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# APPLICATIONS OF ENERGY METHODS TO FINITE-DIFFERENCE SOLUTIONS OF THE PARABOLIC WAVE EQUATION

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**Abstract**—Implicit finite-difference techniques may be applied readily to solve acoustic wave-propagation problems in diverse ocean environments. For only the most simple cases, however, can the stability of such schemes be established. In this paper the method of energy inequalities is applied to examine the stability and step-size requirements for solving the parabolic approximation to the wave equation.

## 1. INTRODUCTION

The parabolic approximation to the wave equation (PE) was first applied by Tappert[1], who used the split-step Fourier algorithm[2] to effect its solution. McDaniel[3] noted that implicit finite-difference (IFD) solutions offered greater flexibility in the range of ocean environments to which the PE could be applied. Lee *et al.*[4] developed an IFD solution to the PE which was subsequently extended[5, 6] to include a treatment of the boundary between media of differing densities and indices of refraction, such as that which occurs at the water–sediment interface. Using the Fourier transform, we have demonstrated the stability (and hence the convergence) of the solutions obtained for the case where the propagation media show no dependence on range. However, when applied to media having range-dependent refractive indices, the Fourier transform method fails.

In this paper the method of energy inequalities developed by Samarsky and Gulin[7] is used to demonstrate the stability of IFD solutions to the PE in range-dependent environments. For both continuous media and diverse media having horizontal interfaces, the energy method may be used to establish stability, and it also has applications for selecting range step sizes to bound the error in the norm of the field. In the case of diverse media separated by nonhorizontal interfaces, the energy method fails to establish stability. The energy method also fails when applied to an alternative scheme for treating irregular interfaces.

For clarity, the problem addressed in this paper is that of propagation in a two-dimensional ocean, where the medium is assumed lossless and Dirichlet boundary conditions are assumed to hold at the surface and at a deep interface within the bottom. In Section 2 the basic equation of interest to us is introduced. Its IFD solution is presented, and the failure of the Fourier transform method to establish stability is demonstrated. In Section 3 the method of energy inequalities is applied to the stability of IFD solutions for propagation in a continuous range-dependent medium and in Section 4 is extended to the horizontal interface. Section 5 considers the stability of IFD solutions for diverse media with nonhorizontal interfaces, and Section 6 addresses the implications of the results for selecting range step increments.

## 2. THEORETICAL BACKGROUND

Propagation in a two-dimensional continuous ocean medium is described by the reduced wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + k_0^2\, \epsilon(x, z)p = 0, \tag{1}$$

where $p$ is the pressure, $k_0$ is an arbitrarily chosen reference wavenumber, and $\epsilon$, the square of the index of refraction, is dependent on both range $x$ and depth $z$. The medium is lossless: $k_0$ and $\epsilon$ are real. To obtain the parabolic approximation, $p = u(x, z) \exp(ik_0x)$ is substituted



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into Eq. (1). With the assumption that $u(x, z)$ is weakly dependent on $x$, so that $\partial^2 u/\partial^2 x$ may be neglected, the result is

$$2ik_0 \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial z^2} + k_0^2 (\epsilon - 1)u = 0. \tag{2}$$

If eq. (2) is discretized on a range and depth grid $(nk, mh)$, its solution[4] is given by

$$(I - iA^{n+1})\, \mathbf{u}^{n+1} = (I + iA^n)\, \mathbf{u}^n, \tag{3}$$

where $A$ is a real symmetric tridiagonal matrix and $\mathbf{u}$ is a column vector. The diagonal elements of $A$ are

$$A^n_{mm} = a^n_m = k[(k_0/4)(\epsilon^n_m - 1) - 1/2h^2k_0],$$

and the off-diagonal elements are $b = k/4h^2k_0$.

To prove stability using the Fourier transform method, we expand $u^n_m$ as a Fourier series:

$$u^n_m = \sum_s \psi^n_s \exp{(ihms)}, \tag{4}$$

and the stability of the difference scheme is established for each Fourier component $s$ by using the ratio test. Substituting $\psi^n_s \exp{(imhs)}$ into eq. (3) and performing the matrix operations yield

$$|\psi^{n+1}_s/\psi^n_s| = |(1 + id^n_m)/(1 - id^{n+1}_m)|, \tag{5}$$

where $d^n_m = a^n_m + 2b \cos{(hs)}$. The finite-difference scheme, in this case, is stable if $|\psi^{n+1}_s/\psi^n_s| \le 1$. If $\epsilon^n_m = \epsilon^{n+1}_m$, the method is stable. However, for a medium having a range-dependent index of refraction $\epsilon^n_m \ne \epsilon^{n+1}_m$, the Fourier transform method fails to establish stability. When a nonhorizontal interface separates media of different densities, the Fourier transform method is inapplicable because the basis functions used in Eq. (4) are not a complete set of functions obeying the boundary conditions at the interface.

## 3. THE METHOD OF ENERGY INEQUALITIES

To apply the method of energy inequalities[7], we expand the field $\mathbf{u}^n$ on a local set of eigenvectors of $A^n$:

$$\mathbf{u}^n = \sum_s \phi^n_s\, v^n_s, \tag{6}$$

where

$$A^n \mathbf{v}^n_s = \lambda^n_s\, \mathbf{v}^n_s. \tag{7}$$

Because $A$ is a real symmetric matrix, the eigenvalues $\lambda_s$ are real, and the eigenvectors $\mathbf{v}_s$ are complete and orthonormal:

$$\sum_s v^*_{sm}\, v_{sj} = \delta_{jm}, \tag{8}$$

$$\sum_m v^*{}_{sm}\, v_{rm} = \delta_{sr}, \tag{9}$$

where $v_{sj}$ is the $j$th element of the $s$th eigenvector, and * denotes the complex conjugate of the transposed matrix.

Substituting Eq. (6) into Eq. (3) and making use of Eq. (7) then yield

$$\sum_s \phi_s^{n+1}(1 - i\lambda_s^{n+1})\mathbf{v}_s^{n+1} = \sum_s \phi_s^n(1 + i\lambda_s^n)\mathbf{v}_s^n. \tag{10}$$

The transposed complex conjugate of Eq. (10) is

$$\sum_r \mathbf{v}_r^{*n+1}(1 + i\lambda_r^{n+1})\phi_r^{*n+1} = \sum_r \mathbf{v}_r^{*n}(1 - i\lambda_r^n)\phi_r^{*n}. \tag{11}$$

Multiplying, respectively, the right and left sides of Eq. (10) by those of Eq. (11) and using Eq. (9) yield

$$\sum_s \phi_s^{*n+1}\phi_s^{n+1}|1 + i\lambda_s^{n+1}|^2 = \sum_s \phi_s^{*n}\phi_s^n|1 + i\lambda_s^n|^2. \tag{12}$$

From Eqs. (6) and (9) the norm of the field $\|\mathbf{u}^n\|$ is given by

$$\|\mathbf{u}^n\|^2 = \sum_m u_m^{*n}u_m^n = \sum_s \phi_s^{*n}\phi_s^n. \tag{13}$$

Because the $\lambda$ are real,

$$\sum_s \phi_s^{*n}\phi_s^n \leq \sum_s \phi_s^{*n}\phi_s^n|1 + i\lambda_s^n|^2,$$

so that, for a fixed step size $k$,

$$\|\mathbf{u}^n\|^2 \leq \sum_s \phi_s^{*q}\phi_s^q|1 + i\lambda_s^q|^2 = M \tag{14}$$

for all $q$ and $n$. Thus, if the initial data ($q = 0$) and $\lambda_s^0$ are bounded (which is easily arranged in practice), $M$ will be finite, and the norm of the field will remain finite for all $n$. Thus, for bounded input data, the difference scheme is stable[7]. The method of energy inequalities does not require a range-independent medium for stability.

## 4. THE HORIZONTAL INTERFACE

The energy method may be readily extended to demonstrate the stability of IFD solutions to the PE when a horizontal interface separates two media of differing densities and indices of refraction, as shown in Fig. 1. The density $\rho$ in each medium is assumed to be independent of

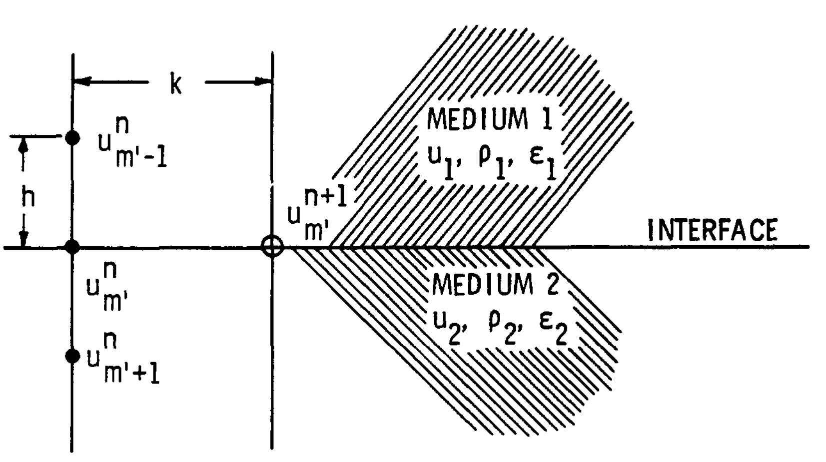


Fig. 1. The horizontal interface between media of different densities and indices of refraction.

range, whereas the square of the refractive indices $\epsilon_1(x, z)$ and $\epsilon_2(x, z)$ may vary arbitrarily. In this case the IFD solution[5] is given by

$$(I - i\,C^{n+1})\mathbf{u}^{n+1} = (I + i\,C^{n})\mathbf{u}^{n}, \tag{15}$$

where $C$ is a real tridiagonal matrix.

Away from the interface, $m \neq m'$, $C_{mj} = A_{mj}$. On the interface,

$$\begin{aligned} C^{n}_{m',m'-1} &= 2\rho_2 b/(\rho_1 + \rho_2), \\ C^{n}_{m',m'} &= (\rho_1 a^{n}_{m'2} + \rho_2 a^{n}_{m'1})/(\rho_1 + \rho_2), \\ C^{n}_{m',m'+1} &= 2\rho_1 b/(\rho_1 + \rho_2), \end{aligned} \tag{16}$$

where the $\rho_1 > 0$. For $\rho_1 \neq \rho_2$, $C$ is not symmetric. To symmetrize $C$, we introduce the diagonal positive definite matrix $B$:

$$B_{mm} = \begin{cases} 1/\rho_1, & m < m', \\ (\rho_1 + \rho_2)/2\rho_1\rho_2, & m = m', \\ 1/\rho_2; & m > m'. \end{cases} \tag{17}$$

With this, Eq. (15) may be written as

$$B^{-1/2}(I - iD^{n+1})\, B^{1/2}\mathbf{u}^{n+1} = B^{-1/2}(I + iD^{n})\, B^{1/2}\mathbf{u}^{n}, \tag{18}$$

where $D = B^{1/2}\, C\, B^{-1/2}$ is a real symmetric matrix. Defining

$$\mathbf{w} = B^{1/2}\,\mathbf{u} \tag{19}$$

and multiplying both sides of Eq. (18) by $B^{1/2}$ then yield

$$(I - iD^{n+1})\,\mathbf{w}^{n+1} = (I + iD^{n})\,\mathbf{w}^{n}. \tag{20}$$

Using the methods of Section 3, we can project $\mathbf{w}$ onto the eigenvectors of $D$ and readily establish that

$$\|\mathbf{w}^{n}\|^2 \geq \sum_{s} \phi^{*q}_{s}\phi^{q}_{s}\,|1 + i\lambda^{q}_{s}|^2 = M, \tag{21}$$

where the $\lambda_s$ are, in this case, the real eigenvalues of $D$. Thus, for bounded input data the difference scheme represented by Eq. (21) is stable. The finite-difference scheme given by Eq. (15) is said to be stable in the $B$-norm[8].

$$\|\mathbf{u}^{n}\|^2_B = \sum_{m} u^{*n}_{m} B u^{n}_{m} \leq M, \tag{22}$$

where $M$ is finite.

Note that the assumption that $B$ was constant, requiring constant $\rho_1$ and $\rho_2$, was necessary to cast Eq. (18) into a form [Eq. (20)] to which the method of energy inequalities is applicable. Without this assumption the energy method fails, as will become apparent in the treatment of sloping interfaces.

## 5. SLOPING INTERFACES

Lee and McDaniel[6] developed an IFD technique to handle irregular interfaces between media of different densities. The geometry considered is shown in Fig. 2, where the irregular

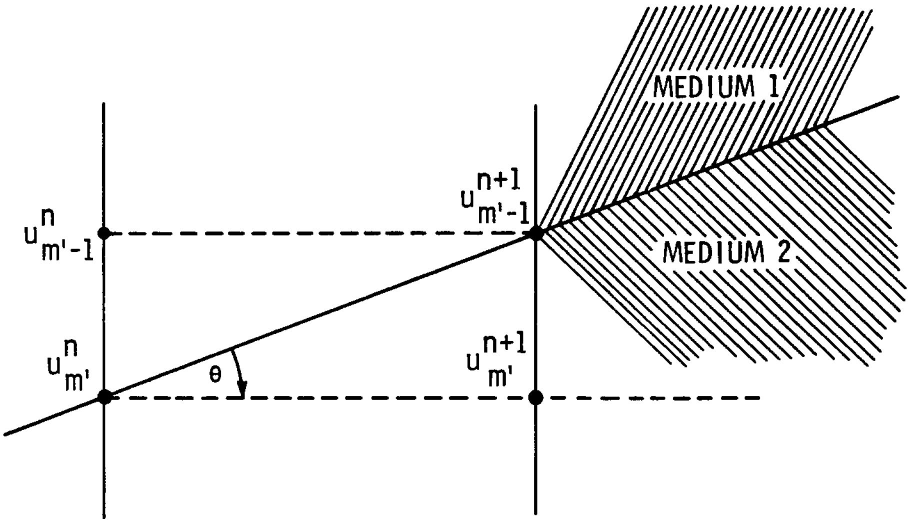


Fig. 2. The sloping interface between two media.

boundary is approximated by an interface of constant slope passing through the grid points. For propagation in a two-dimensional ocean, the scheme of Ref. [6] takes the form

$$(I - i\,E^{n+1})\mathbf{u}^{n+1} = (I + i\,E^{n})\mathbf{u}^{n}, \tag{23}$$

where $E$ is a real tridiagonal matrix:

$$\begin{aligned} E^{n}_{mj} &= A^{n}_{mj} \quad \text{for } m \neq m', \quad \text{and} \quad E^{m}_{m',m'-1} = \frac{k\gamma_2}{2i}, \\ E^{n}_{m',m'} &= \frac{k(\alpha_2 - \beta_2 - \gamma_2)}{2i}, \quad E^{n}_{m',m'+1} = \frac{k\beta_2}{2i}. \end{aligned} \tag{24}$$

For

$$\begin{aligned} &m \neq m' - 1,\ E^{n+1}_{mj} = A^{n+1}_{mj} \quad \textit{and} \quad E^{n+1}_{m'-1,m'-2} = k\gamma_1/2i, \\ &E^{n+1}_{m'-1,m'-1} = k(\alpha_1 - \beta_1 - \gamma_1)/2i, \quad E^{n+1}_{m'-1,m'} = k\beta_1/2i. \end{aligned} \tag{25}$$

Expressions for $\alpha_1$, $\beta_1$ and $\gamma_i$ may be obtained from Ref. [6] by substituting $ik_0$ for $v_r/v$.

From Eqs (24) and (25) it is clear that the matrices $E^n$ and $E^{n+1}$ may be symmetrized by a transformation similar to that used for the horizontal interface. It is also, however, clear that the same matrix will not symmetrize both $E^n$ and $E^{n+1}$. Hence, as anticipated, the method of energy inequalities fails to establish the stability of the scheme represented by Eq. (23) because it cannot be cast into the required form.

If the sloping interface is approximated by a stairstep, Fig. 3, and the stable finite-difference scheme of Section 4 is employed, the stability of the resulting solution cannot be established by energy methods. To show this, we must recognize that a complete step, in this case, consists of two substeps. The first substep is the solution of Eq. (20) for an intermediate value $\mathbf{w}_i^{n+1}$, and the second is the assignment of new densities to $\mathbf{w}_i^{n+1}$ to yield $\mathbf{w}^{n+1}$ in preparation for the next application of Eq. (20).

Thus, the first substep yields

$$\mathbf{w}_i^{n+1} = (I - iD^{n+1})^{-1}(I + iD^{n})\,\mathbf{w}^{n}. \tag{26}$$

The second substep may be represented by

$$\mathbf{w}^{n+1} = H^{n+1}\,\mathbf{w}_i^{n+1}. \tag{27}$$

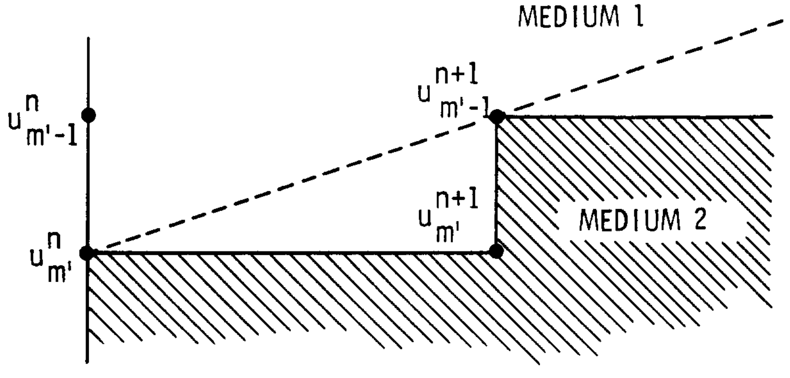


Fig. 3. The "stairstep" interface between diverse media.

where $H$ is a diagonal matrix having elements $H_{mm} = 1$, for $m \neq m' - 1, m'$, and

$$H_{m'-1,m'-1} = [(\rho_1 + \rho_2)/2\rho_2]^{1/2}, \quad H_{m',m'} = [2\rho_1/(\rho_1 + \rho_2)]^{1/2}. \tag{28}$$

Combining Eqs (27) and (28) then yields

$$(I - iD^{n+1})(H^{n+1})^{-1}\,\mathbf{w}^{n+1} = (I + iD^{n})\,\mathbf{w}^{n}, \tag{29}$$

for which the energy method fails. Although the stability of neither method considered for treating sloping interfaces can be demonstrated, the scheme of Eq. (23) is preferable because it includes an exact treatment of the boundary conditions at the interface.

## 6. STEP-SIZE SELECTION

Both the range step size and the reference wavenumber $k_0$ may be selected to bound the norm of the field. To develop expressions for the dependence of $\|\mathbf{u}\|$ on $k$ and $k_0$, the fact that the eigenvectors of $A$ are those of the local acoustic normal modes will be used. Thus, for $A$,

$$\lambda_s = \frac{k}{2}\,\frac{[k_0^2 - k_s^2]}{2k_0}, \tag{30}$$

where $k_s$ is the solution of

$$(v^n_{s,m+1} - 2v^n_{s,m} + v^n_{s,m-1})/h^2 + (k_0^2\epsilon^n_m - k_s^2)v^n_{s,m} = 0, \tag{31}$$

subject to the appropriate boundary conditions. Thus, if our input data consist of a single normal mode, the $r$th for example, $\phi^0_s = 0$ for $s \neq r$, and with the choice $k_0 = k_r$, Eq. (14) becomes

$$\|\mathbf{u}^n\|^2 \leq \phi^{*0}_r\phi^0_r = \|\mathbf{u}^0\|^2. \tag{32}$$

In this case the norm of the propagated field will be bounded by the norm of the input field.

In most underwater applications, however, the input field consists of a sum over several modes to approximate an omnidirectional acoustical source. At progressive range steps the modal content of the field decreases until at long ranges only a few dominant lower-order modes remain. This pattern suggests that the norm of the field may be suitably bounded with the choice of range steps in the vicinity of the source shorter than those used at long ranges. The bound on the norm of the field is, in this case, weaker than that obtained in Eq. (14):

$$\|\mathbf{u}^n\|^2 \leq \sum_s \phi^{*q'}_s\phi^{q'}_s\,|1 + i\lambda^{q'}_s|^2. \tag{33}$$

where $q'$ denotes the interval having the *greatest* range increment, and $n \geqslant q'$. A similar bound is obtained for the norm of $\mathbf{w}$ when horizontal interfaces are present. An analysis of the phase errors[3] that arise from the local truncation error also indicates that it is desirable to use small range steps at short ranges and to select $k_0$ to correspond closely to the eigenvalues of the dominant normal modes.

## CONCLUSIONS

The method of energy inequalities has been applied to establish the stability of IFD solutions to the PE for propagation in a range-dependent ocean. Whereas the energy method succeeds when horizontal interfaces separate media of different densities, it fails for nonhorizontal interfaces. The results obtained imply a step size and reference wavenumber choice that is in qualitative agreement with that obtained from a consideration of the local truncation error.

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