otimes 1_{a_1} & & 0 \\ & \ddots & \\ & & \ddots & \\ 0 & & \operatorname{diag}(w_{n_0 - n_{r-1} + 1}, \dots, w_{n_0}) \otimes 1_{a_r} \end{bmatrix} \tag{9}$$

in $H_{(\rho)}$, where we use $\operatorname{diag}(w_{i+1}, \dots, w_j)$ to denote the $(j-i) \times (j-i)$ diagonal matrix with the (h, h)-entry given by w_{i+h} for any $1 \le h \le (j-i)$.

As H acts on an element of N by simultaneous conjugation of the elements matrix components, we can decompose N into its weight spaces relative to the weights of $T_{(\rho)}$. For any $1 \le i, j \le n_0$, let

$$N_{i,j} = \left\{x \in N | (w_1; \cdots; w_{n_0}) \cdot x = w_i w_j^{-1} x \qquad \text{for all } (w_1; \cdots; w_{n_0}) \in T_{(\rho)}\right\} \tag{10}$$

and $N_0 = \{x \in N | \tau \cdot x = x \text{ for all } \tau \in T_{(\rho)}\}$. Then $N = [\bigoplus_{i \neq j} N_{i,j}] \oplus N_0$ (see, for example, [8, Section II.2]).

Now let $\lambda: \mathbf{G}_m \to H_{(\rho)}$ be a 1-PSG whose image is contained in $T_{(\rho)}$. Then we can write $\lambda(t) = (\lambda_1(t); \cdots; \lambda_{n_0}(t))$ for every $t \in \mathbf{G}_m$. As λ must be a group homomorphism, for each $1 \le i \le n_0$, we get $\lambda_i(t) = t^{q_i}$ for some $q_i \in \mathbb{Z}$. So given any closed point $y \in N$, we can write y uniquely as

$$y = y_0 + \sum_{i \neq j} y_{i,j} \tag{11}$$

where $y_0 \in N_0$ and $y_{i,j} \in N_{i,j}$ for all $i \neq j$. Then

$$\begin{aligned} \lambda(t) \cdot y &= y_0 + \sum_{i \neq j} \lambda_i(t)\big(\lambda_j(t)\big)^{-1} y_{i,j} \\ &= y_0 + \sum_{i \neq j} t^{q_i - q_j} y_{i,j}. \end{aligned} \tag{12}$$

Therefore $\lim_{t \to 0}(\lambda(t) \cdot y) = 0$ if and only if $y_0 = 0$ and $y_{i,j} = 0$ whenever $q_i \leq q_j$.

Choose a $\sigma \in S_{n_0}$ (where we use S_{n_0} to denote the symmetric group on n_0 letters) such that

$$q_{\sigma^{-1}(1)} \geq q_{\sigma^{-1}(2)} \geq \cdots \geq q_{\sigma^{-1}(n_0)}. \tag{13}$$

Then from our work above, $\lim_{t \to 0} \lambda(t) \cdot y = 0$ if and only if

$$y \in \bigoplus_{q_{\sigma^{-1}(i)} > q_{\sigma^{-1}(j)}} N_{i,j} \subseteq \bigoplus_{\sigma(i) < \sigma(j)} N_{i,j}. \tag{14}$$

So for any $\sigma \in S_{n_0}$, let

$$Y_\sigma = \bigoplus_{\sigma(i) < \sigma(j)} N_{i,j} \quad \text{and} \quad Y = \bigcup_{\sigma \in S_{n_0}} Y_\sigma. \tag{15}$$

Therefore, if λ is a 1-PSG whose image is contained in $T_{(\rho)}$ and $y \in N$ is a closed point such that $\lim_{t \to 0} \lambda(t) \cdot y = 0$, then $y \in Y$.

Conversely, if $y \in Y$ is a closed point, then $y \in Y_\sigma$ for some $\sigma \in S_{n_0}$. Define $\lambda_\sigma(t) = (t^{-\sigma(1)}; \cdots; t^{-\sigma(n_0)}) \in T_{(\rho)}$ for all $t \in k^\times$. Then it follows from the above paragraph that $\lim_{t \to 0} \lambda_\sigma(t) \cdot y = 0$, hence the reduced induced structure on Y defines a closed $T_{(\rho)}$-stable subvariety of the reduced induced variety of $\pi_N^{-1}(\pi_N(0))$.

Now let $\lambda \colon \mathbf{G}_m \to H$ be an arbitrary 1-PSG. Then, as mentioned above, the image of λ is contained in some maximal torus T of H. Since H is connected, T must be H-conjugate to $T_{(\rho)}$ by [7, Corollary 21.3.A]. Let $h \in H$ be such that $T_{(\rho)} = hTh^{-1}$. Then $h\lambda(t)h^{-1} \in T_{(\rho)}$ for all $t \in \mathbf{G}_m$. Therefore, if $y \in N$ is a closed point having the property that $\lim_{t \to 0} \lambda(t)y = 0$, then $\lim_{t \to 0} h\lambda(t)h^{-1}hyh^{-1} = 0$ hence $y \in h^{-1}Yh$ by our results for 1-PSG's whose image is contained in $T_{(\rho)}$.

Conversely, if $y \in H \cdot Y$ is a closed point, where $H \cdot Y$ is the image of $H \times Y$ under the H-action $H \times N \to N$, then there exists an $h \in H$ such that $h^{-1}yh \in Y$. As $h^{-1}yh \in Y$, there exists a $\sigma \in S_{n_0}$ such that $\lim_{t\to 0}\lambda_\sigma(t)\cdot(h^{-1}yh) = 0$, so the 1-PSG given by $\lambda'(t) = h\lambda_\sigma(t)h^{-1}$ has the property that $\lim_{t\to 0}\lambda'(t)\cdot y = 0$. Therefore y is a closed point of the reduced induced variety of $\pi_N^{-1}(\pi_N(0))$, thus we get that the reduced induced variety of $\pi_N^{-1}(\pi_N(0))$ is equal to the reduced induced variety defined by $H\cdot Y$. As Y is a finite union of affine k-subspaces of N, our result follows. Q.E.D.

COROLLARY 2.3. *If $H_{(\rho)}$ is a torus* (*hence* $\rho = (1\_a_1,\dots,1\_a_r)$), *then the reduced induced structure of $\pi_N^{-1}(\pi_N(0))$ is isomorphic to Y where Y, a finite union of k-subspaces of N, was defined in Eq.* (15).

Proof. In this proof we keep the notation used in the proof of Proposition 2.2. Since $\rho = (1\_a_1,\dots,1\_a_r)$ it follows from (7) and (8) that $H = H_{(\rho)} = T_{(\rho)}$. Therefore every 1-PSG λ of H has image contained in $T_{(\rho)}$. Hence, by the treatment of this special case in the proof of Proposition 2.2, a closed point $y\in N$ is in $\pi_N^{-1}(\pi_N(0))$ if and only if $y\in Y$ where Y is defined in (15). Q.E.D.

PROPOSITION 2.4. *Let $\xi \in V_{m,n}$ have representation type $\rho = (n_1\_a_1,\dots,n_r\_a_r)$ and let $n_0 = \sum_{i=1}^r n_i$. Then $\pi_X^{-1}(\xi)$ has at most $n_0!/(n_1!\cdots n_r!)$ irreducible components.*

Proof. Let $\xi\in V_{m,n}$ have representation type ρ as stated. Let $\phi \in \pi_X^{-1}(\xi)$ be semisimple with $PGL_n(k)$ stabilizer $H = H_{(\rho)}$. Define $N = N_\phi$, $T = T_{(\rho)}$, $N_{i,j}$, Y and $\{Y_\sigma | \sigma\in S_{n_0}\}$ as in the proof of Proposition 2.2. Since H is connected and Y_σ is irreducible for any $\sigma \in S_{n_0}$, we know $H\cdot Y_\sigma$ must be an irreducible subvariety of $H\cdot Y$.

Let $K = S_{n_1}\times S_{n_2}\times\cdots\times S_{n_r}$. Then we can identify K with a subgroup of S_{n_0} by letting $(\sigma_1,\dots,\sigma_r)(j) = n_{i-1} + \sigma_i(j-n_{i-1})$ whenever $i>1$, $n_{i-1} < j \le n_i$, and $(\sigma_1,\dots,\sigma_r)(j) = \sigma_1(j)$ whenever $j\le n_1$. Let $C_H(T)$ and $N_H(T)$ denote respectively the centralizer and the normalizer of T in H. We can also identify K with a subgroup of $W(H,T) = N_H(T)/C_H(T)$ by identifying $(\sigma_1,\dots,\sigma_r)$ with the image of the matrix

$$\begin{bmatrix} h_{\sigma_1}\otimes 1_{a_1} & & 0 \\ & \ddots & \\ & & \ddots & \\ 0 & & h_{\sigma_r}\otimes 1_{a_r} \end{bmatrix} \tag{16}$$

in H where we use h_{σ_i} to denote the permutation matrix corresponding to σ_i in $GL_{n_i}(k)$ for all i. We claim that for any $\sigma,\sigma'\in S_{n_0}$, if $\sigma^{-1}\sigma'\in K$, then $H\cdot Y_\sigma = H\cdot Y_{\sigma'}$.

Proof of Claim. Note that it is sufficient to show that if $\sigma^{-1}\sigma' \in K$, then $Y_\sigma \subseteq H \cdot Y_{\sigma'}$. We then use a symmetric argument to show $Y_{\sigma'} \subseteq H \cdot Y_\sigma$ and our claim follows.

Let $\tau = \sigma^{-1}\sigma' \in K \subseteq W(H,T)$ and let $h_\tau \in N_H(T)$ be a representative of τ. Let $T_\tau = h_\tau T h_\tau^{-1}$. Then $h_\tau^{-1} N_{i,j} h_\tau \subseteq N_{\tau^{-1}(i),\tau^{-1}(j)}$. Indeed, for any $p = (p_1; \cdots ; p_{n_0}) \in T$, $p(h_\tau^{-1} N_{i,j} h_\tau)p^{-1} = h_\tau^{-1} p_\tau N_{i,j} p_\tau^{-1} h_\tau = p_{\tau^{-1}(i)} p_{\tau^{-1}(j)}^{-1} h_\tau^{-1} N_{i,j} h_\tau$ where $p_\tau = h_\tau T h_\tau^{-1} = (p_{\tau^{-1}(1)}; \cdots ; p_{\tau^{-1}(n_0)})$. So $h_\tau^{-1} N_{i,j} h_\tau \subseteq N_{\tau^{-1}(i),\tau^{-1}(j)}$ by the definition of the $N_{i,j}$'s.

Finally,

$$\begin{aligned} h_\tau^{-1} Y_\sigma h_\tau &= \bigoplus_{\sigma(i)<\sigma(j)} h_\tau N_{i,j} h_\tau^{-1} \\ &\subseteq \bigoplus_{\sigma(i)<\sigma(j)} N_{\tau^{-1}(i),\tau^{-1}(j)} \\ &= \bigoplus_{\sigma'(i)<\sigma'(j)} N_{i,j} \\ &= Y_{\sigma'} \end{aligned} \tag{17}$$

So $Y_\sigma \subseteq h_\tau Y_{\sigma'} h_{\tau^{-1}}$ and our claim is proved.

To conclude the proof of the proposition, for each $\sigma \in S_{n_0}$, $H \cdot Y_\sigma$ is irreducible. By our claim, $H \cdot Y_{\sigma'} = H \cdot Y_\sigma$ if $\sigma^{-1}\sigma' \in K$. Therefore, for each irreducible component of $H \cdot Y$, there is at least one left coset of K in S_{n_0} that can be associated to it. But the number of left K cosets in S_{n_0} is $q = n_0!/(n_1! \cdots n_r!)$. So $H \cdot Y$ has at most q irreducible components, each of which is H-stable. Therefore it follows that $Y' = \pi_N^{-1}(\pi_N(0))$ has at most q irreducible components. As $G = PGL_n(k)$ is also irreducible, $G \times Y'$ has at most q irreducible components, so $(G \times Y')//H$ also has at most q irreducible components. Using the isomorphism (6), we get that $\pi_X^{-1}(\xi)$ also must have at most q irreducible components. Q.E.D.

COROLLARY 2.5. *Let $\xi \in V_{m,n}$ be of representation type $\rho = (n_1\_a_1, \dots, n_r\_a_r)$ and let $n_0 = \sum_{i=1}^r n_r$. Then BS_ξ has at most $n_0!/(n_1! \cdots n_r!)$ irreducible components.*

Proof. From Proposition 2.4, $\pi_X^{-1}(\xi)$ has at most $q = n_0!/(n_1! \cdots n_r!)$ irreducible components. Therefore $\pi_X^{-1}(\xi) \times \mathbb{P}(V)$ must have at most q irreducible components. By definition, $B_{m,n}$ is the dense open subscheme of Brauer-stable points in $Y_{m,n} = X_{m,n} \times \mathbb{P}(V)$ and $B_\xi = [\pi_X^{-1}(\xi) \times \mathbb{P}(V)] \cap B_{m,n}$, therefore B_ξ must have at most q irreducible components. Hence $BS_\xi = B_\xi//PGL_n(k)$ also has at most q irreducible components. Q.E.D.

LEMMA 2.6. *If $\xi \in V_{m,n}$ is of representation type $\rho = (1\_a_1, \dots, 1\_a_r)$ then $\pi_X^{-1}(\xi)$ is reduced. Furthermore, if $\phi \in \pi_X^{-1}(\xi)$ is semisimple with*

$PGL_n(k)$-stabilizer $H = H_{(\rho)}$ and N is an H-stable complement of $T_\phi(G \cdot \phi)$ in $T_\phi(X_{m,n})$ (where $G = PGL_n(k)$), then $\pi_N^{-1}(\pi_N(0))$ is also reduced where $\pi_N \colon N \to N//H$ is the canonical quotient morphism.

Proof. Let $\phi \in \pi_X^{-1}(\xi)$ be semisimple with stabilizer $H = H_{(\rho)}$ (as defined in (7)) and $N = N_\phi$ be an H-stable complement of $T_\phi(G \cdot \phi)$ in $T_\phi(X_{m,n})$. Let $\pi_N \colon N \to N//H$ be the canonical quotient morphism. Note that the existence of the étale isomorphism (6) tells us that it is sufficient to show that $\pi_N^{-1}(\pi_N(0))$ is reduced. Indeed, if $\pi_N^{-1}(\pi_N(0))$ is reduced, then $G \times \pi_N^{-1}(\pi_N(0))$ is reduced, which in turn gives us that $[G \times \pi_N^{-1}(\pi_N(0))]//H$ is a reduced étale covering of $\pi_X^{-1}(\xi)$, therefore $\pi_X^{-1}(\xi)$ must be reduced.

By (7), when ξ has representation type ρ, H is a torus (hence $n_0 = r$) and by [8, p. 159] N is H-isomorphic to

$$N' = M_n(k) \oplus \cdots \oplus M_n(k) \oplus \begin{pmatrix} k \cdot 1_{a_1} & & & \\ & k \cdot 1_{a_2} & 0 & \\ & 0 & \ddots & \\ & & & k \cdot 1_{a_r} \end{pmatrix} \tag{18}$$

where there are $m - 1$ copies of $M_n(k)$ in the above sum and H acts on N' via simultaneous conjugation by H. Here we use 1_a to denote the $a \times a$ identity matrix, $k \cdot 1_a = \{c1_a | c \in k\}$. Let $\pi_{N'} \colon N' \to N'//H$ be the canonical quotient morphism.

As $k[N'//H] = k[N']^H$, let us describe $k[N']$ as a polynomial ring and compute the H-invariants. For any $1 \le i, j \le n$ and $1 \le l \le m - 1$, let $z_{i,j}^{(l)}$ denote the coordinate function on N' corresponding to the (i, j)th entry of the lth summand in N'. Also, for any $1 \le \beta \le r$, let z'_β be the coordinate function corresponding to projection onto the mth summand of N' followed by projection onto the coefficient of the 1_{a_β} block diagonal entry. Then

$$k[N'] = \left[z_{i,j}^{(l)}, z'_\beta | 1 \le i,\ j \le n,\ 1 \le l \le (m-1),\ 1 \le \beta \le r \right] \tag{19}$$

is a polynomial ring in $(m-1)n^2 + r$ variables.

Now we can partition the set $\{1, \ldots, n\}$ into subsets $P_1 = \{1, \ldots, a_1\}$, $P_2 = \{(1 + a_1), \ldots, a_2\}, \ldots, P_r = \{(1 + \sum_{w=1}^{r-1} a_w), \ldots, a_r\}$. Then if $i \in P_w$ and $j \in P_{w'}$ for some $1 \le w, w' \le r$, the induced H-action on $k[N']$ (e.g., [11, pg. 45]) gives that $(h_1; \cdots; h_r) \cdot z_{i,j}^{(l)} = h_w^{-1} h_{w'} z_{i,j}^{(l)}$ for any $1 \le l \le (m - 1)$ and $1 \le w \le r$. Furthermore, $h \cdot z'_\beta = z'_\beta$ for all $h \in H$ and for all β. Therefore we see that a monomial $z_{i_1,j_1}^{(l_1)} \cdots z_{i_t,j_t}^{(l_t)} (z'_{\beta_1})^{d_1} \cdots (z'_{\beta_q})^{d_1}$ is in $k[N']^H$ if and only if $t = 0$ or there exists a $\tau \in S_t$ such that $i_s \in P_w \Leftrightarrow j_{\tau(s)} \in P_w$ for all $1 \le s \le t$ and all $1 \le w \le r$. Then the set of all such monomials union $\{1_k\}$ forms a k-basis of $k[N']^H$ as a k-vector space. Also the set of all non-constant H-invariant monomials form a k-basis for the maximal ideal M defining $\pi_{N'}(0)$. Therefore $k[N']M$ is the ideal of $k[N']$ that defines $\pi_{N'}^{-1}(\pi_{N'}(0))$. Then we claim that $k[N']M$ is equal to its own radical, hence $\pi_{N'}^{-1}(\pi_{N'}(0))$ is reduced. This implies that $\pi_N^{-1}(\pi_N(0))$ is also reduced.

Proof of Claim. Let $p = \sum_{w=1}^{t} m_t$ where the set $\{m_1, \dots, m_t\}$ is a set of linearly independent monomials. Assume $p^q \in k[N']M$ for some positive integer q. We will show that $p \in k[N']M$ using induction on t.

First let us assume p is a monomial in $k[N']$. If z'_l is a factor of p for any $1 \le l \le r$, then $p \in k[N']M$. Therefore we can assume no z'_l is a factor of p. Then no z'_l can be a factor of p^q. As $p^q \in k[N']M$, there exists a minimal monomial factor f of p^q such that $f \in M$. Then if we write $f = z_{i_1, j_1}^{(l_1)} \cdots z_{i_d, j_d}^{(l_d)}$, there would exist a $\tau \in S_d$ such that $i_s \in P_w \Leftrightarrow j_{\tau(s)} \in P_w$ for all $1 \le s \le d$ and all $1 \le w \le r$. If the triples $(i_1, j_1, l_1), \dots, (i_d, j_d, l_d)$ are all distinct, then f must be a factor of p hence $p \in k[N']M$. So assume that there exist $1 \le s < s' \le d$ such that $(i_s, j_s, l_s) = (i_{s'}, j_{s'}, l_{s'})$. As f is a minimal invariant factor of p^q, τ must be a cycle in S_d. So there exists a $1 \le c \le (d-1)$ such that $\tau^c(s) = s'$. Then $z_{i_s, j_s}^{(l_s)} z_{i_{\tau(s)}, j_{\tau(s)}}^{(l_{\tau(s)})} \cdots z_{i_{\tau^{c-1}(s)}, j_{\tau^{c-1}(s)}}^{(l_{\tau^{c-1}(s)})}$ is a proper factor of f that is H-invariant, hence contradicts the minimality of f. Therefore the triples (i_s, j_s, l_s) must be distinct, hence $p \in k[N']M$.

Now assume that for any k-independent set of monomials $\{u_1, \dots, u_{t-1}\}$ that $(\sum_{w=1}^{t-1} u_w)^q \in k[N']M$ implies $(\sum_{w=1}^{t-1} u_w) \in k[N']M$ for any positive integer q. Let $p = \sum_{w=1}^{t} m_w$ be a sum of t independent monomials $\{m_1, \dots, m_t\}$. If we assign an order to the variables $z_{i_s, j_s}^{(l_s)}$, we can order the set of monomials in $k[N']$ lexigraphically with respect to the powers of the variables $z_{i_s, j_s}^{(l_s)}$. Therefore, if we write $p = \sum_{w=1}^{t} m_w$ with $m_1 < m_2 < \cdots < m_t$ (with respect to the lexigraphic ordering), then $p^q = m_1^q + p_+$ where p_+ is a sum of independent monomials that are greater than m_1^q with respect to our lexigraphical ordering. Since we have a k-basis of monomials for $k[N']M$, then $p^q \in k[N']M$ if and only if when p^q is written as a sum of independent monomials, each monomial in that sum is also in $k[N']M$. As m_1 is minimal in p, m_1^q must be the minimal monomial in the expression for p^q. Therefore $m_1^q \in k[N']M$ which we already proved implies that $m_1 \in k[N']M$. Therefore $p^q = u \cdot m_1 + (\sum_{w=2}^{t} m_w)^q$ for some $u \in k[N']$, so $(\sum_{w=2}^{t} m_w)^q \in k[N']M$. Hence, by our induction hypothesis, $(\sum_{w=2}^{t} m_w) \in k[N']M$. This gives us that $p \in k[N']M$ and our claim is proved, which in turn proves our lemma. Q.E.D.

THEOREM 2.7. *Let $\xi \in V_{m,n}$ be a closed point of representation type $\rho = (1\_a_1, \dots, 1\_a_r)$. Then BS_ξ has exactly $r!$ smooth irreducible components that meet transversally. Furthermore, the dimensions of the irreducible* *components are equal and of dimension*

$$n - r + (m-1)\sum_{i<j} a_i a_j. \tag{20}$$

Proof. Let $\xi \in V_{m,n}$ be of the given representation type and choose $\phi \in \pi_X^{-1}(\xi)$ to be semisimple with $PGL_n(k)$-stabilizer $H = H_{(\rho)}$ (as defined in (7)). Let N be an H-stable (vector space) complement of $T_\phi(G\cdot\phi)$ in $T_\phi(X_{m,n})$ where $G = PGL_n(k)$ and let $\pi_N\colon N \to N//H$ be the canonical quotient morphism. Then, by Lemma 2.6, $\pi_N^{-1}(\pi_N(0))$ and $\pi_X^{-1}(\xi)$ are both reduced. So, by Corollary 2.3, $\pi_N^{-1}(\pi_N(0)) = Y$ where Y is the finite union of k-subspaces of N defined in (15).

As Y is the union of $r!$ subspaces of N (no one of which is contained in any other), Y has $r!$ smooth irreducible components that intersect transversally. As $G = PGL_n(k)$ is connected, $G \times Y$ also has $r!$ smooth irreducible components that intersect transversally. Then since H is connected and acts freely on $G \times Y$, the quotient $[G\times Y]//H$ also has $r!$ irreducible components, but it is not immediately clear that these components are either smooth or intersect transversally.

Now by [9, pp. 86–87], for example, $[G\times Y]//H$ is a principal fiber bundle (in the sense of [16]) with base $G//H$ and fiber type Y. Therefore, there exists an open étale covering $\{\varphi_i\colon U_i \to G//H\}$ such that $U_i \times_{G//H} ([G\times Y]//H) \cong U_i \times Y$. (Note that the condition [16, (NR$'$)] implies that the covering is both flat and unramified, hence étale.) As G is connected, $G//H$ must be smooth and irreducible, hence all the U_i are necessarily smooth and irreducible. Also, since all the φ_i's are étale, we can use the base change property (e.g., [10, Proposition I.3.3]) to conclude that the induced morphisms

$$\tilde{\varphi}_i\colon \tilde{U}_i = U_i \times_{(G//H)} ([G\times Y]//H) \cong U_i \times Y \to [G\times Y]//H \tag{21}$$

are also étale for every i and form an étale open cover of $[G\times Y]//H$. Note that the irreducible components of $\tilde{U}_i \cong U_i \times Y$ are smooth and intersect transversally. If an irreducible component of $[G\times Y]//H$ is not smooth, there would exist an $\tilde{U}_i$ with a non-smooth irreducible component. Therefore the irreducible components of $[G\times Y]//H$ must be smooth. Similarly, if the irreducible components of $[G\times Y]//H$ did not intersect transversally, there would exist a $\tilde{U}_i$ whose irreducible components would not intersect transversally. Therefore the irreducible components of $[G\times Y]//H$ must intersect transversally as well.

Now by the isomorphism (6), $[G\times Y]//H$ is G-isomorphic to $\pi_X^{-1}(\xi)$, therefore $\pi_X^{-1}(\xi)$ has exactly $r!$ smooth irreducible components that meet transversally. This will also hold for $\pi_X^{-1}(\xi)\times \mathbb{P}(V)$. As the scheme of

Brauer-stable points $B_{m,n}$ is an open subscheme of $Y_{m,n} \cong [X_{m,n} \times \mathbb{P}(V)]$ (by our discussion preceding Lemma 1.2) and $B_\xi = B_{m,n} \cap [\pi_X^{-1}(\xi) \times \mathbb{P}(V)]$, we can conclude that B_ξ will have $r!$ smooth irreducible components that meet transversally if no irreducible component of $\pi_X^{-1}(\xi) \times \mathbb{P}(V)$ is contained in the complement of B_ξ. Since ξ has representation type $(1\_a_1, \dots, 1\_a_n)$, for every element of $\phi \in \pi_X^{-1}(\xi)$ there exists at least one element $kv \in \mathbb{P}(V)$ such that (ϕ, kv) is Brauer-stable. Therefore each irreducible component of $\pi_X^{-1}(\xi) \times \mathbb{P}(V)$ has a nontrivial intersection with B_ξ.

Note in the above paragraph that the irreducible components of Y are affine k-spaces of dimension $(m-1)\sum_{i<j} a_i a_j$, so that the irreducible components of $(G \times Y)//H$ are of dimension $\dim_k(G/H) + (m-1)\sum_{i<j} a_i a_j = n^2 - r + (m-1)\sum_{i<j} a_i a_j$ since H is a torus of rank $r-1$. So it follows that the irreducible components of B_ξ all have dimension $\dim_k(B_\xi) = n^2 - r + n - 1 + (m-1)\sum_{i<j} a_i a_j$.

Given that B_ξ is reduced, we conclude that $BS_\xi = B_\xi//PGL_n(k)$ is also reduced. Since $\{V_M | M \text{ is special}\}$ forms a finite affine open cover of $BS_{m,n}$ by Theorem 1.7, the set $\{V_M \cap BS_\xi | M \text{ is special}\}$ forms a finite affine open cover of BS_ξ. Also, $V_M \cong V'_M \subseteq U_M$ where V'_M is a linear subvariety of U_M that is an étale slice for the $PGL_n(k)$ action on U_M by Corollary 1.10. Therefore $BS_\xi \cap V_M \cong B_\xi \cap V'_M$. Since B_ξ is G-saturated, $B_\xi \cap U_M \cong [B_\xi \cap V'_M] \times PGL_n(k)$, by Corollary 1.10. So if the irreducible components of $BS_\xi \cap V_M$ do not meet transversally, then neither will the irreducible components of $B_\xi \cap V'_M$ and this implies that the irreducible components of $B_\xi \cap U_M$ would not intersect transversally. Hence the irreducible components of $BS_\xi \cap V_M$ must meet transversally.

Since $PGL_n(k)$ is connected, every irreducible component of B_ξ is $PGL_n(k)$-stable. Also, every $PGL_n(k)$ orbit of B_ξ is closed (by Lemma 1.2) so there is a one-to-one correspondence between the irreducible components of B_ξ and BS_ξ. Hence BS_ξ has exactly $r!$ irreducible components.

Let Z be an irreducible component of B_ξ and let $\pi\colon B_{m,n} \to BS_{m,n} \cong B_{m,n}//PGL_n(k)$ be the canonical quotient morphism. Then $\pi(Z)$ is the corresponding irreducible component of BS_ξ. Since $PGL_n(k)$ is connected, Z must be $PGL_n(k)$-stable. Then for any special sequence M, $U_M \cong V'_M \times PGL_n(k)$ by Corollary 1.10 and so $Z \cap U_M \cong (Z \cap V'_M) \times PGL_n(k)$. As Z is smooth, so must be $Z \cap V'_M$. But $Z \cap V'_M \cong \pi(Z) \cap V_M$. So, $\pi(Z)$ is smooth on each affine open subset V_M of $BS_{m,n}$ and therefore $\pi(Z)$ must be smooth. Also, as $\dim_k(Z) = n^2 - r + n - 1 + (m-1)\sum_{i<j} a_i a_j$, we get $\dim_k(\pi(Z)) = n^2 - r + n - 1 - (n^2 - 1) + (m-1)\sum_{i<j} a_i a_j = n - r + (m-1)\sum_{i<j} a_i a_j$. Q.E.D.

If $\xi \in V_{m,n}$ is a closed point of representation type $\rho = (1\_a_1, \dots, 1\_a_r)$ for some positive integers $a_1, \dots, a_r$, then the irreducible components of

$\mathrm{Bsev}_n(\mathbb{T}_\xi, k, tr)$ can be described as Brauer−Severi schemes in their own right. We show this by first constructing the corresponding algebra with trace and then showing an isomorphism between the Brauer−Severi scheme of this algebra with an irreducible component of $\mathrm{Bsev}_n(\mathbb{T}_\xi, k, \mathrm{tr})$.

Fix a closed point $\xi \in V_{m,n}$ of representation type ρ and let $\phi \in \pi_X^{-1}(\xi)$ be semisimple with stabilizer $H = H_{(\rho)}$. Let N be an H-stable vector space complement of $T_\phi(G \cdot \phi)$ in $T_\phi(X_{m,n})$. Finally, let $\pi_N\colon N \to N//H$ be the canonical quotient morphism. Then by Corollary 2.3 and Lemma 2.6, $Y = \pi_N^{-1}(\pi_N(0))$ where $Y = \bigcup_{\sigma \in S_r} Y_\sigma$ is defined in (15). Therefore the isomorphism given in (6) tells us $\pi_X^{-1}(\xi) \cong [G \times \bigcup_\sigma Y_\sigma]//H \cong \bigcup_\sigma [G \times Y_\sigma]//H$, since H is a torus which tells us that each Y_σ is H-stable. Also, for each $\sigma \in S_r$, it follows that $[G \times Y_\sigma]//H$ is an irreducible component of $(G \times Y)//H$, hence defines an irreducible component Z_σ of $\pi_X^{-1}(\xi)$.

Let P_σ be the prime ideal of $S_{m,n}$ defining Z_σ. Then the canonical surjection $\tau_\sigma\colon S_{m,n} \to S_\sigma = S_{m,n}/P_\sigma$ induces a surjection $M_n(\tau_\sigma)\colon M_n(S_{m,n}) \to M_n(S_\sigma)$. Let $\mathbb{T}_{\xi,\sigma}$ denote the image of $\mathbb{T}_{m,n}$ under $M_n(\tau_\sigma)$. Note that since $P_\sigma \cap C = \mathbf{m}_\xi$, where $\mathbf{m}_\xi$ is the maximal ideal of C corresponding to ξ, the center of $\mathbb{T}_{\xi,\sigma}$ is $C/\mathbf{m}_\xi \cong k$.

COROLLARY 2.8. *Using the notation of the above discussion, each irreducible component of* $\mathrm{Bsev}_n(\mathbb{T}_\xi, k, \mathrm{tr})$ *is isomorphic to* $\mathrm{Bsev}_n(\mathbb{T}_{\xi,\sigma}, k, \mathrm{tr})$ *for some* $\sigma \in S_r$.

Proof. Let Z be an irreducible component of $\mathrm{Bsev}_n(\mathbb{T}_\xi, k, \mathrm{tr})$. By Lemma 1.2 every G-orbit in B_ξ is closed in $B_{m,n}$. Since G is connected, there is a one-to-one correspondence between the irreducible components of B_ξ and $BS_\xi \cong \mathrm{Bsev}_n(\mathbb{T}_\xi, k, \mathrm{tr})$. As $B_\xi = [\pi_X^{-1}(\xi) \times \mathbb{P}(V)] \cap B_{m,n}$, by our above discussion, we get a G-isomorphism

$$B_\xi \cong \bigcup_{\sigma \in S_r} \big[(Z_\sigma \times \mathbb{P}(V)) \cap B_{m,n}\big] \tag{22}$$

where the $(Z_\sigma) \times \mathbb{P}(V)) \cap B_{m,n}$ are distinct irreducible components of B_ξ. Let $\sigma \in S_r$ be the permutation such that $((Z_\sigma) \times \mathbb{P}(V)) \cap B_{m,n} = \pi_X^{-1}(Z)$.

Consider $\mathbb{T}_{\xi,\sigma}$. For $\phi\colon \mathbb{T}_{\xi,\sigma} \to M_n(k)$ let $\tilde{\phi} = \phi \circ \tau_\sigma$. Then $\tilde{\phi}\colon \mathbb{T}_{m,n} \to M_n(k)$ corresponds to the representation $\psi_\phi\colon S_{m,n} \to k$ defined by $x_{i,j}^{(l)} \mapsto \tilde{\phi}(X_l)_{i,j}$ where we use $\tilde{\phi}(X_l)_{i,j}$ to denote the (i,j)th entry of $\tilde{\phi}(X_l)$ for $1 \le i, j \le n$ and $1 \le l \le m$. Then it follows that ψ_ϕ is a representation in $Z_\sigma = \mathrm{Spec}(S_\sigma)$. So $\mathrm{Bsev}_n(\mathbb{T}_{\xi,\sigma}, k, \mathrm{tr})$ is the set of G-orbits of Brauer-stable pairs $(\phi, kv) \in Z_\sigma \times \mathbb{P}(V)$. Therefore

$$\mathrm{Bsev}_n(\mathbb{T}_{\xi,\sigma}, k, \mathrm{tr}) \cong \big[(Z_\sigma \times \mathbb{P}(V)) \cap B_{m,n}\big]//G \cong Z \tag{23}$$

as desired. Q.E.D.

3. AN EXAMPLE

In this section we develop some intuition about Brauer–Severi schemes of the trace ring both globally and locally by computing these schemes for the $m = 2$, $n = 2$ case when the characteristic of k is zero. In other words, we will concentrate on two-dimensional representations of $F_2 = k\{\mathscr{Y}_1, \mathscr{Y}_2\}$ on a k vector space V. For convenience, let X and Y denote the first and second generic matrices of $\mathbb{T}_{2,2}$.

Our first goal is to calculate $BS_{2,2} = \mathrm{Bsev}_2(\mathbb{T}_{2,2}, C_{2,2}, \mathrm{tr})$. We know that $BS_{2,2}$ is covered by affine subvarieties corresponding to special sequences. In this case, there are only two special sequences which we denote by $M_X = \{(1,1)\}$ and $M_Y = \{(1,2)\}$. The corresponding functions in Q are $h_X = h_{M_X} = [1, X]$ and $h_Y = h_{M_Y} = [1, Y]$. So $BS_{2,2}$ is covered by the two affine subsets $V_X = V_{h_X}$ and $V_Y = V_{h_Y}$, both of which are isomorphic to $\mathbb{A}^6$, the six-dimensional affine space over k. Therefore, the only work is finding out how these two varieties glue together.

From Theorem 1.9, we know $k[V_X] = k[s_1, s_2, s_3, s_4, s_5, s_6]$ where we define

$$\begin{aligned} s_1 &= [X^2, X]h_X^{-1} & s_3 &= [Y, X]h_X^{-1} & s_5 &= [YX, X]h_X^{-1} \\ s_2 &= [1, X^2]h_X^{-1} & s_4 &= [1, Y]h_X^{-1} & s_6 &= [1, YX]h_X^{-1} \end{aligned} \tag{24}$$

and the s_i are algebraically independent over k. Similarly, we let $k[V_Y] = k[t_1, \ldots, t_6]$ where

$$\begin{aligned} t_1 &= [X, Y]h_Y^{-1} & t_3 &= [XY, Y]h_Y^{-1} & t_5 &= [Y^2, Y]h_Y^{-1} \\ t_2 &= [1, X]h_Y^{-1} & t_4 &= [1, XY]h_Y^{-1} & t_6 &= [1, Y^2]h_Y^{-1} \end{aligned} \tag{25}$$

and the t_i are algebraically independent over k.

Finally, let

$$\begin{aligned} X' &= \begin{pmatrix} 0 & s_1 \\ 1 & s_2 \end{pmatrix} & Y' &= \begin{pmatrix} s_3 & s_5 \\ s_4 & s_6 \end{pmatrix} \\ X'' &= \begin{pmatrix} t_1 & t_3 \\ t_2 & t_4 \end{pmatrix} & Y'' &= \begin{pmatrix} 0 & t_5 \\ 1 & t_6 \end{pmatrix} \end{aligned} \tag{26}$$

Then for any $(\phi, kv) \in U_X$, $(\phi(X), \phi(Y))$ is equivalent to $(X'(\phi, kv), Y'(\phi, kv))$ and similarly (X, Y) is equivalent to (X'', Y'') on U_Y.

It is clear from definitions that in $Q_{((h_X h_Y))}$ we get $s_4 t_2 = 1$, $t_1 s_4 = -s_3$, and $t_2 s_3 = -t_1$. Also, $\mathrm{tr}(X) = \mathrm{tr}(X') = s_2$ must equal $\mathrm{tr}(X'') = t_1 + t_4$. Similarly, from $\mathrm{tr}(Y)$ we get $s_3 + s_6 = t_6$. Finally, using $\det(X)$ and $\det(Y)$

we get that $-s_1 = t_1t_4 - t_2t_3$ and $-t_5 = s_3s_6 - s_4s_5$. Solving this system we get the following relationships defining the gluing morphism μ on $V_X \cap V_Y$:

$$\begin{array}{lll} s_1 = t_2t_3 - t_1t_4 & s_3 = -t_1t_2^{-1} & s_5 = t_2t_5 - t_1t_6 - t_1^2t_2^{-1} \\ s_2 = t_1 + t_4 & s_4 = t_2^{-1} & s_6 = t_6 + t_1t_2^{-1}. \end{array} \tag{27}$$

It is not hard to verify that the morphism given by

$$\begin{array}{lll} t_1 = -s_3s_4^{-1} & t_3 = s_1s_4 - s_3s_2 - s_3^2s_4^{-1} & t_5 = s_4s_5 - s_3s_6 \\ t_2 = s_4^{-1} & t_4 = s_2 + s_3s_4^{-1} & t_6 = s_3 + s_6 \end{array} \tag{28}$$

is the inverse of μ. This formulation of $BS_{2,2}$ does not give us the best picture of what the scheme looks like, so we state the following proposition that gives a description of Q as a C-algebra.

PROPOSITION 3.1. *Let $Q' = C[u_1, u_2, u_3]/(f)$ where*

$$\begin{aligned} f = \det(Y)u_1^2 + u_2^2 + \det(X)u_3^2 + (\operatorname{tr}(XY) - \operatorname{tr}(X)\operatorname{tr}(Y))u_1u_3 \\ - \operatorname{tr}(X)u_2u_3 + \operatorname{tr}(Y)u_1u_2 \end{aligned} \tag{29}$$

and let Q' be graded by letting the degree of the u_i be one and the elements of C be of degree zero. Then $B_{2,2} = \operatorname{Proj}(Q')$.

Proof. Note that $\operatorname{Proj}(Q')$ is covered by two open affine subsets, $U_1 = \operatorname{Spec}(Q'_{((u_1))})$ and $U_3 = \operatorname{Spec}(Q'_{((u_3))})$.

First, we claim that $U_1 \cong V_X$. Let $\tau: C \hookrightarrow k[V_X]$ be the natural injection and let $\rho: k[U_1] \to k[V_X]$ be defined by letting $\rho|C = \tau$, $\rho(u_2u_1^{-1}) = -s_3$, and $\rho(u_3u_1^{-1}) = s_4$. In order for ρ to be well-defined, we need to check that $\tilde{\rho}(fu_1^{-2}) = 0$ for the canonical lifting of ρ to $C[u_2u_1^{-1}, u_3u_1^{-1}]$. But this follows from the following calculations:

$$\begin{aligned} &\rho\left(fu_1^{-2}\right) \\ &\quad = \rho(\det(X))s_4^{-2} + s_3^2 + \rho(\det(Y)) + \rho(\operatorname{tr}(XY) - \operatorname{tr}(X)\operatorname{tr}(Y))s_4 \\ &\qquad + \rho(\operatorname{tr}(X))s_3s_4 - \rho(\operatorname{tr}(Y))s_3 \\ &\quad = \tau(\det(X))s_4^{-2} + s_3^2 + \tau(\det(Y)) + \tau(\operatorname{tr}(XY) - \operatorname{tr}(X)\operatorname{tr}(Y))s_4 \\ &\qquad + \tau(\operatorname{tr}(X))s_3s_4 - \tau(\operatorname{tr}(Y))s_3 \\ &\quad = (-s_1)s_4^{-2} + s_3^2 + (s_3s_6 - s_4s_5) + (s_5 + s_1s_4 + s_2s_6 - s_2(s_3 + s_6))s_4 \\ &\qquad + (s_2)s_3s_4 - (s_3 + s_6)s_3 \\ &\quad = 0. \end{aligned} \tag{30}$$

To finish proving our claim, we show ρ is invertible. But we can construct the inverse map by sending s_3 and s_4 to $-u_2u_1^{-1}$ and $u_3u_1^{-1}$, respectively. Then the images of the rest of the indeterminants are determined by $s_1 = -\det(X)$, $s_2 = \mathrm{tr}(X)$, $s_5 = \mathrm{tr}(XY) + \det(X)s_4 - \mathrm{tr}(X)(\mathrm{tr}(Y) - s_3)$, and $s_6 = \mathrm{tr}(Y) - s_3$ as $\rho|C = \tau$.

In a similar way, one can show that there is an isomorphism $\rho'\colon U_3 \cong V_Y$ such that $\rho'|C$ is equal to the canonical injection $C \hookrightarrow k[V_Y]$. Then it is tedious, but straightforward, to show that ρ' can be constructed so that $\mu \circ \rho$ and ρ' agree on $Q'_{((u_1u_3))}$. Therefore, we get an isomorphism between the schemes $B_{2,2}$ and $\mathrm{Proj}(Q')$. Q.E.D.

So we can think of $BS_{2,2}$ as a conic in projective 2-space over $C_{2,2}$ defined by f. We now consider the local Brauer–Severi schemes to investigate the nature of $BS_{2,2}$.

From [5, Lemma 1] it follows that $C_{2,2} = k[\mathrm{tr}(X), \mathrm{tr}(Y), \det(X), \det(Y), \mathrm{tr}(XY)]$ and that the generating set $\{\mathrm{tr}(X), \mathrm{tr}(Y), \det(X), \det(Y), \mathrm{tr}(XY)\}$ is algebraically independent over k. Now given any closed point $\xi \in V_{2,2}$, its corresponding maximal ideal $\mathbf{m}_\xi$ is generated by elements of the form $\mathrm{tr}(X) - \alpha_1, \mathrm{tr}(Y) - \alpha_2, \mathrm{tr}(XY) - \alpha_3, \det(X) - \alpha_4, \det(Y) - \alpha_5$. Then the image of f in $(C/\mathbf{m}_\xi)[u_1, u_2, u_3]$, which we will denote by f_ξ, is

$$f_\xi = \alpha_5 u_1^2 + u_2^2 + \alpha_4 u_3^2 + (\alpha_3 - \alpha_1\alpha_2)u_1u_3 - \alpha_1 u_2 u_3 + \alpha_2 u_1 u_2. \quad (31)$$

After a bit of straightforward algebraic manipulation, it can be shown that f_ξ factors into two linear terms exactly when

$$(\alpha_1^2 - 4\alpha_4)(\alpha_2^2 - 4\alpha_5) - (2\alpha_3 - \alpha_1\alpha_2)^2 = 0. \quad (32)$$

So for generic ξ, f_ξ is irreducible, hence $BS_\xi = \mathrm{Bsev}_2(\mathbb{T}_\xi, k, \mathrm{tr})$ is isomorphic to $\mathbb{P}_k^1$, as expected. Under condition (32), $f_\xi = (u_2 + \beta_1 u_1 + \beta_2 u_3)(u_2 + \gamma_1 u_1 + \gamma_2 u_3)$ where

$$\begin{aligned} \beta_1 &= \tfrac{1}{2}\left\{\alpha_2 + \epsilon_1\sqrt{\alpha_2^2 - 4\alpha_5}\right\} & \beta_2 &= \tfrac{1}{2}\left\{-\alpha_1 + \epsilon_2\sqrt{\alpha_1^2 - 4\alpha_4}\right\} \\ \gamma_1 &= \tfrac{1}{2}\left\{\alpha_2 - \epsilon_1\sqrt{\alpha_2^2 - 4\alpha_5}\right\} & \gamma_2 &= \tfrac{1}{2}\left\{-\alpha_1 + \epsilon_2\sqrt{\alpha_1^2 - 4\alpha_4}\right\} \end{aligned} \quad (33)$$

and $\epsilon_i \in \{1, -1\}$ is chosen so that $\beta_1\gamma_2 + \beta_2\gamma_1 = \alpha_3 - \alpha_1\alpha_2$. It is not unexpected that this is exactly the condition for a two-dimensional representation of F_2 not to be irreducible.

PROPOSITION 3.2. *Let $\phi\colon F_2 \to M_2(k)$ be a representation of F_2. Then ϕ is irreducible if and only if*

$$\big(\operatorname{tr}(\phi(X))^2 - 4\det(\phi(X))\big)\big(\operatorname{tr}(\phi(Y))^2 - 4\det(\phi(Y))\big) - \big(2\operatorname{tr}(\phi(XY)) - \operatorname{tr}(\phi(X))\operatorname{tr}(\phi(Y))\big)^2 = 0. \qquad (34)$$

Proof. Assume that $\phi\colon F_2 \to M_2(k)$ is not irreducible. Then the pair $(\phi(X), \phi(Y))$ is equivalent to the pair $(\bar{X}, \bar{Y})$ given by

$$\bar{X} = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{pmatrix} \qquad \bar{Y} = \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{pmatrix} \qquad (35)$$

for appropriate x_{ij}, y_{ij}. Therefore $\operatorname{tr}(\phi(X)) = \operatorname{tr}(\bar{X})$ and similarly for the rest of our generators for $C_{2,2}$.

Now, for any $A, B \in M_2(k)$, define $a(A, B) = \operatorname{tr}(AB) - \frac{1}{2}\operatorname{tr}(A)\operatorname{tr}(B)$. Then Eq. (34) can be rewritten as

$$4a(\phi(X), \phi(X))a(\phi(Y), \phi(Y)) - \big(2a(\phi(X), \phi(Y))\big)^2 = 0. \qquad (36)$$

But $a(\phi(X), \phi(Y)) = a(\bar{X}, \bar{Y}) = \frac{1}{2}(x_{11} - x_{22})(y_{11} - y_{22})$ and the case is similar for $a(\phi(X), \phi(X))$ and $a(\phi(Y), \phi(Y))$. So Eq. (34) follows when ϕ is not irreducible.

To show the converse, let P be the ideal defining the closed subvariety of $V_{2,2}$ of equivalence class non-irreducible representations. It follows from [8, p. 158] that P is a prime ideal of height 1. By what was done previously, it is clear that $p \in P$ where $p = a(X, X)a(Y, Y) - a(X, Y)^2$. As we have noted in Proposition 3.1, p is irreducible in $C_{2,2}$, so (p) is a nonzero prime ideal of C contained in the height one prime ideal P defining the equivalence classes of non-irreducible representations. Therefore $P = (p)$ and our proposition follows. Q.E.D.

So generically when $\xi \in V_{2,2}$ corresponds to an equivalence class of non-irreducible representations, $\operatorname{Bsev}_2(\mathbb{T}_\xi, k, \operatorname{tr})$ is isomorphic to two projective lines intersecting transversally at a point.

Note that this picture is consistent with what Artin describes in [2, Example 1.5(i)]. In this example, Artin is only considering $\operatorname{Bsev}_n(A, R, \operatorname{tr})$ when R is a Dedekind domain with field of fractions K and A is an R-order in the central simple algebra $A \otimes_R K$. In our example, although $\mathbb{T}_{2,2}$ is a maximal C-order in the corresponding universal division algebra, $UD_{2,2}$, the ring C is certainly not Dedekind. Yet, as the closed subvariety in $V_{2,2}$ of non-irreducible representations is defined by a height one prime ideal, there is a nice analogy to the example of Artin's mentioned above.

Note that when $m > 2$ or $n > 2$ the closed subvariety in $V_{m,n}$ of non-irreducible representations is no longer defined by a height one prime. In these cases, the direct correspondence to Artin's examples breaks down.

Even in our current example, there is an aspect we have yet to examine. In particular, $V_{2,2}$ can be given a Luna stratification of three locally closed, smooth, irreducible subvarieties, say W_1, W_2, W_3, such that $W_3 = \overline{W}_3 \subseteq \overline{W}_2 \subseteq \overline{W}_1 = V_{2,2}$, where $\overline{W}_i$ denotes the Zariski closure of W_i in $V_{2,2}$ (see, for example, [8]). Here W_1 corresponds to the open subvariety of $V_{2,2}$ of ξ of equivalence classes of irreducible representations and $\overline{W}_2$ is the complement of W_1 in $V_{2,2}$. The variety W_3 consists of ξ of equivalence classes of semisimple representations that decompose into an irreducible one-dimensional representation of F_2 occurring with multiplicity two.

PROPOSITION 3.3. *If* $\xi \in W_3$, *then* $\mathrm{Bsev}_2(\mathbb{T}_\xi, k, \mathrm{tr}) = \mathrm{Proj}(C[u_1, u_2, u_3]/(f_0^2))$ *where* $f_0 = u_2 + b_1 u_1 + b_2 u_3$ *for appropriate* $b_1, b_2 \in k$.

Proof. From our discussion preceding Proposition 3.2, $\mathrm{Bsev}_2(\mathbb{T}_\xi, k, \mathrm{tr}) = \mathrm{Proj}(C[u_1, u_2, u_3]/(f_0^2))$ if and only if $\beta_1 = \gamma_1$ and $\beta_2 = \gamma_2$. This happens if and only if $\alpha_2^2 - 4\alpha_5 = 0$ and $\alpha_1^2 - 4\alpha_4 = 0$, so $b_1 = \alpha_2/2$ and $b_2 = -\alpha_1/2$. So let P be the ideal generated by $p = a(X,X)a(Y,Y) - a(X,Y)^2$, $\mathrm{tr}(X)^2 - 4\det(X)$, and $\mathrm{tr}(Y)^2 - 4\det(Y)$. Then it is sufficient to show that P is a prime ideal defining W_3 in $V_{2,2}$.

Let P' denote the prime ideal defining W_3 in $V_{2,2}$. As (p) defines $\overline{W}_2$ and $W_3 \subseteq \overline{W}_2$, it follows that $p \in P'$. Next, let $\xi \in W_3$. Then for any $\phi \in (\pi_X)^{-1}(\xi)$, there exists a $g \in PGL_2(k)$ such that

$$\phi^g(X) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{11} \end{pmatrix} \qquad \phi^g(Y) = \begin{pmatrix} y_{11} & y_{12} \\ 0 & y_{11} \end{pmatrix} \tag{37}$$

for some $x_{11}, x_{12}, y_{11}, y_{12} \in k$. Therefore, $\phi(\mathrm{tr}(X)^2 - 4\det(X)) = \phi^g(\mathrm{tr}(X)^2 - 4\det(X)) = 0$ and similarly $\phi(\mathrm{tr}(Y)^2 - 4\det(Y)) = 0$. So $P \subseteq P'$.

Next, we claim that P is a prime ideal of C. Indeed, it is straightforward to see that $\mathrm{tr}(XY) - \frac{1}{2}\mathrm{tr}(X)\mathrm{tr}(Y) \in P$ so $C/P \cong k[\mathrm{tr}(X), \mathrm{tr}(Y)]$ is a domain and thus P is prime. In particular, P is a prime ideal of height 3. It follows from [8, p. 158] that P' is also of height 3. Therefore, $P = P'$.

Q.E.D.

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